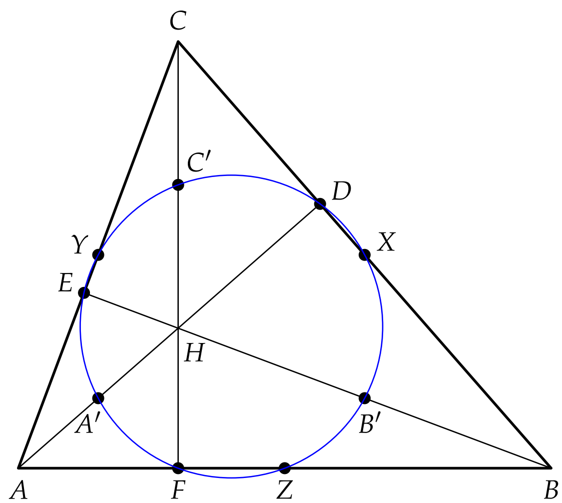
The Nine Point Circle

Now we come to one of the real gems of geometry, a theorem which was discovered more than two thousand years after Euclid's day. Remember that for every three points which do not lie on a line, there is a unique circle which passes through them. For four points, there is very rarely going to be a circle which passes through them all, and if there is, then those four points form a very special type of quadrilateral called a cyclic quadrilateral. Given that it's already quite difficult for four points to lie on a circle, then it must be near impossible to find nine points which lie on a circle. And yet, this is precisely what the nine point circle theorem tells us — we can find nine points which lie on a circle, associated to any particular triangle we choose to think about.

Theorem (Nine Point Circle Theorem)**.** *Let ABC be a triangle with altitudes AD, BE, CF, medians AX, BY, CZ, and orthocentre H. If A', B', C' are the midpoints of AH, BH, CH, respectively, then the nine points $A', B', C', D, E, F, X, Y, Z$ all lie on a circle which is — for obvious reasons — called the* nine point circle *of triangle ABC.*



Proof. There are midpoints galore in this problem — in fact, six of the nine points that we are interested in are defined as midpoints. So it seems like a prime opportunity to use the midpoint theorem. Applied in triangle ABH, we obtain that $A'B'$ is parallel to AB. Applied in triangle ABC, we obtain that XY is also parallel to AB. Applied in triangle ACH, we obtain that $A'Y$ is parallel to CH. Applied in triangle BCH, we obtain that $B'X$ is also parallel to CH.

In summary, XY and $A'B'$ are parallel to each other and to AB. Furthermore, $A'Y$ and $B'X$ are parallel to each other and to CH. However, since CH is perpendicular to AB by assumption, we know that $A'B'XY$ must be a rectangle. The exact same argument can be used to show that $B'C'YZ$ and $C'A'ZX$ are also rectangles.

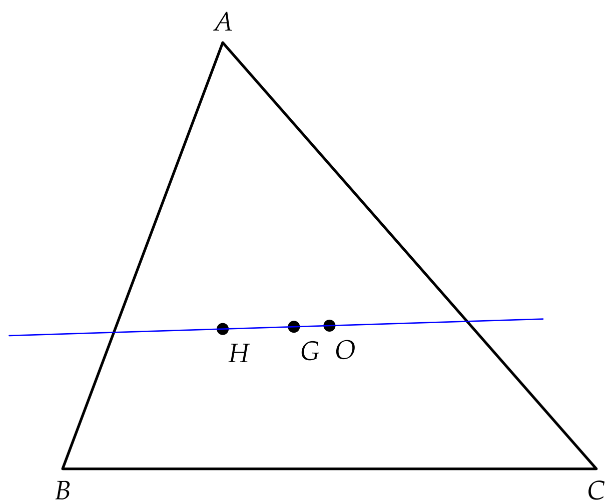
Now let N be the midpoint of $A'X$ and note that this means that N is the centre of the circle passing through the vertices of rectangle $A'B'XY$ as well as the centre of the circle passing through the vertices of $C'A'ZX$, both of which have $A'X$ as diameters. Therefore, the six points A', B', C', X, Y, Z all lie on a circle with centre N. Six points down and three to go...

Since $A'B'XY$ is a rectangle, we have $\angle A'YX = 90^\circ$. But $\angle A'DX = 90^\circ$ as well, so we know that the quadrilateral $A'XDY$ is cyclic. In particular, D lies on the circumcircle of triangle $A'XY$, which we have already seen actually passes through the six points A', B', C', X, Y, Z. The same argument shows that E and F lie on this circle as well. So we can now conclude that the nine points $A', B', C', D, E, F, X, Y, Z$ all lie on a circle. □

The Euler Line and Other Gems

This next theorem tells us that three of the four triangle centres that we've already considered are very closely related — in fact, they always lie on a line. It's amazing that such a simple fact seems to have escaped everybody's notice until Euler, a pretty amazing mathematician whom we'll encounter again later, arrived on the scene in the eighteenth century.

Theorem. *The orthocentre H, the centroid G and the circumcentre O of any triangle lie on a line known as the* Euler line*. Furthermore, we have the equation $HG = 2GO$.*



There are many, many, many, many, many more geometric gems out there and we've really only scratched the surface of Euclidean geometry. For example, there is the following fact which adds the nine point circle centre to the list of points lying on the Euler line.

Theorem. *The orthocentre H, the nine point circle centre N, the centroid G and the circumcentre O of any triangle lie on a line known as the* Euler line*. Furthermore, we have the equation $HN = NO$.*

To be honest, Euclidean geometry is not a thriving area of research mathematics. This is probably because new theorems in Euclidean geometry have limited uses in other areas. But that doesn't mean they aren't interesting, and it doesn't mean that people aren't discovering new geometric gems all the time. Here's a pair of related facts, the first which was discovered just over a century ago, while the second was discovered about ten years ago.

Theorem.

- *Observe that if you take any four points A, B, C, D, then you can draw the nine point circles of the triangles ABC, BCD, CDA and DAB. Amazingly, all four of these circles meet at a single point — let's call this point P.*
- *Now suppose that we consider the feet of the perpendiculars from A to the sides of triangle BCD, where you may have to extend the sides. This will give us three points whose circumcircle we call the A-circle. Similarly, there is a B-circle, a C-circle and a D-circle. Amazingly, all four of these circles — you guessed it — meet at a single point, which happens to coincide precisely with the point P.*

It's amazing — or at least I think it's amazing — that you can build this incredibly intricate world of geometry with just pen, paper and ten little axioms.

Non-Euclidean Geometry

We're now going to go back to where our journey began, way back to Euclid and his axioms. The common notions are pretty much trivial statements with no real geometric content. So let's look at the other five axioms, which are commonly referred to as *postulates*.

A1. You can draw a unique line segment between any two given points.

A2. You can extend a line segment to produce a unique line.

A3. You can draw a unique circle with a given centre and a given radius.

A4. Any two right angles are equal to each other.

A5. Suppose that a line ℓ meets two other lines, making two interior angles on one side of ℓ which sum to less than 180°. Then the two lines, when extended indefinitely, will meet on that side of ℓ.

It didn't take very long at all for mathematicians to notice that one of the postulates sticks out like a sore thumb. Can you guess which one it is? Of course you can… it's the only one which takes more than one line to write down. At a deeper level, it just seems to be conceptually so much more complicated than the rest. So for centuries upon centuries after Euclid announced his axioms, one of the burning questions in mathematics was whether or not the fifth axiom was needed at all, whether or not it could be proved from the other axioms.

Many mathematicians throughout the ages tackled the problem, but to no avail. A small advance was made by showing that you could write down the parallel postulate in the following simpler looking, but actually equivalent, form. In fact, when Euclid's fifth axiom is written in this way, it's often referred to as the *parallel postulate*.

A5′. Given a line and a point not on the line, there exists a unique line through the given point, parallel to the given line.

One common approach was to to assume the opposite of the parallel postulate and keep deducing and deducing, with the aim of finding a contradiction. If a contradiction could be found, then the opposite of the parallel postulate was false, thereby proving that the parallel postulate was true. Unfortunately — or possibly, fortunately — no one was successful in doing so. What they should have done is stop looking for a contradiction, because there are none. If they had done this, then they would have realised that all the statements that they were deducing were theorems in a new type of geometry. Since this geometry comes from taking the opposite of one of Euclid's axioms, it's commonly known as *non-Euclidean geometry*.

There were various people who contributed to the discovery of non-Euclidean geometry, all of whom lived around the turn of the nineteenth century. The mathematicians Bolyai and Lobachevsky probably deserve the most credit, but their ideas are related to work by Saccheri, Gauss and various other people. Note that there are actually two ways to change the parallel postulate so as to create a non-Euclidean geometry.

- Given a line and a point not on the line, there exists *no line* through the given point, parallel to the given line.
- Given a line and a point not on the line, there exists *more than one line* through the given point, parallel to the given line.

Spherical Geometry

Euclidean geometry, for thousands of years, seemed to reflect the world around us and was the only geometry studied by mathematicians. But why should mathematics follow the rules of the real world? Why can't you invent your own rules? And this is precisely what mathematicians decided to do. Nowadays, there are many different types of geometry which people study. And usually, after studying these, people discovered that they turned out to be both interesting and surprisingly useful to the real world. So what makes these seemingly crazy-sounding theories deserve to be called geometry? Loosely speaking, you can think of a geometry as anything where you have objects called points and lines, where points can lie on lines and lines can intersect lines, and so on.

The simplest non-Euclidean geometry is actually rather simple. It's *spherical geometry*, which is the geometry we are used to when we talk about flying around the Earth. Take a sphere — let's say it has radius 1 to be definite — and consider the paths of shortest distance between two points. These are just arcs of great circles, those circles whose centre is the centre of the sphere. We can now define the points in spherical geometry to be the points on the unit sphere, and we also define the lines in spherical geometry to be the great circles. We now have a new type of geometry with points and lines and angles and circles and so forth, but which satisfies the following statement.

- Given a line and a point not on the line, there exists *no line* through the given point, parallel to the given line.

You can see that two lines in spherical geometry can never be parallel, because they always meet at two points on opposite sides of the sphere.

It's possible to prove lots of interesting theorems in spherical geometry — here's just one interesting example. It shows that you can calculate the area of a spherical triangle from its three angles alone, something which is impossible to do in Euclidean geometry.

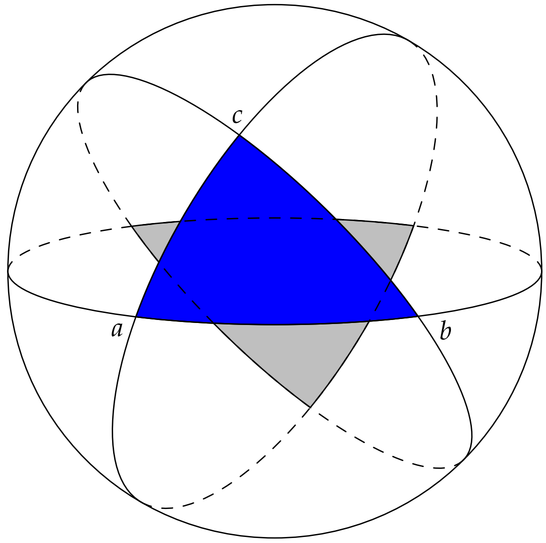
Theorem. *A spherical triangle whose angles measured in radians*[1] *are a, b, c has area $a + b + c - \pi$.*

Proof. Let's think of our sphere as an orange with radius 1 and recall that its surface area is simply 4π. Consider what happens if you slice through the orange twice, with the knife passing through the centre of the orange, such that the two cuts make an angle of a radians with each other. This will create two slices, on opposite sides of the orange, whose total surface area is $\frac{2a}{2\pi}$ times the total surface area of the orange. So these two slices will have surface area $\frac{2a}{2\pi} \times 4\pi = 4a$.

Now take a spherical triangle whose angles measured in radians are a, b, c. If we extend the sides, we obtain a diagram very much like the one below, where the blue spherical triangle is the one whose area we wish to calculate. Note that there is the same triangle, which we've shaded in grey, appearing on the back of the sphere as well.

Cutting along the two lines which form the angle a will create two orange slices with area $4a$. Similarly, cutting along the two lines which form the angle b will create two orange slices with area $4b$. And similarly again, cutting along the two lines which form the angle c will create two orange slices with area $4c$.

[1] I'm hoping that you already know how to measure angles in radians, but if not, then it's easy to learn. All you need to know is that π radians is the same thing as 180°. Now you're probably wondering why on earth someone would decide to measure angles in this way. A better thing to wonder is why on earth someone would decide to split a straight angle into 180 parts and call each one a degree. Radians are very natural, because if you take a slice of pizza of a given angle, then the angle measured in radians is simply the length of the crust divided by the length of one of the cuts — in other words, the length of the arc divided by the radius of the circle. Because this is so natural, many formulas in mathematics look a whole lot nicer when you use radians to measure angles.



Now let's see what happens when we add up all of these areas. Most of the sphere is accounted for exactly once, but the blue spherical triangle has been accounted for three times and the grey spherical triangle has also been accounted for three times. So if we let A denote the area of the blue triangle, or equivalently, the area of the grey triangle, we have the equation

$$4a + 4b + 4c = 4\pi + 4A.$$

Here, the number 4π represents the total surface area of the sphere with radius 1. This equation rearranges to give the desired formula $A = a + b + c - \pi$. $\square$

One very interesting consequence of this theorem is the fact that the angles in a spherical triangle must add to more than $180°$. This should seem intuitively true, because the sides of a spherical triangle seem to "bulge outwards", creating larger angles than a Euclidean triangle. This is one of the major differences between spherical and Euclidean geometry.

Remember that one of our goals is to show that you can have a geometry in which all of Euclid's axioms are true except for the parallel postulate. Certainly, the parallel postulate fails to hold in spherical geometry, but unfortunately it's not the only one which fails. For example, it's not true in spherical geometry that you can draw a unique line segment between any two given points. That's because between the north pole and the south pole of the sphere, there are infinitely many line segments. However, knowing a bit about how spherical geometry works will help us with hyperbolic geometry, a case where all of Euclid's axioms do hold except for the parallel postulate.

Hyperbolic Geometry

We now consider a type of geometry which satisfies the following statement.

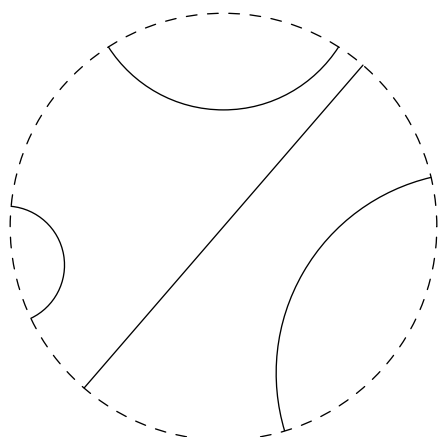
- Given a line and a point not on the line, there exists *more than one line* through the given point, parallel to the given line.

This geometry will be far more difficult to visualise and is conceptually more removed from everyday experience. The geometry we will talk about is called *hyperbolic geometry* and there are many ways to describe it. The particular way that we're going to use is called the *Poincaré disk model*.

In the Poincaré disk model, the hyperbolic plane is an open disk — let's say it has radius 1 to be definite — or, in other words, all of the points which are inside, but not on the circumference of, a circle. Very intuitively, you can think of the hyperbolic plane as a circular pond of quicksand, one where it's easy to walk around when you're close to the middle, but which gets infinitely difficult to travel through when you're close to the edge. So the effort that it takes you to get from point A to point B or equivalently, the *hyperbolic distance* from point A to point B, is warped and not like the normal Euclidean distance between two points. This means that if you were told to walk from point A to point B, you would probably take a curved path, which bends toward the middle of the circular pond of quicksand. Anyway, it's possible to make all of this mumbo jumbo precise and the end result is that hyperbolic geometry works as follows.

- A hyperbolic point is just a normal point inside the hyperbolic plane.
- A hyperbolic line is the arc of a circle which is perpendicular to the boundary of the disk or a diameter of the disk.

You can see some examples of hyperbolic lines drawn in the Poincaré disk model below. Remember that lines are said to be parallel if they never meet. You should now be able to convince yourself that, given a line in hyperbolic geometry and a point not on the line, there exists more than one line through the given point, parallel to the given line.



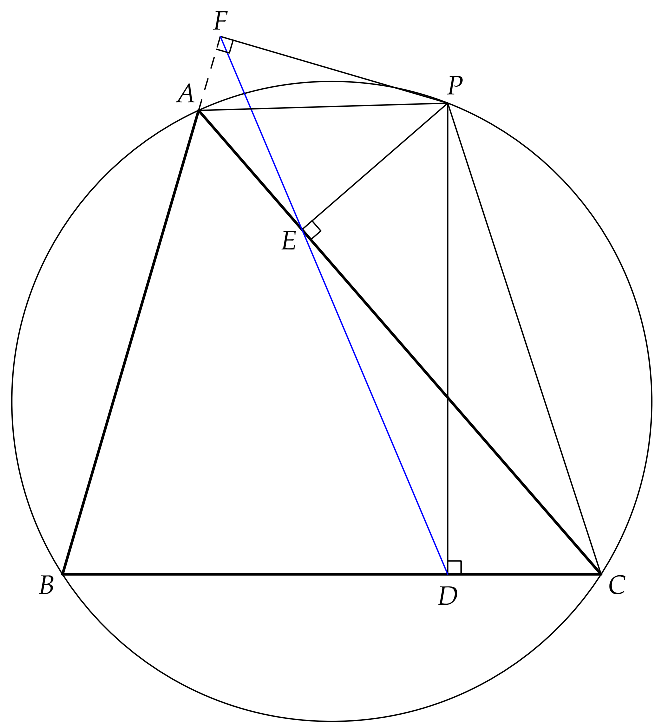
Remember that in spherical geometry, the area of a triangle with angles a, b, c is simply $a + b + c - \pi$. Not only does this mean that you can calculate the area of a triangle from its angles alone, but also that in every spherical triangle, the angles add up to more than 180°. Hyperbolic geometry is, in many ways, the exact opposite of spherical geometry. For example, in every hyperbolic triangle, the angles add up to less than 180°. This should seem intuitively true, because the sides of a hyperbolic triangle seem to "curve inwards", creating smaller angles than a Euclidean triangle. Furthermore, you can calculate the area of a hyperbolic triangle from its angles alone in the following way.

Theorem. *A hyperbolic triangle whose angles measured in radians are a, b, c has area $\pi - a - b - c$.*

Note that there are theorems which will be the same in Euclidean, spherical or hyperbolic geometry. This is simply because there are results which you can prove which don't rely on the parallel postulate at all. Even though hyperbolic geometry seems the most far removed from our everyday experience, it is, in some sense, the most mathematically important and is used in various areas of pure mathematics and theoretical physics.

Problems

Problem. *Consider a point P on the circle which passes through the vertices of triangle ABC. Let D be the point on BC such that PD is perpendicular to BC. Let E be the point on CA such that PE is perpendicular to CA. Let F be the point on AB such that PF is perpendicular to CA. (You may have to extend the lines AB, BC and CA for this to be possible.) Prove that the points D, E and F lie on a line.*[2]



Proof. The first thing to do, as always, is to draw a large accurate diagram. With a big fat circle in the picture, it is hard to miss the fact that the quadrilateral $ABCP$ must be cyclic. This is a very useful piece of information to note down because we're going to use it later.

In general, right angles will often lead to cyclic quadrilaterals, and there are actually three of them that we haven't yet mentioned. One of the easiest ones to see is the quadrilateral $PFAE$, which is cyclic because its opposite angles add up to 180°.

$$\angle PFA + \angle AEP = 90^\circ + 90^\circ = 180^\circ.$$

Another one which is not too difficult to spot is the quadrilateral $PFBD$, which is also cyclic because its opposite angles add up to 180°.

$$\angle PFB + \angle BDP = 90^\circ + 90^\circ = 180^\circ$$

Finally, we consider quadrilateral $PEDC$, which is cyclic because of the hockey theorem applied to the equal angles

$$\angle PEC = \angle PDC = 90^\circ.$$

[2]The line passing through the points D, E and F is actually known as the *Simson line* of the point P and the triangle ABC.

Let's remember exactly what we're trying to prove — namely, that the points D, E and F lie on a line. One way to prove this would be to show that $\angle PEF + \angle PED = 180°$. If we label the angle $\angle PEF = x$, then the hockey theorem applied to the cyclic quadrilateral $PFAE$ tells us that $\angle PAF = x$ as well. The angle next door to this must then be $\angle PAB = 180° - x$. Since the opposite angles in the cyclic quadrilateral $ABCP$ must add up to $180°$, this in turn yields the fact that $\angle BCP = x$. And now we use the fact that the opposite angles in the cyclic quadrilateral $PEDC$ must add up to $180°$ to tell us that

$$\angle PED = 180° - \angle DCP = 180° - \angle BCP = 180° - x.$$

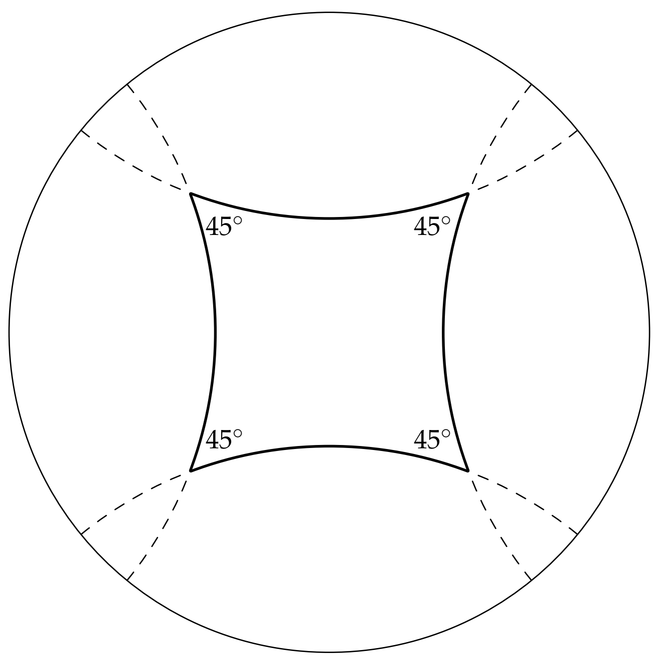
If you've labelled all of these angles on your diagram, then you will have noticed that we have now labelled both $\angle PEF$ and $\angle PED$. This is most useful, because we now have

$$\angle PEF + \angle PED = x + (180° - x) = 180°.$$

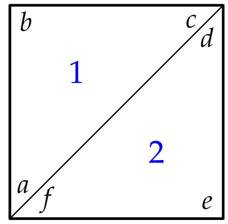
And this is exactly what we're aiming to prove, because it implies that the points D, E and F lie on a line. □

Problem. *In the hyperbolic plane, there exist quadrilaterals all of whose angles are equal to* $45°$. *Sketch one example of such a quadrilateral in the hyperbolic plane using the Poincaré disk model. What is the area of this quadrilateral?*

Proof. The large circle below represents the Poincaré disk model of the hyperbolic plane. The four dotted curves represent four hyperbolic lines. As you can see from the diagram, they form a quadrilateral all of whose angles are equal to $45°$.



Now consider the following schematic diagram for the quadrilateral, where the hyperbolic lines are represented by normal straight lines. We have divided the quadrilateral into two triangles — numbered 1 and 2 — and labelled every single angle in the diagram.



Since each angle of the quadrilateral is equal to 45°, we have four equations that these labelled angles must satisfy. Here, we have labelled the angles using radians rather than degrees.

$$a + f = \frac{\pi}{4}, \quad b = \frac{\pi}{4}, \quad c + d = \frac{\pi}{4}, \quad e = \frac{\pi}{4}$$

One of the advantages of using radians is that the area of a hyperbolic triangle is much easier to determine. In fact, we know that the area of triangle 1 is $\pi - a - b - c$ and the area of triangle 2 is $\pi - d - e - f$. Adding up the areas of these two triangles, we deduce that the area of the quadrilateral is

$$\begin{aligned} & (\pi - a - b - c) + (\pi - d - e - f) \\ = \; & 2\pi - (a + b + c + d + e + f) \\ = \; & 2\pi - (a + f) - (b) - (c + d) - (e) \\ = \; & 2\pi - \frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{4} \\ = \; & \pi. \qquad \square \end{aligned}$$

Gauss

Johann Carl Friedrich Gauss — who was born in 1777 and died in 1855 — was a German mathematician who contributed significantly to a variety of subjects. The book *Men of Mathematics* by Eric Temple Bell and published in 1937 — which apart from the sexist title, is a decent read — has a chapter on Gauss which begins "Archimedes, Newton and Gauss, these three, are in a class by themselves among the great mathematicians, and it's not for ordinary mortals to attempt to range them in order of merit." Hopefully this convinces you that Gauss is regarded among the best mathematicians who ever walked the earth.

Gauss was known to be a child prodigy and there is a famous story which tells of his primary school teacher asking the class to add up the numbers from 1 up to 100. The very young Gauss produced the correct answer in seconds, to the astonishment of his teacher. Presumably, his method was to pair the numbers 1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101, and so on, obtaining fifty pairs, each with sum 101. Hence, the answer is $50 \times 101 = 5050$.

It seems that Gauss may have discovered the possibility of non-Euclidean geometry but, for some strange reason, decided not to publish it. In fact, when a younger mathematician by the name of János Bolyai discovered non-Euclidean geometry and published his work in 1832, Gauss wrote that "To praise it would amount to praising myself. For the entire content of the work...coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years."