Polygon Models

We're going to look at how to very simply draw a surface on a piece of paper, and the idea is inspired by Pac-Man world. Recall that when you glue up the sides of Pac-Man world, you obtain a torus. Furthermore, if you look at the seams of your gluing, they form a map on the torus and, most importantly, this map has exactly one face. This face corresponds precisely to the rectangle that made up Pac-Man world in the first place.

So, if you could find a map on a surface which has one face, then you could cut the surface open along the edges of the map to give a polygon. The surface can be reconstructed from this polygon as long as you have the right gluing instructions. When you can do this, the resulting polygon and its gluing instructions are referred to as a *polygon model* for the surface. Note that a polygon model must always have an even number of sides which are glued together in pairs, without any edges left unglued. Fortunately, every surface can be turned into a polygon model, as the following result shows.

Theorem.

- *Every surface corresponds to a polygon model.*
- *Every polygon model corresponds to a surface.*

Proof.

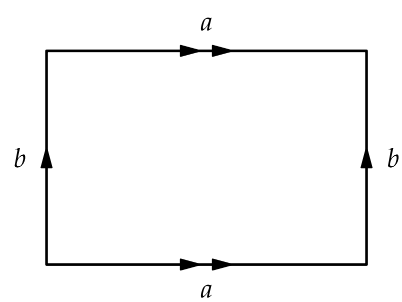
- The idea here is to use the simple sounding though difficult to prove theorem which states that every surface can be cut up into triangles. So suppose you have a surface and that you cut it up into triangles and lay them down on a table. We can label each side of a triangle with a gluing instruction which tells you how the triangles can be glued back together to recreate the surface. Now just take two triangles which are supposed to be glued together and glue them together. Find another triangle which can be glued onto these two and just go ahead and glue it to them. If you keep gluing more triangles onto the shape that you have, they will all end up making one piece which will form a polygon — in fact, the gluing instructions on the unglued edges of the polygon will ensure that it's a polygon model for the surface.
- A polygon model is really just a polygon with $2n$ sides which are glued together in n pairs. You can check that after performing such a gluing, the resulting object is indeed a surface. The main thing to check is that around every point, there is a little disk — this is certainly true for points inside the polygon. And for points on the edges, there will be half of a disk protruding from one edge and half of a disk protruding from the edge that it's glued to — these two halves of disks glue together to make a whole disk. Finally, you need to check that if you take a point which is a vertex of the polygon model, then it's surrounded by a disk. But this is easy to check, since all of the vertices which glue together will have a little pizza slice of a disk and these pizza slices glue together to make a whole disk. □

Edge Words

Imagine that someone calls you up on the phone and asks you to describe your favourite surface — how can you do it? One way is to cut up your surface into triangles, glue the triangles together until they form a polygon, and then describe the resulting polygon and its gluing instructions — remember that this is called a polygon model. These gluing instructions are usually given by writing the same letter on each pair of sides which get glued together and placing arrows on the sides to show in which direction they get glued.

So you can describe a surface by reading around its polygon model — usually in a counterclockwise fashion — calling out x if the side is labelled x and the arrow is facing in the right direction and calling out x^{-1} if the side is labelled x and the arrow is facing in the wrong direction. This sequence of letters, some with inverses and some without, is called an *edge word* for a surface. Note that one surface can be described by many many different edge words. Hopefully it's clear that once you tell someone an edge word to your favourite surface, then they can reconstruct the surface that you're talking about.

Example. Let's consider the Pac-Man world example from earlier, which we know to be the torus. Starting in the bottom-left corner and reading around counterclockwise, the first letter is a, the second is b, the third is a^{-1} and the fourth is b^{-1}, which takes us back to where we started. So an edge word for the torus is $aba^{-1}b^{-1}$.

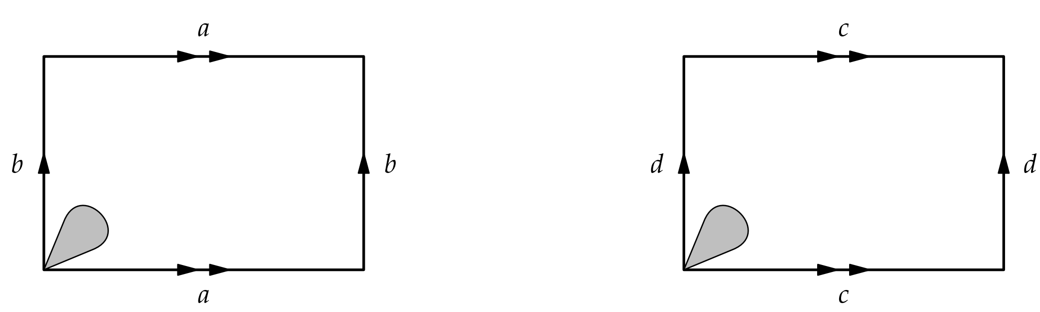


Example. Some other useful edge words to keep in mind are aa, which represents the projective plane, and $aba^{-1}b$ which represents the Klein bottle.

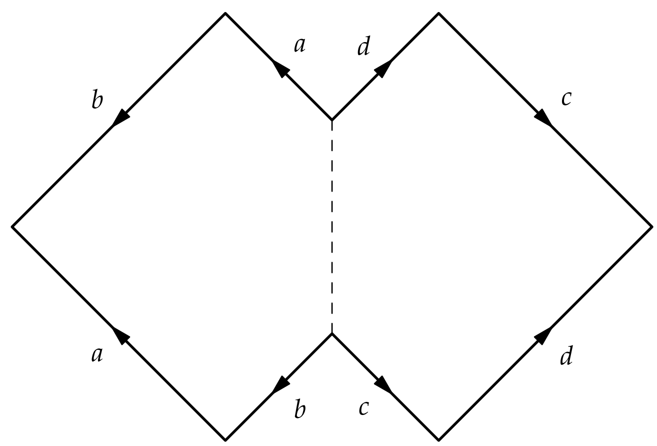
Playing with Edge Words

What we'd like to do is translate some of the constructions to do with surfaces into the world of edge words. This will allow us to play with surfaces by just playing with edge words — and that is a much simpler game, because edge words don't involve any crazy pictures, imagining things in four-dimensional space, and so on.

First, let's consider what happens to edge words when we take connected sums. We earlier discussed that when you take connected sums, it doesn't matter in the slightest where you decide to cut out your holes from each surface. So if we want to take the connected sum of two tori, for example, we could describe one by $aba^{-1}b^{-1}$, the other by $cdc^{-1}d^{-1}$, and decide to cut out holes from each one as shown in the diagram below.



If you open each of these holes up, the rectangles look more like pentagons, where four of the edges are labelled and the unlabelled edges in each pentagon will get glued together. The result will be an octagon whose edge word is simply $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.



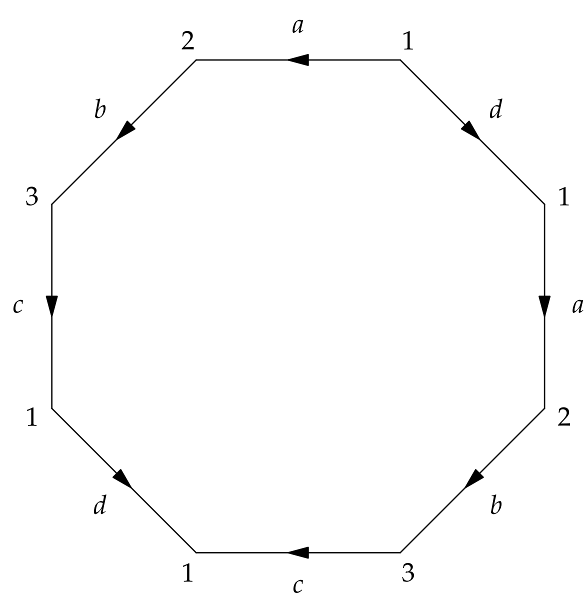
It is a simple matter to generalise this reasoning and the final result is the following fact.

Proposition. *Given polygon models for the surfaces S_1 and S_2 with edge words W_1 and W_2, an edge word for the connected sum $S_1 \# S_2$ is W_1W_2, where the words W_1 and W_2 have simply been written one after the other.*

If you analyse what happens when you slide a counterclockwise arrow around a polygon model, then you'll find that nothing happens to the direction of the arrow unless you happen to slide over an edge and emerge from the edge that it is glued to. In fact, the direction of the arrow doesn't change unless these two edges are facing in the same direction around the polygon. In other words, we have the following fact which allows us to deduce the orientability of a surface very simply from its edge word.

Proposition. *In an edge word for an orientable surface, every letter appears with its inverse. In an edge word for a non-orientable surface, there exists at least one letter which does not appear with its inverse.*

Finally, let's consider how to calculate the Euler characteristic from an edge word, for example the edge word $abcdc^{-1}b^{-1}a^{-1}d^{-1}$. Once you've seen how to calculate the Euler characteristic for this edge word, you'll no doubt be able to calculate the Euler characteristic for just about any edge word I give you.



Above we have a diagram of the polygon model corresponding to the edge word. Of course, to work out the Euler characteristic of the corresponding surface, we'd like to determine the number of vertices, edges and faces. The easiest of these is the number of faces, since we always have $F = 1$ for a polygon model. The next easiest is the number of edges, since the eight edges will get glued into four pairs, so the resulting map on the surface will have $E = 4$. And finally we can calculate the number of vertices as $V = 3$. We do this by determining which vertices get matched up with each other after the gluing occurs. In the diagram, the three different vertices are labelled 1, 2 and 3. The strategy for finding this labelling is to make sure that the two vertices at the tip of the edges labelled a correspond to the same vertex and the two vertices at the tail of the edges labelled a correspond to the same vertex. If you do this for every edge label, then you should find that the vertices of the polygon divide into the three groups indicated, each group corresponding to one of the three vertices of the resulting map. Now that we have the number of vertices, edges and faces, it's a simple matter to calculate $\chi = V - E + F = 3 - 4 + 1 = 0$. Furthermore, we know that this surface is orientable because, in the edge word $abcdc^{-1}b^{-1}a^{-1}d^{-1}$, every letter appears with its inverse.

The Classification of Surfaces

Finally, we're in a position where we can classify the surfaces — that is, write a list of every different surface possible. A lot of effort by mathematicians goes into classifying mathematical objects. However, surfaces is one of the few examples where we've been successful and achieved a complete classification. In fact, this was all worked out back in the early twentieth century.

Theorem (The classification of surfaces)**.** *Every surface is homeomorphic to the sphere, a connected sum of tori or a connected sum of projective planes. Furthermore, no two of these surfaces are homeomorphic to each other.*

Sketch of the proof. The proof of this theorem, although reasonably elementary, is rather involved. Rather than get bogged down in all the gory details, I'll just give an overview of the proof, which relies heavily on the fact that surfaces can be described by edge words and that every edge word describes a surface.

- Take any edge word — our goal is to show that that it corresponds to a sphere, a connected sum of tori or a connected sum of projective planes.
- The idea is to apply simplifying moves to the edge word which do not change the surface. Such moves can be applied until the edge word can be recognised as the sphere or a connected sum of tori and projective planes. The only moves you require to perform this task are the following — you should check that they do not change the surface.

  - $\sigma(aXa^{-1}Y) \cong \sigma(b^{-1}XbY)$
  - $\sigma(XY) \cong \sigma(YX)$
  - $\sigma(X) \cong \sigma(X^{-1})$
  - $\sigma(aa^{-1}X) \cong \sigma(X)$
  - $\sigma(aXaY) \cong P \,\#\, \sigma(XY^{-1})$
  - $\sigma(aWbXa^{-1}Yb^{-1}Z) \cong T \,\#\, \sigma(WZYX)$

  The idea is that a sequence of such moves allows you to break an edge word into a piece which is a projective plane or a torus connected sum with a surface whose edge word is shorter. Here, I am using a and b to represent letters, and W, X, Y, Z to represent words. Furthermore, P and T represent the projective plane and the torus, respectively. And finally, the word X^{-1} is what you get if you read the word X backwards. For example, if X was $aba^{-1}cdc^{-1}$, then reading it backwards would give you $cd^{-1}c^{-1}ab^{-1}a^{-1}$. Here, we are using the notation $\sigma(W)$ to denote the surface which corresponds to the edge word W.

- So you can use the aforementioned simplifying moves to reduce the surface to a connected sum of tori and projective planes. However, connected sums of tori and projective planes do not actually appear in our classification of surfaces. But that's all right, because when you have a mix of the two appearing in a connected sum, you can replace the torus with two projective planes. Another way to say this is that $T \# P \cong P \# P \# P$. The proof of this statement is something that you can try in the comfort of your own home — it's simply equivalent to the statement $\sigma(aba^{-1}b^{-1}cc) = \sigma(aabbcc)$.
- These tricks can be used to reduce any edge word until it looks like it represents the sphere, a connected sum of tori or a connected sum of projective planes. All that remains is to show that these are all different from each other. However, this is quite easy, because we can use orientability and the Euler characteristic to tell all of these surfaces apart. For more details, you can consult the following useful table. □

SURFACE	ORIENTABILITY	EULER CHARACTERISTIC
sphere	orientable	2
connected sum of g tori	orientable	$2-2g$
connected sum of n projective planes	non-orientable	$2-n$

So we now have a very useful corollary — it took a lot of work to get to this point in time, but we can now say that we essentially understand everything there is to understand about surfaces.

Corollary. *In order to identify a surface, all you need are its orientability and its Euler characteristic.*

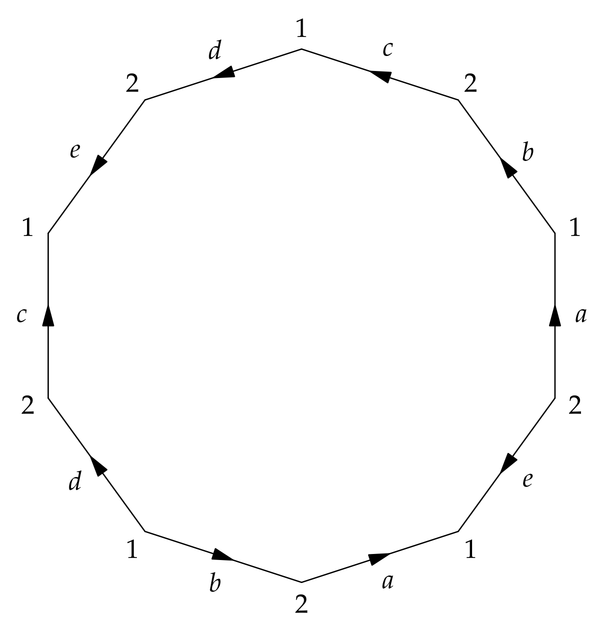
Problems

Problem. *Identify the surface corresponding to the edge word* $abcdc^{-1}b^{-1}a^{-1}d^{-1}$.

Proof. We earlier calculated that this surface has Euler characteristic equal to zero and is orientable. Now we can consult the table above and identify the surface as a connected sum of one torus — namely, a torus. □

Problem. *Identify the surface corresponding to the edge word* $abcdec^{-1}d^{-1}bae^{-1}$.

Proof. The following is a diagram of the polygon model, with the vertices of the polygon labelled 1 and 2. All of the vertices labelled 1 will coincide after the gluing is performed and all of the vertices labelled 2 will coincide after the gluing is performed. Therefore, the Euler characteristic is simply $\chi = V - E + F = 2 - 5 + 1 = -2$.



It is easy to see from the edge word that the surface is non-orientable, since the letter a does not appear with its inverse. Therefore, the classification of surfaces tells us that it must be a connected sum of projective planes. In fact, the Euler characteristic tells us that it must be a connected sum of four projective planes, which we can write as $P \# P \# P \# P$. □

Noether

Amalie Emmy Noether was a German mathematician who was born into a Jewish family in 1882 and died from surgery complications in 1935 at the age of fifty-three. She is known for her groundbreaking contributions to abstract algebra and to theoretical physics, despite many obstacles in her life. For example, her decision to attend the University of Erlangen was unconventional since the Academic Senate of the university had declared that allowing coeducation would "overthrow all academic order". She was one of only two female students in a university of nearly one thousand and was only allowed to audit classes with the permission of each individual professor.

After completing her dissertation, Noether worked at the Mathematical Institute of Erlangen, but had to do so without pay for seven years. Finally in 1915, she was invited by the great mathematicians David Hilbert and Felix Klein to join the mathematics department at the University of Göttingen, a world-renowned centre of mathematical research. However, when the Nazis came to power in 1933, Noether was forced to leave her job due to her Jewish background. She accepted the decision calmly and was fortunately offered a job at Bryn Mawr College near Philadelphia, which provided a welcoming home for her during the last two years of her life.

Noether was highly respected for her teaching. Apparently, she did not follow a lesson plan for her lectures, which frustrated some students. Instead, she used her lectures as a spontaneous discussion time, to think through and clarify important cutting-edge problems in mathematics. Once, when the mathematics department was closed for a state holiday, she gathered the class on the steps outside, led them through the woods, and lectured at a local coffee house. Later, after she had been dismissed by the Third Reich, she invited students into her home to discuss their future plans and mathematical concepts. In addition to her own publications, Noether was generous with her ideas and is credited with several lines of research published by other mathematicians, even in fields far removed from her main work.

Noether was described by Albert Einstein and others as the most important woman in the history of mathematics. She revolutionised abstract algebra and discovered Noether's theorem in physics, which explains the fundamental connection between symmetry and conservation laws — this tells us why we can expect conversation of physical properties such as energy, momentum and angular momentum. Her theorem has been called "one of the most important mathematical theorems ever proved in guiding the development of modern physics".