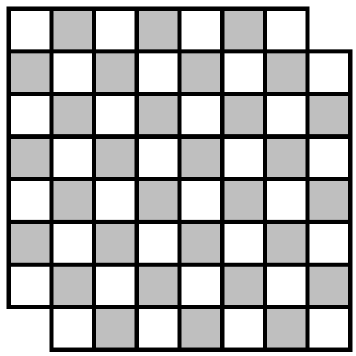
The Mutilated Chessboard

One of the most famous of tiling conundrums is the following, a problem which almost every mathematician must have encountered at one time or another.

> Consider an 8×8 chessboard, where the top-right and bottom-left squares have been removed. Is it possible to tile this mutilated chessboard with 2×1 dominoes?



The first thing you should do is take out some pen and paper, draw a mutilated chessboard, and try to tile it with 2×1 dominoes. However, I can tell you right now that you'll fail — not because your tiling skills are inadequate, but because the task is impossible. The answer to this problem should seem surprising to an unsuspecting audience.

Prior to removing the two squares, there is a myriad of ways to perform such a domino tiling — actually, $3604^2 = 12988816$ ways to be precise, but that's another story. So why should such a trivial alteration of the board reduce this number to zero? The argument is stunning in its simplicity and the key to the solution lies in the seemingly unimportant colouring of the chessboard into black and white squares. This colouring is such that the placement of any domino on the board will cover exactly one square of each colour. So if it's possible to tile the board with dominoes, then it must be the case that there is an equal number of black and white squares. However, a quick count reveals that the mutilated chessboard has 30 black squares and 32 white squares. A slicker way to see that there are unequal numbers of black and white squares is to notice that we removed two squares of the same colour from a board that previously had equal numbers of each. From this disparity, we are led to the conclusion that the mutilated chessboard cannot be tiled by dominoes, no matter how hard one might try.

Colouring Arguments

We were lucky with the mutilated chessboard problem, because the standard 8×8 chessboard came with a colouring which helped our cause, free of charge. But sometimes, as in the next problem, you have to invent your own colouring.

> Is it possible to tile a 10×10 square with 4×1 rectangles?

Once again, try as you might, you'll find that it's impossible to tile a 10×10 board with 4×1 rectangles. So it seems like a good idea to generalise the colouring trick which worked for the mutilated chessboard so that it works for this problem too. And if it works for this problem too, then who knows how many other problems this colouring trick will work for?

The crucial aspect of the chessboard colouring that we used is the fact that any domino placed on the board occupied one square of each colour. So the idea here is to find a colouring of the 10×10 square such that any 4×1 rectangle placed on the board occupies one square of each colour. Of course, this means that we require four colours, which we will call 0, 1, 2 and 3. Working along the top row of the board, we may as well label the first four squares 0, 1, 2 and 3, in that order. In order to satisfy the property that any 4×1 rectangle placed on the board occupies one square of each colour, the next square along must be labelled 0. And the next square along must be labelled 1, and the next square along must be labelled 2, and the next square along must be labelled 3, and so on. So we see that every square in the first row will be coloured according to the repeating pattern $0, 1, 2, 3, 0, 1, 2, 3, \ldots$. Since this seems to work along the first row, we can use the same trick to fill the first column. And after a little trial and error, you should find the following very pretty looking colouring of the 10×10 square.

0	1	2	3	0	1	2	3	0	1
1	2	3	0	1	2	3	0	1	2
2	3	0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2	3	0
0	1	2	3	0	1	2	3	0	1
1	2	3	0	1	2	3	0	1	2
2	3	0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2	3	0
0	1	2	3	0	1	2	3	0	1
1	2	3	0	1	2	3	0	1	2

I like to call this a *modulo 4 colouring*, because if we label the rows and columns $0, 1, 2, \ldots$, then the square in row i and column j is coloured $i + j$ modulo 4. If you have no idea what I'm talking about, then that's fine, because the colouring is easy to describe without all of this jargon. You simply cycle through the colours 0, 1, 2, 3 along the first row, and every other row is the same as the previous one, but shifted to the left by one. Hopefully, I don't need to tell you that for other problems, you might need to use a modulo k colouring for some positive integer k.

This colouring certainly obeys the rule that a 4×1 rectangle on the board always occupies one square of each colour. You can easily check this for a 4×1 rectangle placed in the first row which means that it's also true for a 4×1 rectangle placed in any other row or any column. Of course, we're hoping that there are not the same number of squares of each colour, so that we can deduce that it's impossible to tile the board. One way to verify this is to simply count them and you would indeed find that this is true — there are twenty-five squares coloured 0, twenty-six squares coloured 1, twenty-five squares coloured 2, and twenty-four squares coloured 3. However, that's rather pedestrian, so let's use a slicker, more stylish, approach.

We simply note that it's possible — and quite easy to demonstrate — a tiling of the entire board except for the 2×2 square in the top-left corner. This is because any $4 \times n$ rectangle is very easy to tile with 4×1 rectangles. So you can tile the bottom four rows of the grid, leaving a 6×10 rectangle. Then you can tile the bottom four rows of this grid, leaving a 2×10 rectangle. Now you can tile the rightmost four columns of this grid, leaving a 2×6 rectangle. And then you can tile the rightmost four columns of this grid, leaving a 2×2 square.

The part of the board which we've covered in tiles must certainly contain the same number of squares of each colour, otherwise we wouldn't have been able to tile it. Since the remaining part of the board does not — there is one square coloured 0, two squares coloured 1, one square coloured 2 and zero squares coloured 3 — there cannot be the same number of squares of each colour on the entire board. We conclude that a 10×10 square cannot be tiled with 4×1 rectangles.

These colouring arguments are extremely useful and are most commonly applied with a modulo k colouring for some positive integer k. Hopefully you can imagine what such a colouring might look like and how such an argument might work, but if not, then we'll see an example very soon. One thing to keep in mind is that a colouring argument can be used to prove that a tiling is impossible, but it can never ever be used to prove that a tiling is possible. Anyway, to prove to someone that a particular tiling is possible is usually easy — you just have to demonstrate it to them.

Tiling Rectangles with Skinny Rectangles

Since a colouring argument was so successful for the previous problem, we may as well try to solve the following far more general problem.

> For which values of m, n and k is it possible to tile an $m \times n$ rectangle with $k \times 1$ rectangles?

If someone gives you a problem like this, the very first thing you should do with it is experiment with various values of m, n and k. One thing you should realise very quickly is that if m is a multiple of k, then the tiling is really easy to find. And that's because the first column consists of m squares and can be tiled with $\frac{m}{k}$ rectangles. Once you can tile the first column this way, then you can tile every column on the board in this way. Similarly, the tiling is very easy to find if n is a multiple of k. But what happens if neither m nor n is a multiple of k? Well, you should find that the task is impossible, and that's precisely what we're going to prove.

Theorem. *It's possible to tile an $m \times n$ rectangle with $k \times 1$ rectangles if and only if m is a multiple of k or n is a multiple of k.*[1]

Proof. We've already shown that if m is a multiple of k or n is a multiple of k, then the tiling is easy to find. So let's now assume that m and n are not multiples of k and prove that the tiling is impossible.

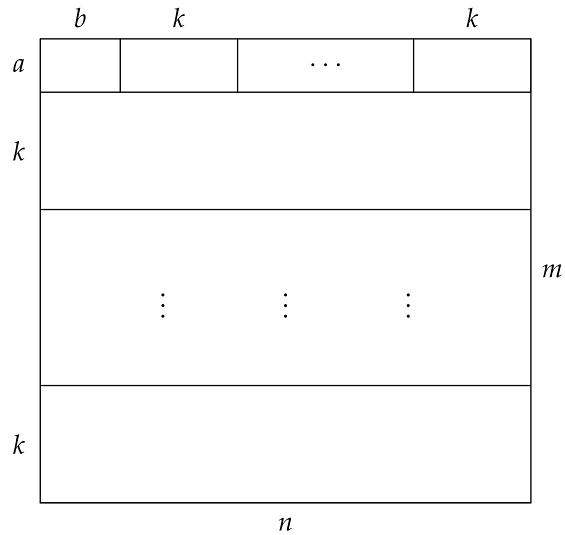
This is where our coloured pencils come to the rescue. Hopefully, you haven't forgotten the problem we solved earlier about tiling with 4×1 rectangles. We solved that one by using a colouring which repeats every four squares. For this problem, we simply use a colouring which repeats every k squares. The strategy here is the same — fill out the first row by cycling through the colours $0, 1, 2, \ldots, k-1$ and let every other row be the same as the previous row one, but shifted to the left by one. I just happen to be using the colours $0, 1, 2, \ldots, k-1$ because that's what I'm used to. You're more than welcome to use the colours $1, 2, 3, \ldots, k$ or even real colours — whatever takes your fancy.

Now if m is not a multiple of k, then it must leave some remainder a after you divide by k. Similarly, if n is not a multiple of k, then it must leave some remainder b after you divide by k. And these numbers a and b can't be any old numbers — they must be positive integers which lie strictly between 0 and k.

[1] In mathematics, when we say A or B, we allow the possibility that both A and B could happen. It's not like when you go to a friend's house and they ask you whether you want tea or coffee, in which case they are usually excluding the fact that you might want both.

We'll also make the assumption that $a \leq b$ which we certainly can do, because if it wasn't true, then we could just switch the names of m and n and the names of a and b to make it true.[2]

Note that it's possible — and quite easy to demonstrate — a tiling of the entire board except for the $a \times b$ square in the top-left corner. This is because you can tile the bottom k rows of the grid, leaving an $(m-k) \times n$ rectangle. Then you can tile the bottom k rows of this grid, leaving an $(m-2k) \times n$ rectangle. And you can keep tiling the bottom k rows of the grid, until you are left with an $a \times n$ rectangle. Now you can tile the rightmost k columns of this grid, leaving an $a \times (n-k)$ rectangle. Then you can tile the rightmost k columns of this grid, leaving an $a \times (n-2k)$ rectangle. And you can keep tiling the rightmost k rows of the grid, until you are left with an $a \times b$ rectangle.



Remember that our goal is to show that there aren't equal numbers of squares of each colour on the entire board. The trick we've used here is to tile a large part of the board, which tells us that the tiled part definitely does have equal numbers of squares of each colour. In other words, we've reduced the problem to showing that there aren't equal numbers of squares of each colour in the $a \times b$ rectangle in the top-left corner.

All we have to do now is note the following two things.

- Note that the colour $a-1$ appears in the bottom-left corner. In fact, it has to appear in every row of the $a \times b$ rectangle. This is because it also appears in the square one up and one right of the bottom-left corner, and in the square one up and one right from that one, and in the square one up and one right from that one, and so on. Since $a \leq b$, this means that the colour $a-1$ appears at least once in every row. The fact that $b < k$ tells us that no colour can appear more than once in a row. So, in summary, colour $a-1$ actually appears exactly a times, once in each row.

0	1	2	$\cdots$	$b-2$	$b-1$
1	2	3	$\cdots$	$b-1$	b
2	3	4	$\cdots$	b	$b+1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a-2$	$a-1$	a	$\cdots$		
$a-1$	a	$a+1$	$\cdots$		

[2]Often, mathematicians will say that we can make an assumption of this sort *without loss of generality*. It basically means that the assumption is allowed, and that you are still covering all possible cases, even though it might not at first appear to be so. In fact, such arguments are often referred to as WLOG arguments.

- Note that the colour 0 appears in the top-left corner but it definitely does not appear in the second row — and this is because the second row looks like $1, 2, 3, \ldots$. So if it did appear, then the second row would have at least k squares, clearly in contradiction of our assumption that $b < k$. This means that the colour 0 definitely does not appear in the second row and yet, we already discussed the fact that no colour can appear more than once in a row. So, in summary, the colour 0 appears fewer than a times.

This tells us that the colour 0 and the colour $a - 1$ do not appear the same number of times on the entire board. And this is precisely what we wanted to prove, because we can now deduce that the tiling is impossible. □

Tiling Rectangles with Rectangles

Thus far, we've considered only the case of tiling with skinny rectangles — in other words, those of the form $k \times 1$. Let's now broaden our horizons and consider the more general case of tiling with $a \times b$ rectangles, where a and b are positive integers. Of course, we can start by making the simplifying assumption that a and b have no common factors greater than 1, since other cases reduce to this after scaling the size of the tiles and the board down. For example, if a and b were both even and we wanted to know whether an $m \times n$ rectangle can be tiled with $a \times b$ rectangles, then this is the same problem as determining whether an $\frac{m}{2} \times \frac{n}{2}$ rectangle can be tiled with $\frac{a}{2} \times \frac{b}{2}$ rectangles. Obviously, the question we would like to answer is the following.

> For which values of m, n, a and b is it possible to tile an $m \times n$ rectangle with $a \times b$ rectangles?

Before we state the answer, let's consider three instructive cases.

- *Can you tile a* 12×15 *rectangle with* 4×7 *rectangles?*
  No, of course not, since the area of each tile does not divide the area of the board.
- *Can you tile a* 17×28 *rectangle with* 4×7 *rectangles?*
  The answer is again in the negative, although for a more subtle reason. It turns out that 4×7 rectangles cannot even be used to cover the first column of a 17×28 rectangle. For if such a tiling is possible, we must certainly be able to write the number 17 as a sum of 4's and 7's. A quick check shows that this is not the case.
- *Can you tile an* 18×42 *rectangle with* 4×7 *rectangles?*
  It is not actually possible to carry out this task. If you could, then you could certainly tile the 18×42 rectangle with 4×1 rectangles, by tiling each 4×7 rectangle with seven 4×1 rectangles. But our earlier result — which we proved using a colouring argument — tells us that you can't tile an 18×42 rectangle with 4×1 rectangles because neither 18 nor 42 are multiples of 4.

These arguments can be generalised to prove the following theorem, which gives a complete answer to our original problem.

Theorem. *Let a and b be positive integers with no common factors greater than 1. A tiling of an $m \times n$ rectangle with $a \times b$ rectangles exists if and only if*

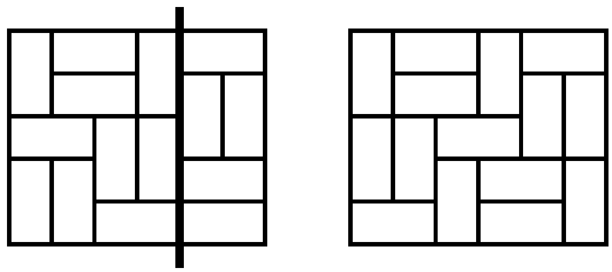
- *both m and n can be written as a sum of a's and b's; and*
- *either m or n is a multiple of a and either m or n is a multiple of b.*

Faulty Tilings

Let's now turn our attention to the following beautiful tiling problem.

> A 6×6 square is tiled with 2×1 dominoes. Prove that it's possible to cut the board into two smaller rectangles with a straight line which doesn't pass through any of the dominoes.

Given a tiling, let's call a line which cuts the board into two pieces and yet does not pass through any of the tiles a *fault line*. For example, the diagram below shows two tilings of a 5×6 rectangle with dominoes, one which has a fault line and one which doesnt. This particular problem asserts that every possible domino tiling of the 6×6 square must have a fault line.



In order to obtain a contradiction, let's suppose that we have a domino tiling of the 6×6 square which has no fault line. Consider any one of the ten potential fault lines — five horizontal and five vertical — and, without loss of generality, we may assume that it is vertical. Since our tiling has no fault line, at least one domino must cross this vertical. However, it cannot be the only such domino, since otherwise, an odd number of squares would remain to the left of the line and this part of the board cannot be tiled with dominoes. So at least two dominoes must cross the given vertical line. The same argument applies for all ten potential fault lines, so at least two dominoes must cross each of the ten potential fault lines. Since a domino may cross only one such line, we conclude that the tiling must involve at least $10 \times 2 = 20$ dominoes. However, 20 dominoes cover an area of 40 squares, more than the area of the board in question. This contradiction implies that every tiling of the 6×6 square with dominoes must have a fault line.

Having solved this question, it's only natural to ask the following more general question.

> When can an $m \times n$ rectangle be tiled with $a \times b$ rectangles without any fault lines?

Despite first appearances, there is a natural answer to this problem as described by the following result. Interestingly enough, the case of tiling a 6×6 rectangle with dominoes which had such an elegant proof, is the only exception to the rule.

Theorem. *Let a and b be positive integers with no common factors greater than 1. A faultless tiling of an $m \times n$ rectangle with $a \times b$ rectangles exists if and only if*

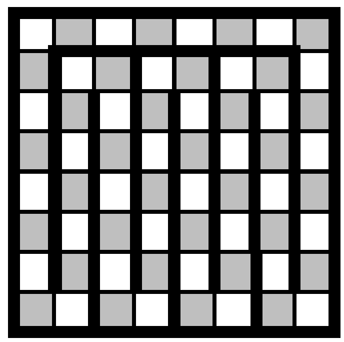
- *both m and n can be written as a sum of a's and b's in such a way that at least one a and one b is used;*
- *either m or n is a multiple of a, and either m or n is a multiple of b; and*
- *for the case where the tiles are dominoes, the rectangle is not* 6×6.

More Mutilated Chessboards

We started with the mutilated chessboard problem — from such humble beginnings, we began our journey into the amazing world of tiling. The mutilated chessboard problem spawns a further interesting question whose answer is not quite so well-known.

> Which pairs of squares may be removed from the regular 8×8 chessboard so that the remaining board can be tiled with dominoes?

Of course, the colouring argument we used to solve the mutilated chessboard problem implies that any such pair of squares must be of opposing colours. But if we remove two such squares, is it always possible to tile the remaining board with dominoes? The answer is in the affirmative and the simplest proof requires us to consider the chessboard as a labyrinth, as pictured below. This labyrinth is hardly the design that might be used for a hedge maze, since it not only has no entrance and exit, but also consists simply of a tour which traverses all of the 64 squares. All that's required now is to note that the removal of two squares of opposite colours divides the path now into two shorter paths, one of which may be empty. Furthermore, these two paths are of even length, so it's easy to tile them both.

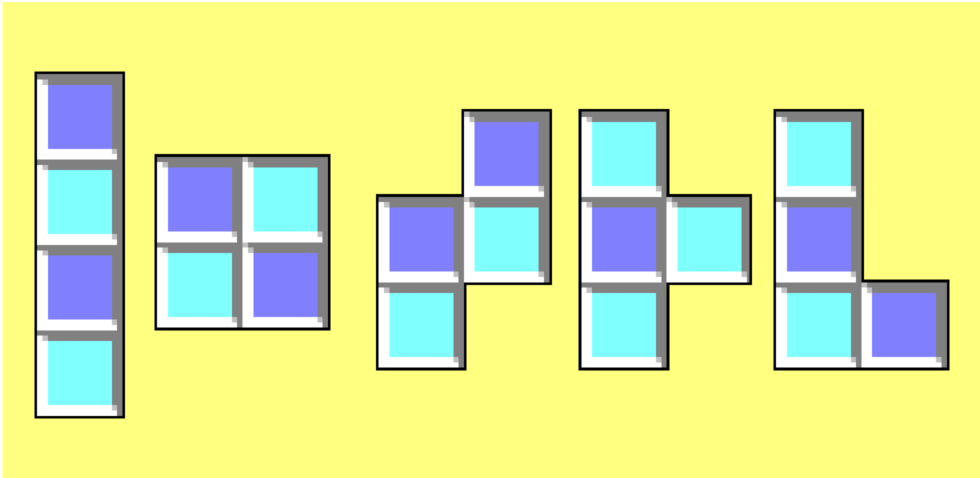


Trominoes, Tetrominoes and Polyominoes

We'll end with some interesting questions which can be solved with the help of colouring arguments and other tiling tricks. But first, we have to introduce trominoes, shapes which can be made by gluing together three unit squares edge to edge. You should be able to see that there are essentially two distinct trominoes — one looks like a 3×1 rectangle while the other looks like an L-shape. We already know which rectangles can be tiled by 3×1 rectangles, so it's natural to ask which rectangles can be tiled by L-trominoes.

And once you've mastered trominoes, of course, you would move on to tetrominoes — shapes which can be made by gluing together four unit squares edge to edge. Anyone who's played the excellent computer game Tetris before will be well-acquainted with them, but even if you haven't, you should be able to see that there are essentially the five different types as shown in the diagram below. With these simple shapes, you have a myriad of tiling problems that you can try, such as the following.

- Can the five tetrominoes tile a rectangle of area 20?
- Can two copies of each of the five tetrominoes tile a rectangle of area 40?
- For each tetromino, determine which rectangles can be tiled by them.



And, of course, once you've mastered tetrominoes, there are many other shapes you can play with. In general, any shape which can be made by gluing together unit squares edge to edge is called a *polyomino*. For any polyomino, you can try to determine which rectangles can be tiled by them.

Problems

Problem. *The* 8×8 *chessboard can be tiled with twenty-one* 3×1 *rectangles and one* 1×1 *square. Determine all possible locations for the* 1×1 *square and prove that these are the only ones possible.*

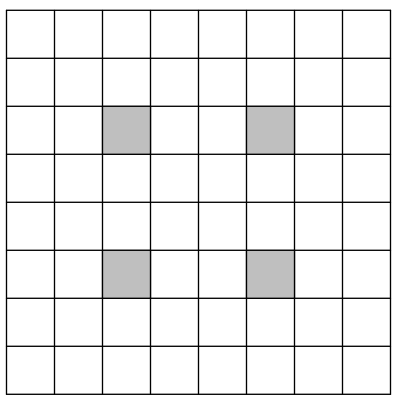
Proof. Our approach will, of course, use a modulo 3 colouring like the one pictured in the diagram below. Recall that the great thing about this colouring is the fact that any 3×1 rectangle placed on the board will cover precisely one square of each colour. This means that twenty-one 3×1 rectangles must cover exactly twenty-one squares of colour 0, twenty-one squares of colour 1 and twenty-one squares of colour 2. But you can plainly see from the diagram that there are actually twenty-one squares of colour 0, twenty-two squares of colour 1 and twenty-one squares of colour 2. What this means is that the 1×1 square must definitely be on a square which has the colour 1. Unfortunately, it's not true that if you put the 1×1 square on a square which has the colour 1 that you can actually tile the remainder of the board with twenty-one 3×1 rectangles.

0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0
0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0
0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2

The trick here is to observe that there are actually two different modulo 3 colourings that we could have tried. In the previous colouring, each row is equal to the previous row, but shifted by one square to the left. Now we simply use a colouring where each row is equal to the previous row, but shifted by one square to the right. Again, the great thing about this colouring is the fact that any 3×1 rectangle placed on the board will cover precisely one square of each colour. This means that twenty-one 3×1 rectangles must cover exactly twenty-one squares of colour 0, twenty-one squares of colour 1 and twenty-one squares of colour 2. But you can plainly see from the diagram that there are actually twenty-two squares of colour 0, twenty-one squares of colour 1 and twenty-one squares of colour 2. What this means is that the 1×1 square must definitely be on a square which has the colour 0.

0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0
1	2	0	1	2	0	1	2
0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0
1	2	0	1	2	0	1	2
0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0

So what we've deduced is that the 1×1 square must be on a square with the colour 1 in the first colouring and on a square with the colour 0 in the second colouring. And there are only four such squares on the 8×8 chessboard — the ones indicated in the diagram below.



I'll leave it as an exercise for you to show that, whichever one of these squares you decide to put the 1×1 square on, you can tile the remainder of the chessboard with 3×1 rectangles. □

Erdős

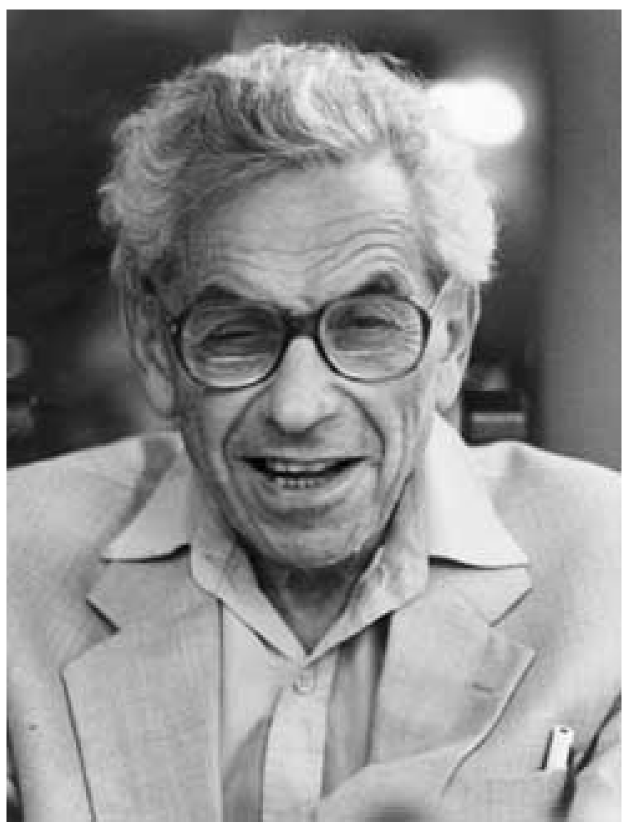
Paul Erdős was a Hungarian mathematician famous for being incredibly prolific but also incredibly eccentric. He has published more mathematics papers than anyone else in history, even more so than Euler, although Euler published more pages. He wrote nearly 1500 articles in his lifetime, in collaboration with over 500 different people. This is due to Erdős' philosophy that mathematics is a social activity.

Erdős was born in 1913 to Jewish parents who were both mathematics teachers. He learnt much from them as a child and supposedly, at the age of three, could calculate how many seconds his friends had lived for. After receiving a doctorate in mathematics at the age of twenty-one, he moved first to England and then to the United States, to escape the growing anti-Semitic sentiment in Europe. At this time, he began to develop the habit of travelling from campus to campus and staying with friends. He would typically show up at a colleague's doorstep, announce that "my brain is open", and stay long enough to collaborate on a few papers before moving on a few days later. Possessions meant very little to Erdős and most of his belongings would fit in a suitcase. Awards and other earnings were generally donated to people in need and various worthy causes. He kept up this vagabond lifestyle until his death in 1996.

His colleague Alfréd Rényi once said that "a mathematician is a machine for turning coffee into theorems". Erdős certainly drank copious amounts of coffee but later in life also started to take amphetamines. At one stage, a friend and colleague bet him $500 that he couldn't stop taking the drug for a month. Erdős won the bet, but complained during his abstinence that mathematics had been set back by a month: "Before, when I looked at a piece of blank paper my mind was filled with ideas. Now all I see is a blank piece of paper." Needless to say, after he won the bet, he promptly resumed his amphetamine habit.

Erdős had his own idiosyncratic vocabulary — he spoke of "The Book", an imaginary book in which God had written down the most elegant proofs for every mathematical theorem. Children were referred to as "epsilons", women were "bosses", men were "slaves", people who stopped doing math had "died", people who physically died had "left", alcoholic drinks were "poison", music was "noise", people who had married were "captured", people who had divorced were "liberated", to give a mathematical lecture was "to preach", and to give an oral exam to a student was "to torture". For his epitaph he suggested, "I've finally stopped getting dumber."

He contributed to many areas of mathematics — most notably combinatorics, graph theory, number theory, analysis, approximation theory, set theory, and probability theory. As a teenager, Erdős managed to give a very nice proof of Bertrand's postulate, which states that there is always a prime number between n and $2n$. He discovered the first elementary proof of the prime number theorem, which states that the number of primes less than n is approximately $\frac{n}{\log n}$. Erdős also proved new results in several fields which were of little interest to him, such as topology.



Erdős is perhaps most well-known for his application of the probabilistic method to extremal combinatorics, particularly Ramsey theory. Ramsey theory is the branch of mathematics concerned with problems of the following type.

> How many people do you need at a party to guarantee that there exist at least n people who all know each other or n people who all don't know each other?

We've already seen that the answer in the case $n = 3$ is six, and we write this as $R(3) = 6$. It turns out that $R(4) = 18$ and that the value of $R(5)$ is known only to be between 43 and 49 inclusive. These so-called Ramsey numbers are incredibly difficult compute, as evidenced by the following story that Erdős used to tell. Imagine that a large alien force, vastly more powerful than us, lands on Earth and asks for the value of $R(5)$ within a year or they will destroy our planet. In that case, we should gather together all of our humanpower and technology to try and find the value. On the other hand, if they ask for $R(6)$ instead, then we should simply gather together all of our humanpower and technology to try and launch a preemptive attack on the aliens.

Erdős' friends created a humorous tribute for him, defining the Erdős number of a mathematician. Erdős himself is the only mathematician with Erdős number zero. Anyone who has written a paper with him has Erdős number one, anyone who has written a paper with someone who has written a paper with him has Erdős number two, and so on. Some have estimated that ninety percent of the world's active mathematicians have an Erdős number smaller than eight. At least twice, there have been eBay auctions offering the chance to collaborate on a paper with someone in order to gain a small Erdős number. I have an Erdős number of two and, unfortunately, it will never ever decrease.