Classifying the Platonic Solids

A *Platonic solid* is a convex polyhedron whose faces are all congruent regular polygons, with the same number of faces meeting at each vertex. In some sense, these are the most regular and most symmetric polyhedra that you can find. Our goal now will be to classify the Platonic solids — in other words, hunt them all down.

Theorem (Classification of Platonic solids)**.** *There are exactly five Platonic solids.*

Proof. The following geometric argument is very similar to the one given by Euclid in the Elements.

- Let's say that the regular polygons have n sides and that d of them meet at every vertex. It should be clear that to form a vertex, you need $d \geq 3$.
- Because the Platonic solids are convex by definition, at each vertex of the solid, the sum of the angles formed by the faces meeting there must be less than 360°. If the sum was equal to 360°, then the faces which meet at the vertex would all lie in the same plane and there wouldn't be a vertex at all. And if the sum was more than 360°, then it would be impossible for the vertex to be convex. By this reasoning, the angle in the regular polygon with n sides must be less than $360° \div 3 = 120°$.
- Note that a regular hexagon has angles of 120° so that a regular polygon with more than six sides has angles which are greater than 120°. Therefore, we only need to consider the cases when n is equal to 3, 4 or 5.
  - When $n = 3$, we have faces which are equilateral triangles, all of whose angles are 60°. Using the fact that the sum of the angles formed by the faces at a vertex must be less than 360°, we obtain that d must be equal to 3, 4 or 5.
  - When $n = 4$, we have faces which are squares, all of whose angles are 90°. Using the fact that the sum of the angles formed by the faces at a vertex must be less than 360°, we obtain that d must be equal to 3.
  - When $n = 5$, we have faces which are regular pentagons, all of whose angles are 108°. Using the fact that the sum of the angles formed by the faces at a vertex must be less than 360°, we obtain that d must be equal to 3.
- So, in summary, we know that there are only five possibilities for the pair of integers (n, d) — namely, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$ and $(5, 3)$. We will simply state the fact here that for each of these possibilities, there is exactly one Platonic solid. To see that there is at least one Platonic solid corresponding to a pair (n, d), all you need to do is construct it out of regular polygons with n sides, with d of them meeting at every vertex. On the other hand, to see that there is at most one Platonic solid corresponding to a pair (n, d) is a more difficult matter, but should seem believable. This is because if you start to glue together regular polygons with n sides, with d of them meeting at every vertex, then you don't have any choice in what the resulting shape will look like. □

Constructing the Platonic Solids

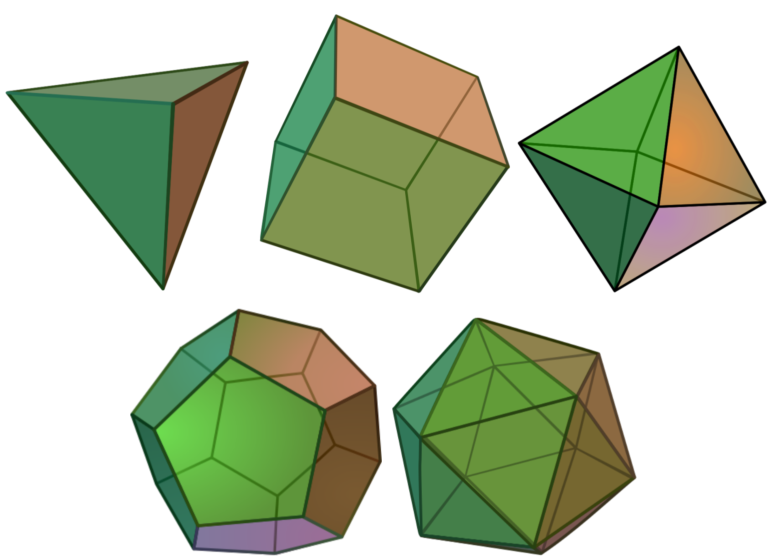
Of course, if you want to build a Platonic solid — the one corresponding to $(n, d) = (5, 3)$, for example — then it helps to know how many vertices, edges and faces are required. So let's have a look at how you can determine these numbers. If you want to work out the three unknown quantities V, E and F, then it makes sense to look for three relations that these numbers obey. One relation will come from using the handshaking lemma, another will come from using the handshaking lemma on the dual, and another is given to us by Euler's formula.

- Since three faces meet at every vertex, we know that every vertex in the polyhedron must have degree three. The handshaking lemma asserts that the sum of the degrees is equal to twice the number of edges, so we have the equation $3V = 2E$ or equivalently, $V = \frac{2}{3}E$. This means that if we know the value of E, then we also know the value of V.
- Since every face is a pentagon, we know that every vertex in the dual must have degree five. The handshaking lemma on the dual asserts that the sum of the numbers of edges around each face is equal to twice the number of edges, so we have the equation $5F = 2E$ or equivalently, $F = \frac{2}{5}E$. This means that if we know the value of E, then we also know the value of F.
- Now we can use Euler's formula $V - E + F = 2$ and substitute for V and F. If you do this properly, you should obtain
$$\frac{2}{3}E - E + \frac{2}{5}E = 2 \quad \Rightarrow \quad E = 30.$$
From this, we can easily deduce that $V = \frac{2}{3}E = 20$ and $F = \frac{2}{5}E = 12$.

You can and should use this method to determine the number of vertices, edges and faces for each of the Platonic solids and, if you do so, you'll end up with a table like the following.

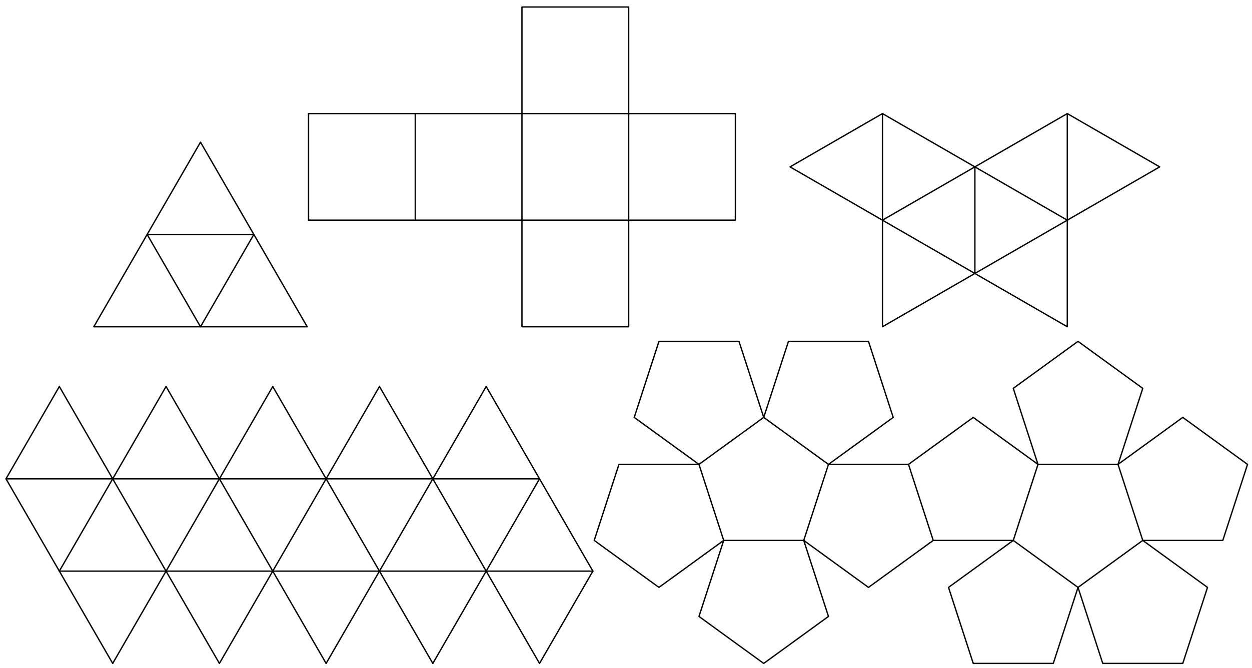
polyhedron	n	d	V	E	F
tetrahedron	3	3	4	6	4
cube	4	3	8	12	6
octahedron	3	4	6	12	8
dodecahedron	5	3	20	30	12
icosahedron	3	5	12	30	20

The following diagram shows the five Platonic solids — they are called the tetrahedron, the hexahedron, the octahedron, the dodecahedron and the icosahedron.[1] The first three are easier to imagine, because they are simply the triangular pyramid, the cube, and the shape obtained from gluing two square pyramids together.



[1]These names come from the ancient Greek and simply mean four faces, six faces, eight faces, twelve faces and twenty faces, respectively. Of course, we almost always refer to the hexahedron more affectionately as the cube.

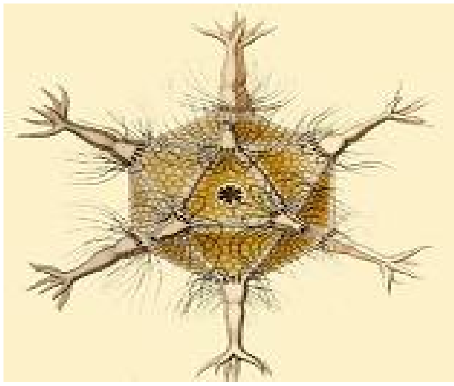
Those of you who are fans of role-playing games such as *Dungeons & Dragons* — you know who you are — may be able to visualise these more easily. This is because such games involve d4's, d6's, d8's, d12's and d20's, where a dn is simply an n-sided die. Another way to visualise a polyhedron is via a net, a figure which you can cut out of cardboard and then fold to create the polyhedron. The following diagram shows nets for all of the Platonic solids.



Platonic Solids in Nature

The aesthetic beauty and symmetry of the Platonic solids have made them a favourite subject of geometers for thousands of years. They are named for the ancient Greek philosopher Plato who thought that the classical elements — earth, water, air, fire and ether — might be constructed from Platonic solids. In the sixteenth century, the German astronomer Johannes Kepler attempted to find a relation between the five known planets at that time — excluding Earth — and the five Platonic solids. In the end, Kepler's original idea had to be abandoned, but out of his research came the realization that the orbits of planets are not circles and the discovery of Kepler's laws of planetary motion.

Each of the five Platonic solids occurs regularly in nature, in one form or another. The tetrahedron, cube, and octahedron all occur in crystals, as does a slightly warped version of the dodecahedron. In the early twentieth century, it was observed that certain species of amoeba known as radiolaria possessed skeletons shaped like Platonic solids — the picture on the right gives an icosahedral example. Also, the outer protein shells of many viruses form regular polyhedra — for example, the HIV virus is enclosed in an icosahedron.



Scientists have also discovered new types of carbon molecule, known as fullerenes, which have very symmetric polyhedral shapes. The most common is C60, which has the shape of a soccer ball, though there are others which possess the shape of Platonic solids.

Symmetries of Platonic Solids

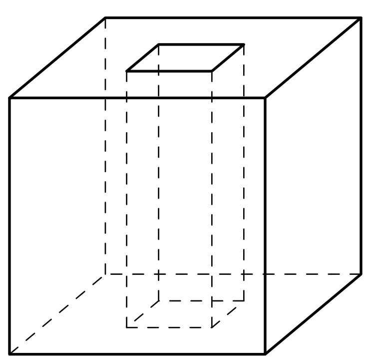
It's definitely worth mentioning that the whole symmetry game we played for subsets of the Euclidean plane can also be played for subsets of Euclidean space. We can define three-dimensional isometries and symmetries of figures in space in an entirely analogous way, although some details are slightly more difficult. For example, the result that any isometry in the plane is a product of three reflections becomes the result that any isometry in space is a product of four reflections. Furthermore, there are more types of isometry than just translation, rotation, reflection and glide reflection — they are called twists and rotatory reflections — and you can try to imagine what these might do.

There is also a three-dimensional analogue of Leonardo's Theorem, a result which classifies all of the possible symmetry groups in Euclidean space. Essentially, the result says something like the only symmetry groups you can get are cyclic groups, dihedral groups, certain slightly more complicated versions of the cyclic and dihedral groups, and symmetry groups of Platonic solids.

As an example, let me tell you about the symmetry group of the tetrahedron. An isometry has to take a particular face of the tetrahedron to one of the four faces of the tetrahedron. And once you decide where one of the equilateral faces ends up, there are still six possibilities, one for each element of D_3. This means that the symmetry group of the tetrahedron has 24 elements. But we just happen to know a group with 24 elements and that group is the symmetric group S_4. And if you guess that the symmetry group of the tetrahedron is isomorphic to S_4, then you'd be right. Now you should try to determine how many elements are in the symmetry groups of the remaining four Platonic solids using the same idea.

What's Wrong with Euler's Formula?

We're now going to leave Platonic solids behind and embark on a new adventure. Recall that when we defined what a polyhedron was, we were very careful to say that it "had no holes". One of the reasons for doing this is because it doesn't seem that Euler's formula works when there are holes. For example, consider the following geometric object which is made by gluing together polygons.



If you count the number of vertices, edges and faces, then you'll find that $V - E + F$ does not equal 2 at all. Remember to be very careful if you do this, because the regions shown on the top and bottom in the diagram are not actually faces, since they have holes in the middle of them. All you need to do is divide these regions into bona fide faces with the help of some extra edges.

Since Euler's formula doesn't work for this shape, there are two things we can do — start to cry or try to make it work. Mathematicians would generally prefer the latter approach. To make Euler's formula work for more general shapes, you simply need to note that $V - E + F = 0$ in this particular case and, in fact, $V - E + F = 0$ for any shape which has one hole through the middle of it. In fact, if you try the same thing for shapes with g holes, you'll eventually discover that

$$V - E + F = 2 - 2g.$$

So $V - E + F$ seems to change when you talk about very different geometric objects — for example, objects with different numbers of holes — but seems to be the same when you talk about similar geometric objects — for example, objects with the same number of holes. Another observation is that if we take the geometric object pictured above and draw it so that it looks a bit curvier, then that doesn't change the number of vertices, edges and faces, so $V - E + F$ doesn't change at all. In fact, we can bend, stretch, warp, morph or deform it and the value of $V - E + F$ wouldn't change.

The Earth

Suppose that you lived a really really long time ago. Then you would probably believe, as did most people, that the surface of the Earth is a big flat plane. And what makes you think that? Well, it's simply due to the fact that everywhere you stand, you notice that there's a pretty flat piece of earth immediately surrounding your feet. And since a big flat plane seems to have this same property, you've simply jumped to the conclusion that the Earth must be big flat plane. But, as you know, this is rather foolish thinking. There are many geometric objects apart from the plane which have this property. The sphere is just one more example, and we're going to explore other shapes that can arise. This idea leads us to study things called *surfaces* which we'll soon talk about.

Topology

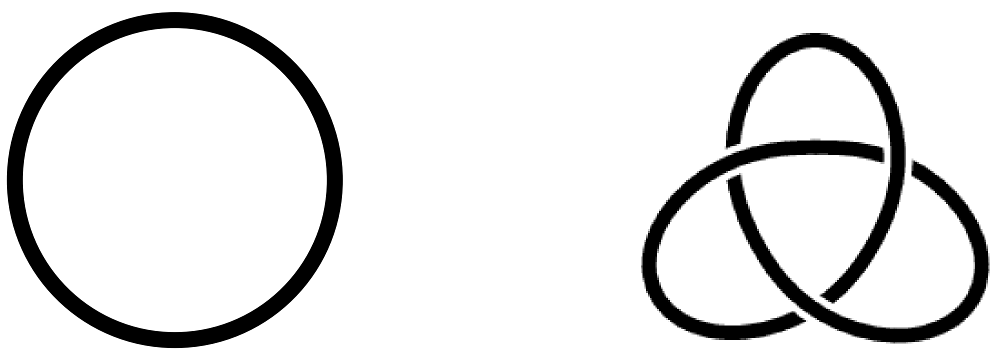
The two stories above — one about Euler's formula and one about the Earth — motivate us to consider *topology*, which very roughly studies intrinsic fundamental properties of geometric objects and which doesn't care about length, size, angle, and so forth. When you study objects in mathematics, you always need to have some notion of when two of those objects are the same. For example, in Euclidean geometry we have congruence, in group theory we have isomorphism, in graph theory we have isomorphism, and in topology we have the notion of homeomorphism. Intuitively speaking, two geometric objects are *homeomorphic* if it's possible to bend, stretch, warp, morph or deform one so that it becomes the other one. Note that you are not allowed to cut and glue here.

A more mathematically precise definition of homeomorphism is as follows. Two geometric objects are homeomorphic if there exists a bijection — in other words, a one-to-one correspondence — from one to the other which is continuous and has a continuous inverse. If two geometric objects A and B are homeomorphic, then we write $A \cong B$, and we call a bijection from A to B which is continuous and has a continuous inverse a *homeomorphism*.[2]

Example. The sphere and the cube are certainly homeomorphic to each other. This is because you can take the sphere and squash it into a box until it looks like a cube. Or, on the other hand, you could take a cube and blow it up like a balloon until it looks like a sphere. These sorts of deformations are certainly allowed and show that the two shapes are homeomorphic. For the same sort of reason, a disk and a square are also homeomorphic to each other. You can probably see why topology is sometimes informally called rubber sheet geometry.

An explicit homeomorphism in this case isn't too difficult to describe and, if you were really keen, you could even write down an equation for one. The idea is to stick the sphere inside of the cube and suppose that the sphere is a light bulb with a source of light at its centre. Any point on the sphere now casts a shadow on the cube, and this gives a map from the sphere to the cube which is a continuous bijection with a continuous inverse.

Example. Consider an unknotted piece of string like the one shown below left and a knotted piece of a string like the one shown below right.



Believe it or not, these two are homeomorphic to each other. Sure, you need to cut the knot open and glue the ends together again to make the unknotted loop — but that's only if you happen to live in three dimensions. Topology, unlike we mere mortals, doesn't live in any number of dimensions. Intuitively, you can deform the unknotted loop into a knot if you make those deformations in four-dimensional space. The reason being that lines can just move past each other in four-dimensional space without hitting. It's just the higher dimensional analogue of the fact that two points on the line can't move past each other without hitting yet in two dimensions, they can do so with ease.

If this doesn't convince you, then you can always go back to the mathematical definition of homeomorphism. Suppose that you wrote the numbers from 1 to 100 along the unknotted loop, in order. You could also do the same thing around the knotted loop and convince yourself that there's a function from one to the other which matches up the numbers. It's easy to see that such a function can be made to be a bijection which is continuous and has a continuous inverse — hence, the two are homeomorphic.

[2]The word homeomorphism comes from the ancient Greek words meaning similar shape.

Example. A sphere and a torus — the surface of a donut — are not homeomorphic, and this is a little difficult to see. And that's because if you want to show that two geometric objects are homeomorphic, then you just go ahead and find the deformation or homeomorphism that makes them so. On the other hand, if you want to show that two geometric objects are not homeomorphic, then you need some tricks. This should be reminiscent of the fact that if you want to show that two groups are isomorphic, then you just go ahead and find the isomorphism but if you want to show that two geometric objects are not isomorphic, then you need some tricks. We're going to learn some of these tricks later on.

Problems

Problem. *Show that at any party, there are always at least two people with exactly the same number of friends at the party.*

Proof. For this problem, we need to assume that if A is friends with B, then B is friends with A.[3] The problem can obviously be translated into graph theoretic terms as follows.

> Show that in any graph — without loops or multiple edges — there are always two vertices with the same degree.

Just to be concrete, let's suppose that we're dealing with a graph which has seven vertices. The degree of a vertex in such a graph can only be 0, 1, 2, 3, 4, 5 or 6 since there are no multiple edges or loops, by assumption. Note that there are seven vertices in total as well as seven possibilities for their degrees. This means that if there aren't two vertices with the same degree, then there must be exactly one vertex with degree 0, exactly one vertex with degree 1, exactly one vertex with degree 2, and so on, up to exactly one vertex with degree 6. However, this situation simply cannot arise, since a graph with seven vertices can't have a degree 0 vertex as well as a degree 6 vertex. This is because a degree 0 vertex is connected to no others by an edge, while a degree 6 vertex is connected to all others by an edge. In party terms, you can't have a loner who is friends with nobody as well as a social butterfly who is friends with everybody. This contradiction means that there are always two vertices with the same degree. Of course, our argument can be generalised to graphs with any number of vertices. □

Problem. *Does there exist a polyhedron with exactly thirteen faces, all of which are triangles?*

Proof. The first thing you should do is try to draw such a polyhedron as a planar graph, keeping in mind that the outside face has to be a triangle. You can try doing this all day, but what you'll find is that the task is impossible. And hopefully, you'll also find that you can have such polyhedra with exactly ten faces or twelve faces or fourteen faces. What this tells you is that there is something to do with oddness and evenness going on in this problem and that suggests that we are going to use the handshaking lemma

So now let's suppose that there does exist a polyhedron with exactly thirteen faces, all of which are triangles. The trick here is to use the handshaking lemma on the dual graph. This asserts that the sum of the numbers of edges around each face is equal to twice the number of edges. In our case, this means that 13×3 is equal to twice the number of edges or, in other words, that the number of edges must be $19\frac{1}{2}$. Of course, this is a contradiction so we can deduce that there does not exist a polyhedron with exactly thirteen faces, all of which are triangles. □

[3] As you probably know, such an assumption isn't always true in the real world.

Problem. *If a planar graph with E edges divides the plane into F faces, prove that* $F \leq \frac{2E}{3}$.

Proof. For this problem, we need to assume — as is usual — that the graph contains no loops or multiple edges. Note that the dual graph has F vertices and, since every face of the original graph has at least three sides, every vertex of the dual graph has degree at least three. So the sum of the degrees of the vertices in the dual graph is at least $3F$. However, the number of edges in the dual graph is equal to E. So we can now invoke the handshaking lemma to deduce that $2E \geq 3F$, which rearranges to give the desired result. □

Problem. *Consider a polyhedron all of whose faces are triangles such that four faces meet at every vertex. Determine the number of vertices, edges and faces of the polyhedron.*

Proof. If you want to work out the three unknown quantities V, E and F, then it makes sense to look for three relations that these numbers obeys. One relation will come from using the handshaking lemma, another will come from using the handshaking lemma on the dual, and another is given to us by Euler's formula.

- Since four faces meet at every vertex, we know that every vertex in the polyhedron must have degree four. The handshaking lemma asserts that the sum of the degrees is equal to twice the number of edges, so we have the equation $4V = 2E$ or equivalently, $V = \frac{1}{2}E$. This means that if we know the value of E, then we also know the value of V.
- Since every face is a triangle, we know that every vertex in the dual must have degree three. The handshaking lemma on the dual asserts that the sum of the numbers of edges around each face is equal to twice the number of edges, so we have the equation $3F = 2E$ or equivalently, $F = \frac{2}{3}E$. This means that if we know the value of E, then we also know the value of F.
- Now we can use Euler's formula $V - E + F = 2$ and substitute for V and F. If you do this properly, you should obtain
$$\frac{1}{2}E - E + \frac{2}{3}E = 2 \quad \Rightarrow \quad E = 12.$$
From this, we can easily deduce that $V = \frac{1}{2}E = 6$ and $F = \frac{2}{3}E = 8$.

In fact, an example of such a polyhedron is given by the Platonic solid known as the octahedron. □

Newton

Sir Isaac Newton was born on Christmas Day way back in 1642 and lived a most productive eighty-four years until his death in 1727. He was an English mathematician who was also a great physicist, accomplished natural philosopher, prolific theologian and dedicated alchemist. His Philosophiæ Naturalis Principia Mathematica is considered to be among the most influential books in the history of science. In this work, Newton described how gravity works and produced three laws of motion, which gave the most accurate description of how the universe works for the following three centuries until around the time of Einstein. Famously, Newton formed his theory of gravitation after seeing an apple fall from a tree. The French mathematician Joseph–Louis Lagrange often said that Newton was the greatest genius who ever lived, and once added that he was also "the most fortunate, for we cannot find more than once a system of the world to establish".

Newton's contributions to mathematics and science are incredibly diverse. In mathematics, he is most famously remembered for his discovery of calculus, independently to the German mathematician Leibniz, but at about the same time. According to Newton's friends, he had worked out his method years before Leibniz, but published almost nothing about calculus until 1693. Meanwhile, Leibniz began publishing his account of calculus up to fifteen years earlier. It didn't take long for scientists from the Royal Society — of which Newton was a member — to accuse Leibniz of plagiarism. Thus began an ugly, bitter and controversial dispute between Newton and Leibniz, which marred the lives of both until the latter's death.

Some people believe that Newton was quite modest about his own achievements, writing that "If I have seen a little further, it is by standing on the shoulders of giants". However, it is more likely that Newton was not modest at all and that this quote was an attack on the scientist Robert Hooke, who was short and hunchbacked. Modest or not, Newton is considered by many to be the greatest scientist who ever lived. A survey of scientists from Britain's Royal Society deemed that Newton has had more impact on the history of science than Einstein.

To be a great mathematician or scientist, I think that you have to be motivated by curiosity, which is why I like the following quote from Newton himself: "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."