More Facts about Finite Symmetry Groups

Last time we proved that a finite symmetry group cannot contain a translation or a glide reflection. We also discovered that in any finite symmetry group, either every isometry is direct or there is an equal number of direct and opposite isometries. Now we just need a few more facts before we can put the pieces together to obtain Leonardo's Theorem.

Lemma. *In a finite symmetry group, every rotation must have the same centre.*

Proof. Suppose that our finite symmetry group has a rotation R_1 with centre O_1 and a rotation R_2 with centre O_2. Obviously, our goal is to prove that $O_1 = O_2$. Consider the composition $R_2^{-1} \circ R_1^{-1} \circ R_2 \circ R_1$ — it must be a direct isometry and, due to the way that rotations compose together, it must actually be a translation.[1] But we already know that our group can't contain translations which aren't the identity — obviously this means that the composition $R_2^{-1} \circ R_1^{-1} \circ R_2 \circ R_1$ must in fact be the identity.

Therefore, we know that $R_2^{-1} \circ R_1^{-1} \circ R_2 \circ R_1(O_1) = O_1$ from which it follows that $R_2^{-1} \circ R_1^{-1} \circ R_2(O_1) = O_1$. If we apply R_2 to both sides, the R_2 and the R_2^{-1} on the left hand side knock each other out and the equation simplifies to $R_1^{-1} \circ R_2(O_1) = R_2(O_1)$. Now we apply R_1 to both sides so that the R_1 and the R_1^{-1} on the left hand side knock each other out and the equation simplifies yet again to

$$R_1 \circ R_2(O_1) = R_2(O_1).$$

Another way to say the same thing is that $R_2(O_1)$ is a fixed point of R_1. However, since R_1 is a rotation, we know that it has a unique fixed point, which we have called O_1. So we can deduce that $R_2(O_1) = O_1$. Another way to say the same thing is that O_1 is a fixed point of R_2. However, since R_2 is a rotation, we know that it has a unique fixed point, which we have called O_2. So we've deduced that $O_1 = O_2$. □

A simple consequence of the previous result is the following.

Lemma. *In a finite symmetry group, every mirror of a reflection passes through the same point.*

Proof. If there is only one reflection in the finite symmetry group, then there is nothing to prove. If there are at least two mirrors, then you can compose the corresponding reflections to obtain rotations. By the previous lemma, we know that all of these rotations have the same centre O.

Now suppose that two mirrors meet at a point P. Then the composition of the corresponding reflections is a rotation about P. But we've already stated that this rotation, since it belongs to our finite symmetry group, must have centre O. Therefore, the two mirrors must have met at O and it follows that every mirror passes through O. □

One final result that we'll need to use is the following little lemma. We'll leave the proof of this as an exercise for the enthusiastic reader.

Lemma. *In a finite symmetry group, the rotations are of the form $I, R, R^2, R^3, \ldots, R^{n-1}$ for some rotation R.*

[1] Expressions like $R_2^{-1} \circ R_1^{-1} \circ R_2 \circ R_1$ which take the form $a \cdot b \cdot a^{-1} \cdot b^{-1}$ are known as *commutators* and are incredibly useful in group theory and other areas of mathematics.

The Proof of Leonardo's Theorem

Recall that Leonardo's theorem states that if a subset of the Euclidean plane has finitely many symmetries, then its symmetry group must be the cyclic group C_n or the dihedral group D_n for some positive integer n. Most of the mathematical content in the proof of Leonardo's theorem is in the lemmas we proved above. All we need to do now is fit the pieces of the puzzle together as follows.

Proof.

- Suppose that someone gives you a finite symmetry group. By the lemma proved earlier, we know that there are no translations nor are there glide reflections. In other words, every element of the finite symmetry group must be a rotation or a reflection.
- If there are only rotations, then the last lemma above guarantees that the finite symmetry group must be C_n for some positive integer n. If there are only reflections as well as the identity, then the finite symmetry group must be D_1. This is because there must be an equal number of direct and opposite isometries.
- So we're left with the case that there are both rotations and reflections, in which case there must be equal numbers of each.[2] The first lemma above states that all of the rotations must share the same centre O while the second lemma above states that all of the mirrors of reflections must pass through O.
- We know that the rotations are "evenly spaced" by the previous lemma, so the rotations must be by the angles
$$0^\circ, \quad \frac{360^\circ}{n}, \quad 2 \times \frac{360^\circ}{n}, \quad \ldots, \quad (n-1) \times \frac{360^\circ}{n},$$
for some positive integer n. If the mirrors were not evenly spaced as well, then there must be two of them which create an angle strictly less than $\frac{180^\circ}{n}$. The composition of the reflections through these two mirrors will then yield a rotation of strictly less than $\frac{360^\circ}{n}$ — a contradiction.
- Therefore, we can conclude that the rotations are equally spaced with centre O and that the mirrors are equally spaced and pass through O. But this just means that the finite symmetry group is dihedral. □

Frieze Patterns

Let's turn our attention now to infinite symmetry groups. Since we've already seen that a finite group of symmetries can't contain a translation, an easy way to create an infinite symmetry group is to consider a single translation T. A symmetry group which contains T necessarily contains the infinitely many elements[3]

$$\ldots, T^{-3}, T^{-2}, T^{-1}, I, T, T^2, T^3, \ldots.$$

Keeping this in mind, we define a *frieze pattern* to be a subset of the Euclidean plane whose symmetry group contains a horizontal translation T along with

$$\ldots, T^{-3}, T^{-2}, T^{-1}, I, T, T^2, T^3, \ldots,$$

and no other translations.

[2] Here, we're considering the identity as a rotation through an angle of zero.
[3] Here, the notation T^n stands for the composition of n copies of T, while T^{-n} stands for the composition of n copies of T^{-1}.

A frieze pattern can certainly have other symmetries which aren't translations, but there aren't too many possibilities for such other symmetries. Actually, it shouldn't be too difficult to see that they can only come in four types.

- H = a reflection in a horizontal mirror
- V = a reflection in a vertical mirror
- R = a 180° rotation
- G = a glide reflection along a horizontal axis

So for each frieze pattern, we can give it an $HVRG$ symbol, depending on which of these four symmetries it possesses. For example, a frieze pattern with symbol VRG would have symmetry group which contains a reflection in a vertical mirror, a 180° rotation and a glide reflection along a horizontal axis, but not a reflection in a horizontal mirror. If we decide to classify frieze patterns in this way, then we obtain the following result.

Theorem. *There are exactly seven types of frieze pattern.*

Proof. There are at most sixteen possible HVRG symbols.

$$\text{none}, \quad H, \quad V, \quad R, \quad G, \quad HV, \quad HR, \quad HG, \quad VR, \quad VG, \quad RG,$$
$$VRG, \quad HRG, \quad HVG, \quad HVR, \quad HVRG$$

However, the following four observations allows us to exclude some of these examples.

- If you have H, then you have G.
- If you have V and R, then you have G.
- If you have R and G, then you have V.
- If you have G and V, then you have R.

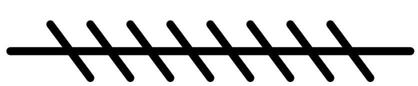
You can prove the first of these observations, for example, by composing H and T to obtain G. The other three facts can be proved in a similar way. If you go ahead and use these observations to eliminate some of the possibilities, then you should find that there are only seven remaining.

$$\text{none}, \quad V, \quad R, \quad G, \quad HG, \quad VRG \quad HVRG$$

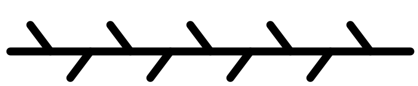
Of course, this only shows that there are at most seven types of frieze. We still have to demonstrate that each of these possibilities actually arises, which we accomplish in the following diagrams. □

none

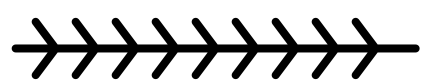
V



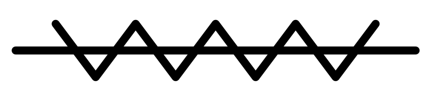
R



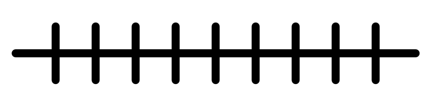
G



HG



VRG

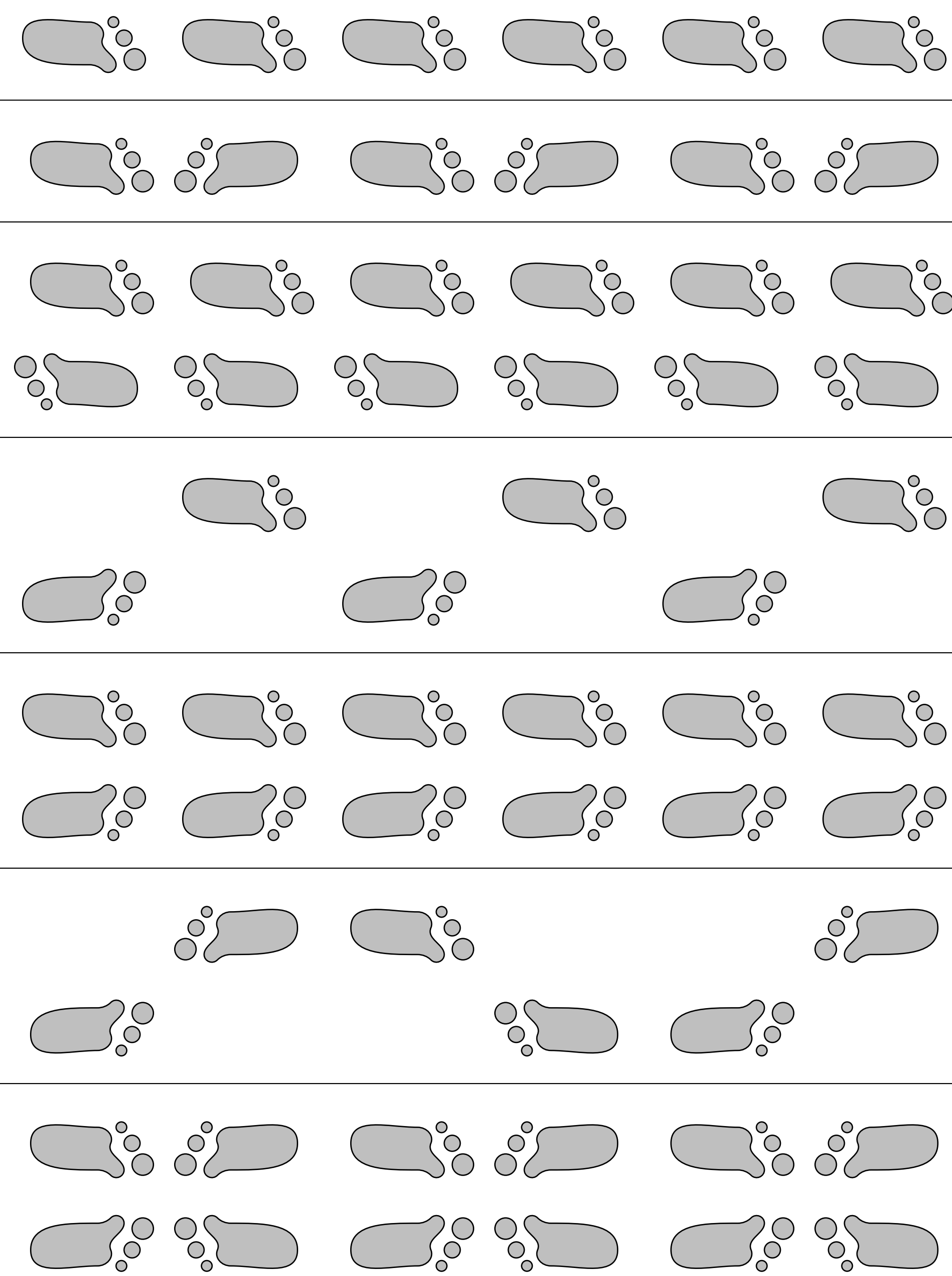


$HVRG$

The mathematician John Conway — who often has his own spin on certain mathematical theorems and proofs — has coined his own set of names for the types of frieze patterns.

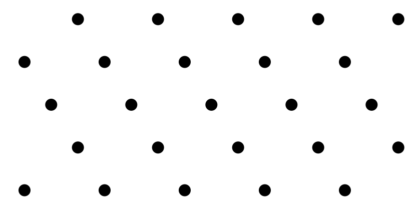
- HOP (none)
- SIDLE (V)
- SPINNING HOP (R)
- STEP (G)
- JUMP (HG)
- SPINNING SIDLE (VRG)
- SPINNING JUMP ($HVRG$)

The following diagrams should hopefully explain Conway's rather strange nomenclature for the frieze patterns.



The Crystallographic Restriction

Consider applying two translations S and T in different directions to a point P. By taking various combinations of S and T, we obtain a set of points in the plane. Any set of points in the plane which you can obtain in this way is called a *lattice*. You can think of a lattice as an equally spaced orchard of apple trees.



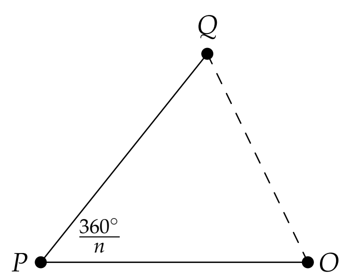
Of course, the translations S and T are symmetries of the lattice, as is any combination involving S and T. Another symmetry of the lattice is a rotation by $180°$ about one of the lattice points. However, it could be possible that there are other rotational symmetries, although there is some restriction as to what those rotational symmetries could be. In order to state the theorem more explicitly, let's define the order of a rotation R to be the smallest positive integer n such that $R^n = I$.

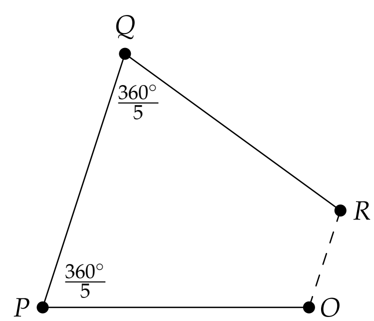
Theorem (Crystallographic Restriction). *If the symmetry group of a lattice contains a rotation, then that rotation must have order 2, 3, 4 or 6.*

Proof. First, we state a simple fact, which the enthusiastic reader is encouraged to prove on their own.

Any group which contains a rotation of order n also contains a rotation by angle $\frac{360°}{n}$.

Let's call a point in the plane *special* if it's the centre of a rotational symmetry of the lattice of order n. Pick any special point O and let P be the closest special point to it. By the fact stated above, a rotation by $\frac{360°}{n}$ about any special point must be a symmetry of the lattice. But, furthermore, any such rotation must be a symmetry of the set of special points. So if we rotate O about P by $\frac{360°}{n}$, we obtain a point Q which is special. For $n \geq 7$, this point Q is closer to O than P, which contradicts our assumption. So we can deduce that there is no rotational symmetry of the lattice of order greater than or equal to 7.





If $n = 5$, then we rotate P about Q by $\frac{360°}{n}$ to obtain another special point R. However, this point R is now closer to O than P, which contradicts our assumption. So we can deduce that there is no rotational symmetry of the lattice of order 5. Hence, we may conclude that any rotational symmetry of the lattice must have order 2, 3, 4 or 6. □

This result derives its name from the fact that it also holds for three-dimensional lattices. In that case, the statement is important in crystallography — the study of crystals — and guarantees that any rotational symmetry of a crystal must have order 2, 3, 4 or 6. It would be nice if the theorem also held in greater than three dimensions, but it just isn't true. This is because rotations in higher dimensions behave in a very different way than in two or three dimensions.

Wallpaper Patterns

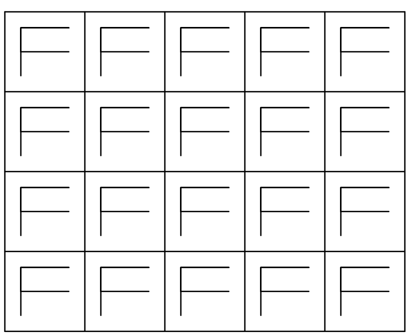
If a subset of the Euclidean plane has a symmetry group whose translations form a lattice, then we call that subset a *wallpaper pattern*. Simply put, they're patterns that cover the whole plane and are repetitive in two different directions. Just as we did for friezes, we can try to classify the types of wallpaper patterns — however, the game is much harder this time. It turns out that there are exactly seventeen different types of wallpaper patterns, although we won't provide a proof here. We'll merely content ourselves with knowing how to identify them. To each wallpaper pattern, we associate an *RMG* symbol consisting of the following three numbers R, M and G.

- R = the maximum order of a rotational symmetry
- M = the maximum number of mirrors which pass through a point
- G = the maximum number of proper glide axes which pass through a point

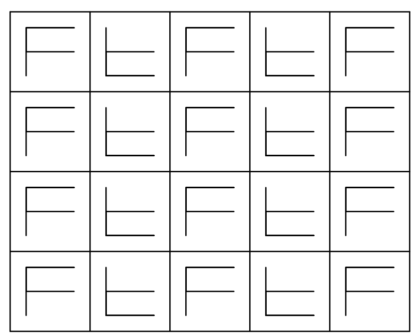
By a proper glide axis, we mean the axis of a glide reflection which is not itself a mirror. The seventeen possible types of wallpaper patterns are pictured below. For each wallpaper pattern, make sure that you can find a centre of a rotational symmetry of order R, a point through which M mirrors pass, and a point through which G proper glide axes pass.

Unfortunately, the *RMG* symbol doesn't quite distinguish all seventeen wallpaper patterns — there are two which are described by 332. So let's call them 332A and 332B and note that they can be distinguished using the following observation.

- For the wallpaper pattern 332A, it's possible to find a centre of rotation which doesn't lie on a mirror.
- For the wallpaper pattern 332B, every centre of rotation lies on a mirror.



100



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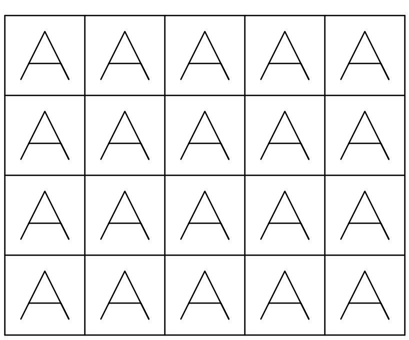

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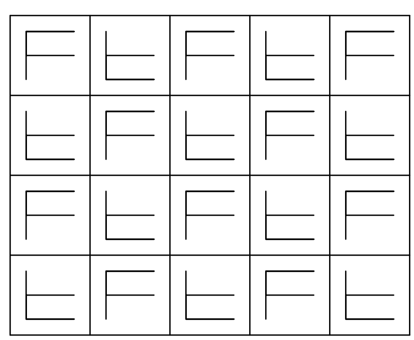

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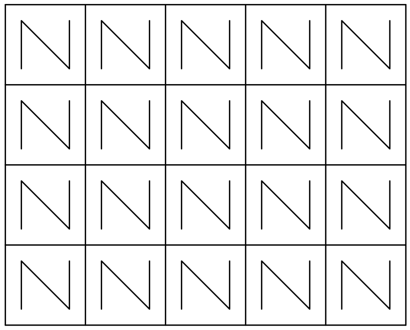

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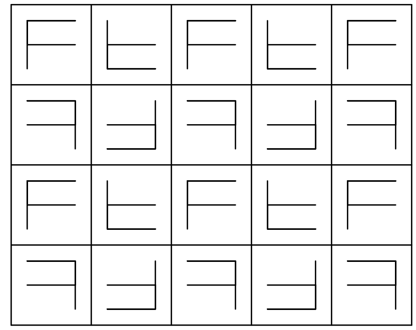
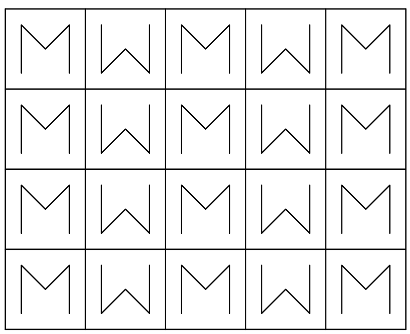


202



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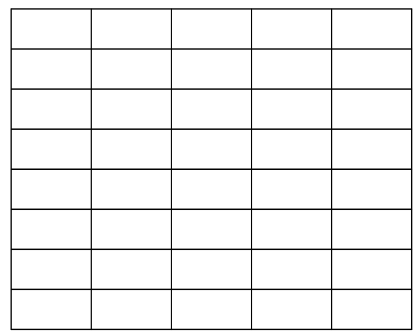

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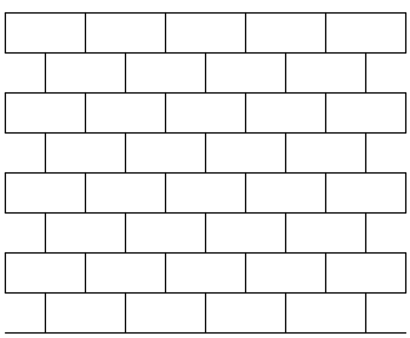
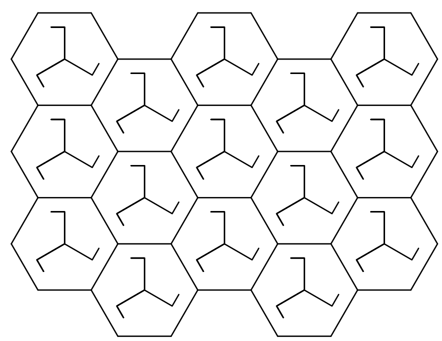


222



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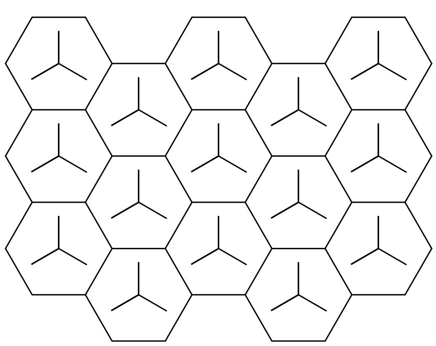
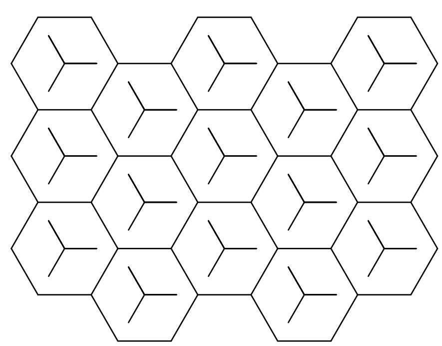
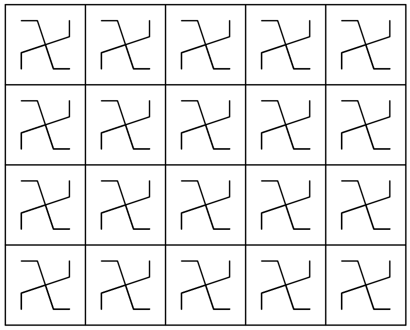


332A



332B



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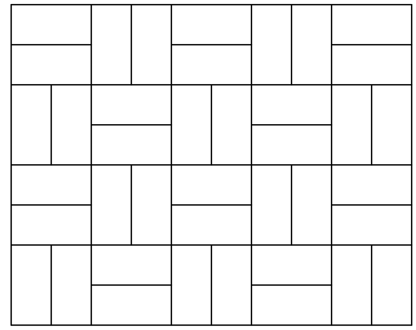
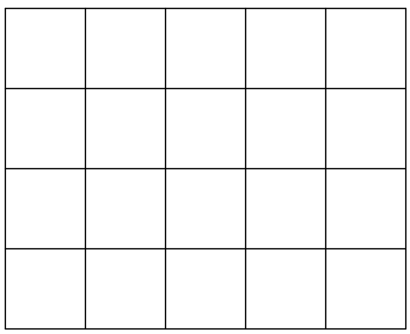


423



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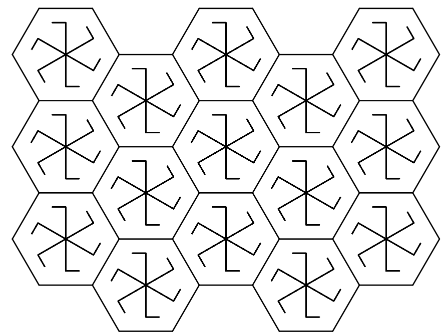
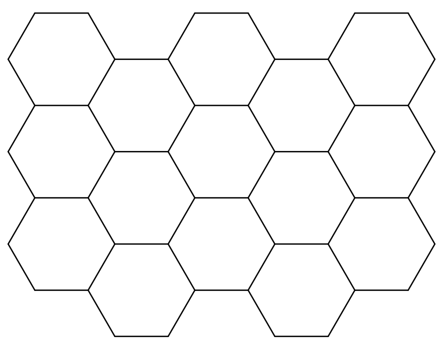


600



664

Problems

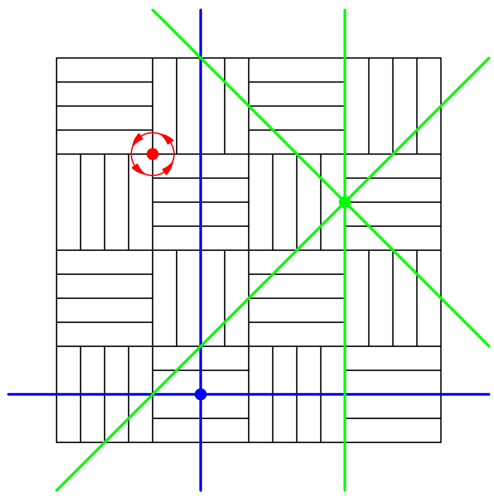
Problem. *Prove that if a symmetry group of a frieze pattern contains a reflection in a vertical mirror and a rotation by* 180°*, then it must also contain a glide reflection.*

Proof. If we denote the reflection in a vertical mirror by V and the rotation by 180° by R, then the symmetry group of the frieze pattern must also contain $R \circ V$. This composition is an opposite isometry and it is easy to see that it is either reflection in a horizontal mirror or a glide reflection along a horizontal axis. (Remember that the centre of rotation for R does not have to lie on the mirror for V.) Since the symmetry group of a frieze pattern contains a horizontal translation by definition, in either case, the symmetry group of the frieze pattern must contain a glide reflection along a horizontal axis.

So how do you know that the composition $R \circ V$ is a reflection in a horizontal mirror or a glide reflection along a horizontal axis? One way to see this is to consider what happens to the picture of a left footprint walking right, above the centre of rotation of R. After applying the reflection V, this becomes a right footprint walking left, above the centre of rotation of R. And then after applying R, this becomes a right footprint walking right below the centre of rotation of R. The only way to start with a left footprint walking right and end up wit ha right footprint walking right is via a reflection in a horizontal mirror or a glide reflection along a horizontal axis. □

Problem. *For each wallpaper pattern pictured above, if it has symbol RMG, find on the diagram a point which is the centre of a rotational symmetry of order R, a point where M mirrors meet, and a point where G proper glide axes meet.*

Proof. For example, the diagram below shows the wallpaper pattern with RMG symbol equal to 423 with a point which is the centre of a rotational symmetry of order 4, a point which 2 mirrors pass through, and a point which 3 proper glide axes pass through. You can do something similar for the other wallpaper patterns. □



Abel

Earlier we spoke about Galois, a mathematician who proved that it was impossibile to write down a formula to solve the quintic equation, before dying at the tender age of twenty. In fact, the first person to give a complete proof of this beat Galois by several years. He was a Norwegian mathematician by the name of Niels Henrik Abel and he was born in 1802 before dying in 1829, living a whole six years longer than Galois.

Abel's proof of the impossibility of solving the quintic equation was not as deep and far-reaching as Galois' proof, but was nonetheless extremely novel. Unfortunately, his work was extremely difficult to read, partly because he had to cut out all of the details to save money on printing. His success in mathematics gave him some finances to travel around Europe and meet some of the more well-known mathematicians. Unfortunately, he contracted tuberculosis while in Paris and became quite ill. During this time, a friend of his had found a prestigious professorship for Abel in Berlin and wrote him a letter to tell him the good news. Unfortunately, the letter arrived two days after Abel died.

The early death of this extremely talented mathematician cut short a career of extraordinary brilliance and promise. Abel had managed to clear some of the prevailing obscurities of mathematics and paved the way for several new fields. His complete works were edited and eventually published by the Norwegian government. The adjective "abelian" is derived from Abel's name and is so commonplace in mathematics that mathematicians don't even bother to capitalise it any more.

Abel is still a relatively big deal in Norway — there have been stamps, banknotes and coins bearing his portrait; there is a crater on the Moon named after him; and there is the prestigious Abel prize for mathematicians, presented by the King of Norway and worth about a million dollars in prize money.