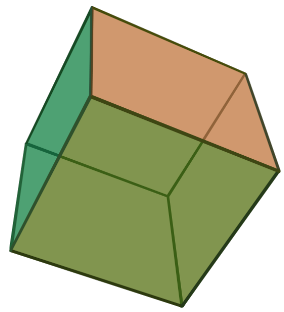
What is a Polyhedron?

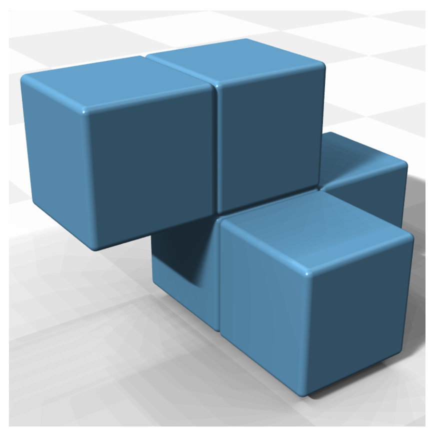
Now that we've covered lots of geometry in two dimensions, let's make things just a little more difficult. We're going to consider geometric objects in three dimensions which can be made from two-dimensional pieces. For example, you can take six squares all the same size and glue them together to produce the shape which we call a cube.



More generally, if you take a bunch of polygons and glue them together so that no side gets left unglued, then the resulting object is usually called a *polyhedron*.[1] The corners of the polygons are called *vertices*, the sides of the polygons are called *edges* and the polygons themselves are called *faces*. So, for example, the cube has 8 vertices, 12 edges and 6 faces.

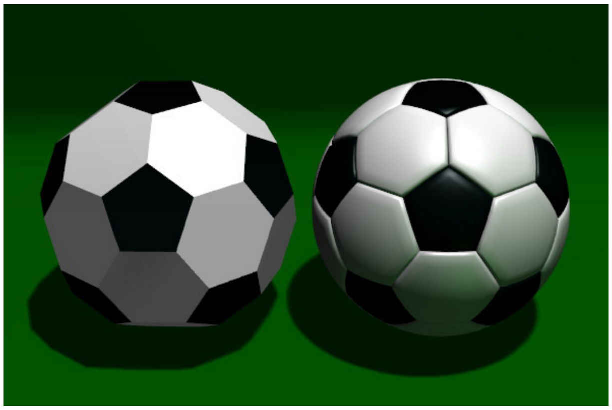
Different people seem to define polyhedra in very slightly different ways. For our purposes, we will need to add one little extra condition — that the volume bound by a polyhedron "has no holes". For example, consider the shape obtained by drilling a square hole straight through the centre of a cube. Even though the surface of such a shape can be constructed by gluing together polygons, we don't consider this shape to be a polyhedron, because of the hole.

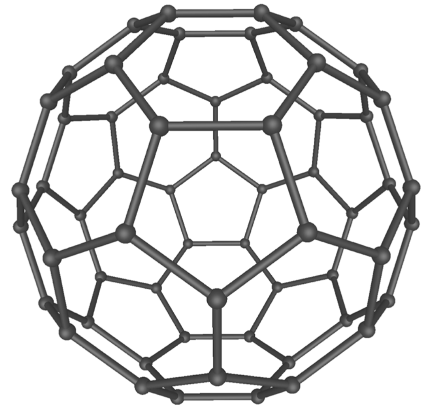
We say that a polyhedron is *convex* if, for each plane which lies along a face, the polyhedron lies on one side of that plane. So, for example, the cube is a convex polyhedron while the more complicated specimen of a polyhedron pictured on the right is certainly not convex. Note that this definition is just a generalisation of the definition of a convex polygon in the plane. One very important thing to keep in mind is the fact that we usually think of a polyhedron as just the outside, the surface of the shape, and not as a solid object carved out of wood.



I'm sure you've probably seen and played with polyhedra many times before. For example, if you've ever picked up a standard soccer ball, you might have noticed that they're usually made by sewing together patches in the shape of pentagons and hexagons. Other examples include the shape formed by the buckminsterfullerene carbon molecule C60 which is found in soot, the Montréal Biodome at Parc Jean Drapeau and the pyramids of Egypt.

[1] Although it's all right to say "polyhedrons", the more common plural to use is "polyhedra". That's because the word polyhedron comes from the ancient Greek words which mean many faces.

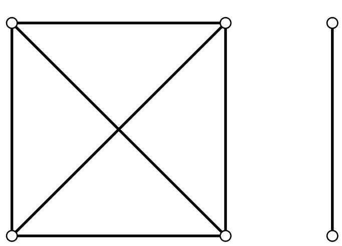




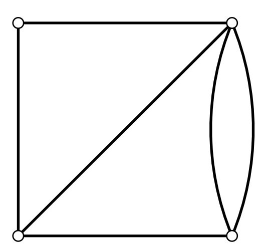
What is a Graph?

It's quite unfortunate that the word graph means at least two completely different things in mathematics. One type of graph that you are no doubt already aware of is the graph of a function or equation, plotted on a set of axes. You probably know that it's possible to plot the graph of an equation like $y = x^2$ and you obtain a shape called a parabola which looks like a large smiley face. We won't talk about these types of graph here at all — so forget you ever heard about them.

For us, a *graph* will always mean a set of points called *vertices*, connected in pairs by lines or curves called *edges*. We make no assumptions about whether the graph is in one connected piece or not. However, we usually won't allow loops — an edge whose endpoints are the same vertex — or multiple edges — more than one edge whose endpoints are the same pair of vertices.

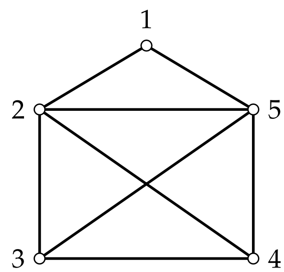


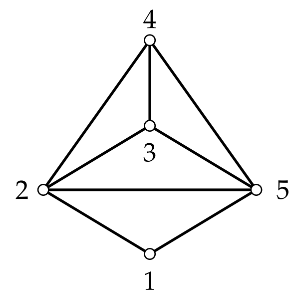
We'll call this a graph.



We won't call this a graph.

For our purposes, it's a good idea to think of a graph as a diagram whose vertices represent people at a party and whose edges represent friendship. So we should consider two graphs to be the same if they represent the same party. In particular, the important thing about a graph is not the way that you draw it on paper, but the relationships that the edges describe. Keeping this in mind, we say that two graphs G and H are *isomorphic* if there is a one-to-one correspondence between their vertices such that two vertices are connected by an edge in G if and only if their two corresponding vertices are connected by an edge in H. For example, the following two graphs are isomorphic and the labelling on the vertices tells you exactly why.





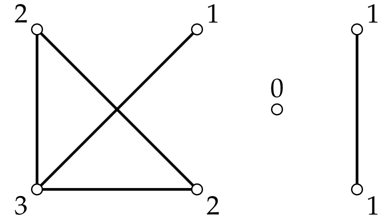
Every Polyhedron is a Graph

The fact that every polyhedron is a graph is a rather simple statement. This is because if you have a polyhedron and simply ignore the faces, then what you have left over is just a bunch of vertices connected in pairs by edges — in other words, a graph.

A more interesting statement is the following fact. Every polyhedron corresponds to a *planar* graph — in other words, a graph which can be drawn in the plane without any of its edges crossing. So why is this true? Well, suppose that your polyhedron is made from some sort of rubbery material, like a balloon. If you pop the balloon by removing one of the faces, then what remains is a rubbery sheet with the vertices and edges still drawn on it. Now just stretch this out flat onto a table and there you have your planar graph. Note that we needed to define our polyhedra to "not have holes" for this to trick to work — donut-shaped and other hole-possessing balloons just don't do the job.

The Handshaking Lemma

The *degree* of a vertex in a graph corresponds to a person's popularity index at a party — it's basically how many friends they have. So the degree of a vertex in a graph is simply the number of ends of edges which meet around that vertex. The following diagram shows a graph with each vertex labelled by its degree.



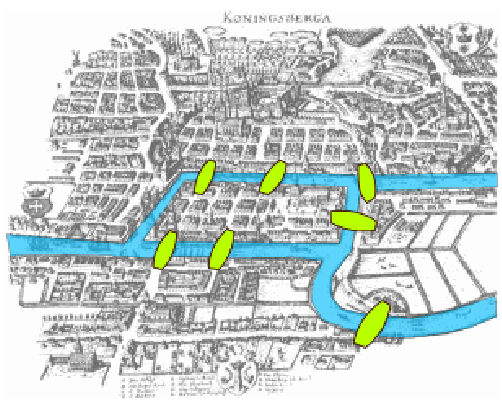
If you give me some numbers and ask me to find a graph with those numbers as its degrees, then the task is often not possible. Obviously, the numbers have to be non-negative integers, but it turns out that other things have to be true as well. The most important relation which these numbers must obey is the following.

Lemma (Handshaking lemma)**.** *At any party, if you ask everyone in the room (including yourself) how many hands they shook, and add up all of the answers, then you will get an even number. In fact, the number you get will be twice the number of handshakes that have occurred during the party. In graph theory terminology, this translates into the following fact — in any graph, the sum of the degrees of all the vertices is equal to twice the number of edges.*

Proof. At the start of the party, everyone has shaken zero hands because no handshakes have taken place. Each time a handshake takes place, the number of hands shaken increases by two, once for each person shaking hands. Therefore, after all handshakes have taken place, the sum of all of the answers must be equal to twice the number of handshakes. In graph theory terms, this means that the sum of the degrees of all the vertices is equal to twice the number of edges in the graph. □

The Bridges of Königsberg

Actually, graph theory began with the mathematician Leonhard Euler in the early eighteenth century. It's pretty amazing that such a simple mathematical construction took so long to be developed. It's now a thriving area of mathematics and, due to its very simplicity, has lots and lots of applications. In fact, the reason why Euler invented graph theory to begin with was to solve a very simple problem, known as the Seven Bridges of Königsberg problem.

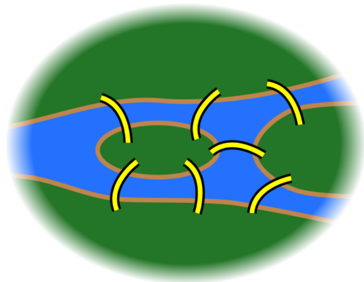


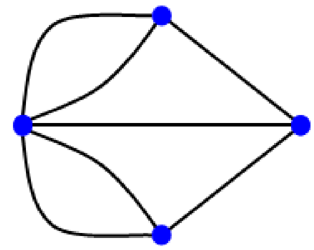
The above map shows what Königsberg — now known as Kaliningrad — looked like back in Euler's day. When the weather was nice, the locals liked to stroll about the town. One thing that many of them tried to do was to walk around Königsberg passing over every bridge exactly once, with no swimming allowed. Since people tried and tried without success, the problem became known as the Seven Bridges of Königsberg problem.

Proposition. *It's impossible to walk around Königsberg, passing over every bridge exactly once.*

Proof. Euler's approach was to pick out only the relevant aspects of the diagram. For example, it didn't really matter that there was a church on the corner of Third Avenue and Fifth Street, nor that there was a troll-like hobo standing on one of the bridges. So if we remove a lot of the irrelevant details, we end up with a picture such as the one below left. However, Euler realised that you could simplify the diagram further by creating one point for every land mass on the map and joining two points each time there is a bridge joining the two land masses. Of course, the result is just the graph pictured below right.[2]

[2]OK, so I said earlier that a graph can't have multiple edges — but we should be allowed to change the rules whenever we like, to suit our application. So for the purposes of solving this problem, let's temporarily allow graphs with multiple edges.





Let's suppose now that it's possible to walk around Königsberg, passing over every bridge exactly once. We'll start with the obvious fact that this walk must start at some vertex of the graph and end at some vertex of the graph. Between the start and end, each time we visit a vertex, we must have walked along two edges incident to it — one going in and one coming out. Even though we can visit a vertex many times, this reasoning tells us that every vertex other than the start and end must have even degree. So to be able to walk around Königsberg, passing over every bridge exactly once, the Königsberg graph above must have at most two vertices of odd degree. But you can see for yourself that all four vertices of the graph have odd degree, an obvious contradiction. □

In honour of Euler, we say that a graph is *Eulerian* if it's possible to walk around the graph, passing over each edge exactly once. The following theorem tells us exactly when a graph is Eulerian.

Theorem. *A connected graph is Eulerian if and only if it has zero or two vertices with odd degree.*

Note that if a graph is Eulerian, then it should consist of one piece and such graphs are called *connected*. Our argument used to solve the Seven Bridges of Königsberg problem tells us that if a connected graph is Eulerian, then it must have at most two vertices with odd degree. However, the handshaking lemma implies that no graph can possibly have exactly one vertex with odd degree. Therefore, if a connected graph is Eulerian, then it must have zero or two vertices with odd degree.

Conversely, it's always possible in such graphs to find a walk which traverses each edge exactly once. The general idea behind the proof is to imagine a drunk person stumbling randomly around the graph, and never passing over an edge twice. Eventually, they have to get stuck. If they've passed over each edge exactly once after they get stuck, then that's great. If not, then they they can always alter their route slightly to pass over more edges — to do this, we need to use the fact that the graph is connected and has zero or two vertices with odd degree. They can keep altering their route in this way until they finally have a route which passes over each edge exactly once.

Six People at a Party

There is a very famous result in graph theory which states that, at any party with six people, there must exist three people who all know each other or three people who all don't know each other. One way to represent this problem is to draw a graph with six vertices, one for each person, and then draw a red edge between two people who know each other or a blue edge between two people who don't know each other. So the theorem can be rephrased in graph theoretic terms in the following way. Prove that if, in a graph with six vertices, every pair of vertices is connected by an edge coloured red or blue, then there must exist a red triangle or a blue triangle. In fact, to make life easier, let's define the *complete graph* K_n to be the graph with n vertices such that every pair of vertices is connected by an edge. Then the six people at a party problem can be stated in the following way.

Proposition. *If every edge of the complete graph K_6 is coloured red or blue, then there must exist a red triangle or a blue triangle.*

Proof. Consider a random partygoer A — of the five edges which are connected to A, note that at least three of them have to be the same colour. So let's just go ahead and assume that there are three edges connected to A, all of which are red. The same argument will apply for the case that there are three edges connected to A, all of which are blue. Suppose that these red edges connect A to the party people B, C and D. If BC is red, then triangle ABC is red; if CD is red, then triangle ACD is red; and if DB is red, then triangle ADB is red. So to avoid a red triangle, the edges BC, CD and DB must be all be blue, which forces triangle BCD to be blue. Therefore, there must exist a red triangle or a blue triangle in the graph. □

Euler's Formula

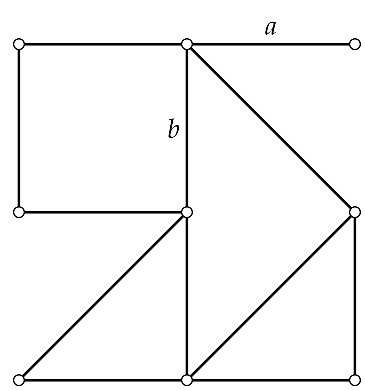
If you ask me to build a polyhedron which has V vertices, E edges and F faces, then sometimes I won't be able to do it — not because I'm incompetent, but because such a polyhedron may not exist. In fact, Euler noticed that the equation $V - E + F = 2$ holds for all polyhedra. For example, you can check that it's true for the following three examples, where a tetrahedron is just a fancy name for a triangular pyramid.

polyhedron	V	E	F
tetrahedron	4	6	4
cube	8	12	6
soccer ball	60	90	32

In fact, since polyhedra correspond to planar graphs, you might expect Euler's formula to work for planar graphs as well — and you'd be right. We have to be a little careful here, because Euler's formula will only work if you include the region outside the graph as a face. This is due to the fact that when we construct a planar graph from a polyhedron by flattening it, we had to remove a face first. Another subtle point is that your planar graph must be connected — otherwise, Euler's formula just cannot hold, and you should check this for yourself.

Theorem (Euler's Formula)**.** *For a connected planar graph with V vertices, E edges and F faces, $V - E + F = 2$.*

Proof. I'm just going to give a sketch proof here, something which is incredibly convincing, but which isn't actually a complete proof. Suppose that we would like to calculate $V - E + F$ for the graph on the right, but we're just too lazy to count the number of vertices, edges and faces. Consider an edge, like the one labelled a, which connects one vertex to the remainder of the graph. If you remove this edge and the vertex attached to it, then you obtain a new graph with one less vertex and one less edge. So this new graph has exactly the same value of $V - E + F$.



Now consider an edge, like the one labelled b, which separates two distinct faces of the graph. If you remove this edge, then you obtain a new graph with one less edge and one less face. So this new graph also has exactly the same value of $V - E + F$. So we can continue removing edges and vertices, without ever changing the value of $V - E + F$, until we are left with the simplest graph possible, consisting of only one vertex and hence, only one face. Now we can calculate the value of $V - E + F$ for this graph and the answer is simply $1 - 0 + 1 = 2$. Therefore, in our original planar graph, we know that $V - E + F = 2$. □

Dual Graphs

Given a planar graph, we can construct a new graph with a vertex for each of the faces of the original graph and an edge between two vertices if the corresponding faces meet along an edge. This new graph is called the *dual graph*. You should draw several examples to convince yourself of the following facts.

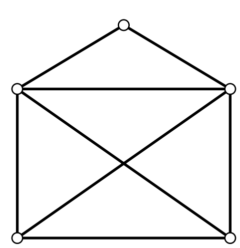
- The dual graph may have loops and multiple edges, but is always planar.
- The number of vertices of the dual graph is equal to the number of faces of the original graph. The number of edges of the dual graph is equal to the number of edges of the original graph. The number of faces of the dual graph is equal to the number of vertices of the original graph.
- The dual graph of the dual graph is the original graph.

One extremely useful thing to do with a polyhedron or a planar graph is apply the handshaking lemma to its dual. To do this, you need to know what the degree of a vertex in the dual graph is. However, a vertex in the dual graph corresponds to a face in the original graph and the degree of a vertex in the dual graph is exactly the number of edges around the corresponding face in the original graph. Therefore, we obtain the following result.

Proposition. *In a planar graph or a polyhedron, the sum of the numbers of edges around each face is equal to twice the number of edges.*

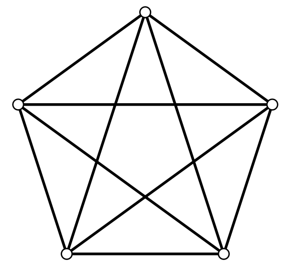
More on Planar Graphs

As fascinating as graph theory is, we're actually here to learn about geometry. And as we saw earlier, one way that geometry makes contact with graph theory is via polyhedra. In particular, we showed that every polyhedron corresponds to a graph and, not just any old graph, but a planar graph. We have to take a little care with the definition of a planar graph, because we say that a graph is planar if it *can* be drawn in the plane without any of its edges crossing, not if it *is* drawn in the plane without any of its edges crossing. So, for example, although the following graph has been drawn with two edges crossing, it actually is planar, as you can check for yourself.



Some graphs just aren't planar, no matter how hard you try to draw them without edges crossing. The one with the smallest number of vertices happens to be the complete graph K_5.

Proposition. *The complete graph K_5 is not planar.*



Proof. To obtain a contradiction, let's suppose that K_5 is planar. Given that $V = 5$ and $E = 10$, Euler's formula tells us that, if we could draw K_5 in the plane without edges crossing, then we would have $F = 7$. But it should be clear that if we could draw K_5 in the plane without edges crossing, then every face must have at least three edges around it — this is because K_5 has no multiple edges or loops. This means that the sum of the numbers of edges around each face is at least $7 \times 3 = 21$. However, by the proposition above concerning the handshaking lemma in the dual graph, we know that the sum of the number of edges around each face coincides with twice the number of edges. In other words, we obtain the inequality $2E \geq 21$ which implies that $20 \geq 21$. Contradictions don't come more blatant than that, so we may now conclude that there is no way to draw K_5 in the plane without edges crossing — in short, K_5 is not planar. □

You can try and prove in an entirely analogous way that $K_{3,3}$ is not planar either, although the proof is a little bit more difficult. A very difficult theorem to prove which is, in my opinion, quite amazing, is the following result which lets you decide whether or not a graph is planar.

Theorem (Kuratowski's Theorem)**.** *A graph is planar if and only if it does not contain a a smaller graph which looks like K_5 or $K_{3,3}$.*

When I say a graph which looks like K_5, I mean a graph which you can make out of K_5 by adding in some essentially useless vertices along its edges. Here's another amazing theorem which involves planar graphs and is also quite difficult to prove.

Theorem (Fáry's Theorem)**.** *Every planar graph can be drawn in the plane without edges crossing so that each edge is represented by a line segment.*

Euler

Leonhard Paul Euler — pronounced "oiler", not "yooler" — was a Swiss mathematician who lived from 1707 to 1783. He is widely regarded as the pre-eminent mathematician of the eighteenth century and one of the greatest of all time. He is the most prolific mathematician to ever have lived, with his collected works filling about 75 quarto volumes. He was also quite prolific in other ways as evidenced by the fact that he fathered at least thirteen children.

I could go on all day about Euler's contributions to mathematics and science. For example, he created graph theory when he solved the Seven Bridges of Königsberg problem; he discovered the Euler line in Euclidean geometry; he discovered the formula $V - E + F = 2$ for polyhedra; he introduced the notation e for the natural base of logarithms, $f(x)$ for a function, $\sum$ for a summation, and i for the square root of negative one; he was the master of the branch of mathematics known as analysis; he discovered many theorems in number theory including a generalisation of Fermat's Little Theorem; his influence was integral in the birth of analytic number theory, a very important branch of mathematics; and he is renowned for work in mechanics, fluid dynamics, optics and astronomy.

Euler discovered many remarkable formulas in his day — for example, he showed that the exponential function e^x could be represented as an infinite polynomial, or Taylor series.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

He solved the famous Basel problem which asked for the sum of the reciprocals of the perfect squares — quite remarkably, the answer is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Euler also established the following formula concerning complex numbers, the exponential function, and trigonometric functions.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This was called "the most remarkable formula in mathematics" by the famous physicist Richard Feynman. In 1988, readers of the Mathematical Intelligencer voted it the most beautiful formula ever and their poll also included two other formulas of Euler in the top five.

Euler's eyesight worsened throughout his life and he eventually became nearly blind in his right eye. His eyesight worsened so much over time that he earned the nickname Cyclops. Later in life, Euler suffered a cataract in his good left eye, rendering him almost totally blind in 1766. Even so, his condition appeared to have little effect on his productivity, as he compensated for it with his mental calculation skills and photographic memory. For example, Euler could repeat the Aeneid of Virgil from beginning to end without hesitation, and for every page in the edition he could indicate which line was the first and which the last. With the aid of his scribes, Euler's productivity on many areas of study actually increased. In fact, he amazingly produced one mathematical paper per week in the year 1775.