FAMILY NAME	______________________
GIVEN NAME(S)	______________________
STUDENT NUMBER	______________________
DATE	any date
START TIME	any time
FINISH TIME	three hours after the start time
EXAMINER	Norman Do

Instructions

1. Fill in your name and student number clearly in the space provided above.
2. Do not remove any pages from this booklet — all of your writing, even rough work, must be handed in.
3. Calculators, books and notes are not permitted.
4. You are encouraged to spend the first few minutes reading through the problems. The problems are not ordered in terms of difficulty and the number of marks available does not indicate the difficulty of the problem.
5. Please write your rough work and solution in the space provided on the page where the question is printed. If you use up all of that space, then you may write on the back of the page or on one of the pages provided at the back of the booklet. If you use up all of that space as well, then please raise your hand and ask the invigilator for another sheet of paper.
6. The examination booklet should consist of this cover, pages 1 through 10 which contain the questions, and pages 11 through 15 which are blank. Please inform the invigilator if your booklet is defective.

1. For each of the following statements, indicate whether it is true or false. If you indicate that a statement is true, then you must explain why it is true and if you indicate that a statement is false, then you must explain why it is false.

   (a) The incentre of a triangle always lies inside the triangle. (Recall that the incentre of a triangle is where the angle bisectors meet.)

   (b) Every isometry can be expressed as the composition of four or fewer reflections.

   (c) If the circumcentre of triangle ABC and the centroid of triangle ABC are the same point, then triangle ABC must be equilateral. (Recall that the circumcentre of a triangle is where the perpendicular bisectors meet and the centroid of a triangle is where the medians meet.)

   (d) In the hyperbolic plane, there exists a triangle with angles 1°, 10° and 100°.

   (e) Let e be the identity in a group G. If g and h are two elements in G such that $g \cdot h = e$, then it is also true that $h \cdot g = e$.

[2 + 2 + 2 + 2 = *10 marks*]

2. Let the incentre of triangle ABC be I and let its incircle touch the sides AB and AC at R and Q, respectively. Let the line segment AI meet the incircle at J.

   (a) Prove that the quadrilateral $AQIR$ is cyclic.

   (b) Prove that $2\angle JRQ = \angle ARQ$.

   (c) Deduce that J is the incentre of triangle ARQ.

[2 + 6 + 2 = *10 marks*]

3. (a) Fill in each entry of the middle column with the word *direct* or *opposite* and each entry of the right column with the word *yes* or *no*.

ISOMETRY	DIRECT OR OPPOSITE	FIXED POINTS
translation		
rotation		
reflection		
glide reflection		

(b) Suppose that $ABCD$ is a square with the vertices labelled counterclockwise. Let

- G_{CB} denote the glide reflection consisting of a reflection in the line CB followed by a translation which takes C to B; and
- M_{CD} denote the reflection in the line CD;
- R_C denote a counterclockwise rotation by $90°$ about C; and
- G_{AC} denote the glide reflection consisting of a reflection in the line AC followed by a translation which takes A to C.

Identify the composition $G_{CB} \circ M_{CD} \circ R_C \circ G_{AC}$.

(c) If X denotes the composition $G_{CB} \circ M_{CD} \circ R_C \circ G_{AC}$, let n be the minimum number of reflections whose composition is equal to X. Determine the value of n and carefully describe n reflections whose composition is equal to X.

[*2 + 4 + 4 = 10 marks*]

4. (a) The following is a *groupoku* puzzle — a Cayley table for the group G, where some of the entries are missing. Use the properties which you know about Cayley tables to fill in all of the missing entries. Give full reasoning only for the first entry of the table that you manage to fill in.

$\circ$	a	b	c	d	e	f
a	*	*	*	c	b	d
b	a	*	*	*	*	*
c	*	*	b	*	f	*
d	f	d	*	*	*	a
e	*	*	d	*	*	c
f	*	*	a	e	*	*

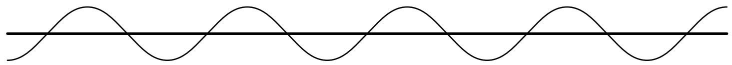
(b) In the lectures, we have seen two groups with six elements — C_6 and D_3 — which are not isomorphic to each other. Prove that G is isomorphic to one of these groups by writing down the explicit isomorphism. How many isomorphisms are there?

(c) Give a subset of the Euclidean plane whose symmetry group is isomorphic to G. Does there exist a finite set of points in the Euclidean plane whose symmetry group is isomorphic to G? If so, determine the minimum number of points in such a set. If not, give a brief explanation of why such a set does not exist.

(d) Why are the cyclic group C_6 and the dihedral group D_3 not isomorphic to each other?

[*3 + 3 + 2 + 2 = 10 marks*]

5. (a) Write down the $HVRG$ symbol for the frieze pattern below.



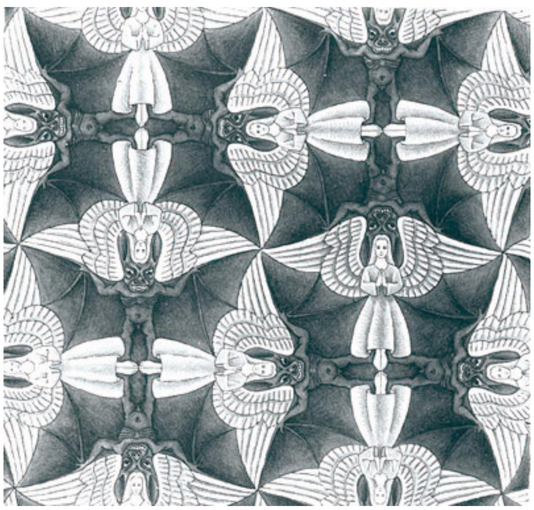
(b) Write down the $HVRG$ symbol for the frieze pattern below.

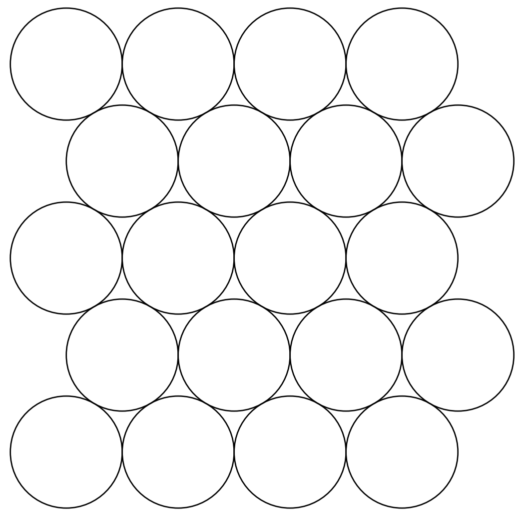
VΛNVΛNVΛNVΛNVΛNVΛNV

(c) Write down the RMG symbol for the wallpaper pattern pictured below left.

(d) Write down the RMG symbol for the wallpaper pattern pictured below right. Mark clearly on the diagram a point which is the centre of a rotational symmetry of order R, a point where M mirrors meet, and a point where G proper glide axes meet.

(e) Prove that if the symmetry group of a frieze pattern contains a rotation by 180° and a glide reflection, then it must also contain a reflection in a vertical mirror.





[*1 + 1 + 2 + 4 + 2 = 10 marks*]

6. Suppose that you have a polyhedron with V vertices, E edges and F faces. You are told that each face is a pentagon or a hexagon and that three faces meet at every vertex. Furthermore, every pentagonal face shares an edge with five hexagonal faces, while every hexagonal face shares an edge with three pentagonal faces and three hexagonal faces.

   (a) Let P be the number of pentagonal faces and H be the number of hexagonal faces of the polyhedron. Prove that the following equations hold.

   $$V = \frac{5}{3}P + 2H \qquad\qquad E = \frac{5}{2}P + 3H$$

   (b) Prove the equation $5P = 3H$.

   (c) Use these equations — along with anything else you know about the polyhedron — to deduce the values of P and H.

   (d) Draw a planar graph corresponding to such a polyhedron and draw an example of such a polyhedron.

   (e) Cut off each vertex of the polyhedron using a plane which passes through the midpoints of the three edges which meet at that vertex. This process produces a convex polyhedron from a convex polyhedron. For example, applying this process to a tetrahedron produces an octahedron. How many vertices, edges and faces does this new polyhedron have? How many triangular faces, quadrilateral faces, pentagonal faces, and hexagonal faces does this new polyhedron have?

   [*3 + 1 + 2 + 2 + 2 = 10 marks*]

7. (a) Fill in each entry of the middle column with the word *orientable* or *non-orientable* and each entry of the right column with a number or mathematical expression for the Euler characteristic.

SURFACE	ORIENTABILITY	EULER CHARACTERISTIC
sphere		
connect sum of g tori		
connect sum of n projective planes		

(b) Determine whether or not the surface corresponding to the edge word $aceb^{-1}deabd^{-1}c^{-1}$ is orientable and calculate its Euler characteristic.

(c) Identify the surface using the table above.

(d) Write down the shortest possible edge word for the surface.

(e) If the surface is homeomorphic to $T \# X$ where T represents a torus and X represents some surface, then what is X?

[*4 + 2 + 2 + 2 = 10 marks*]

8. (a) I would like to tile a 10×10 square with $a \times b$ rectangles, where a and b are positive integers which may or may not be equal. Determine all values of a and b for which this is possible.

   (b) Use a colouring argument to prove that it is not possible to tile a 10×10 rectangle with 4×1 rectangles.

   (c) Hence, deduce that it is not possible to tile a 10×10 rectangle with 4×5 rectangles.

   (d) It is possible to tile a 10×10 rectangle with thirty-three 3×1 rectangles and one 1×1 square. Determine all possible locations for the 1×1 square and prove that these are the only ones possible.

[*3 + 3 + 1 + 3 = 10 marks*]

9. Let G_{XY} denote the glide reflection which is the composition of a reflection in the line XY followed by a translation which takes X to Y.
   (a) If $ABCD$ is a rectangle, show that the composition $G_{DA} \circ G_{CD} \circ G_{BC} \circ G_{AB}$ is equal to the identity. In other words, if you *glide around a rectangle*, then the result is the identity.
   (b) If you glide around an arbitrary quadrilateral, show that the result is not necessarily the identity.
   (c) Which quadrilaterals can you glide around so that the result is the identity?

[*6 + 2 + 2 = 10 marks*]

10. (a) Determine all surfaces which can be obtained by gluing the sides of a quadrilateral in pairs.
    (b) Determine all surfaces which can be obtained by gluing the sides of a hexagon in pairs.

[*6 + 4 = 10 marks*]

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