Triangle Centres

The humble triangle, with its three vertices and three sides, is actually a very remarkable object. For evidence of this fact, just look at the *Encyclopedia of Triangle Centers*[1] on the internet, which catalogues literally thousands of special points that every triangle has. We'll only be looking at the big four — namely, the circumcentre, the incentre, the orthocentre, and the centroid. While exploring these constructions, we'll need all of our newfound geometric knowledge from the previous lecture, so let's have a quick recap.

- *Triangles*
  - *Congruence* : There are four simple rules to determine whether or not two triangles are congruent. They are SSS, SAS, ASA and RHS.
  - *Similarity* : There are also three simple rules to determine whether or not two triangles are similar. They are AAA, PPP and PAP.
  - *Midpoint Theorem* : Let ABC be a triangle where the midpoints of the sides BC, CA, AB are X, Y, Z, respectively. Then the four triangles AZY, ZBX, YXC and XYZ are all congruent to each other and similar to triangle ABC.
- *Circles*
  - The diameter of a circle subtends an angle of $90°$. In other words, if AB is the diameter of a circle and C is a point on the circle, then $\angle ACB = 90°$.
  - The angle subtended by a chord at the centre is twice the angle subtended at the circumference, on the same side. In other words, if AB is a chord of a circle with centre O and C is a point on the circle on the same side of AB as O, then $\angle AOB = 2\angle ACB$.
  - *Hockey Theorem* : Angles subtended by a chord on the same side are equal. In other words, if A, B, C, D are points on a circle with C and D lying on the same side of the chord AB, then $\angle ACB = \angle ADB$.
- *Cyclic Quadrilaterals*
  - The opposite angles in a cyclic quadrilateral add up to $180°$. In other words, if $ABCD$ is a cyclic quadrilateral, then $\angle ABC + \angle CDA = 180°$ and $\angle BCD + \angle DAB = 180°$.
  - If the opposite angles in a quadrilateral add up to $180°$, then the quadrilateral is cyclic.
  - *Hockey Theorem* : If $ABCD$ is a convex quadrilateral such that $\angle ACB = \angle ADB$, then the quadrilateral is cyclic.
- *Tangents*
  - *Ice Cream Cone Theorem* : A picture of an ice cream cone is symmetric so that the two tangents have the same length and the line joining the centre of the circle and the tip of the cone bisects the cone angle.
  - *Alternate Segment Theorem* : Suppose that AT is a chord of a circle and that PQ is a line tangent to the circle at T. If B lies on the circle, on the opposite side of the chord AT from P, then $\angle ABT = \angle ATP$.

These facts should all be burned into your memory, so that you can recall and use them, whenever you encounter a problem in Euclidean geometry.

[1] `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

The Circumcentre

A *perpendicular bisector* of a triangle is a line which passes through the midpoint of one side and is perpendicular to that side. Note that three randomly chosen lines will almost never ever meet at a point and yet, for any particular triangle we choose, we'll see that its three perpendicular bisectors always do.

Proposition. *The three perpendicular bisectors of a triangle meet at a point.*

Proof. Our proof relies crucially on the following lemma, which can be proven using congruent triangles.

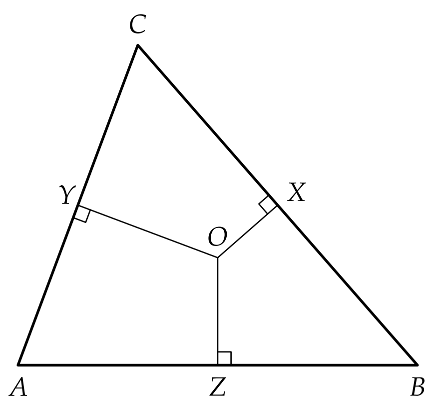
> *Lemma.* A point P lies on the perpendicular bisector of AB if and only if $AP = BP$.

So take a triangle ABC and the perpendicular bisectors of the sides AB and BC. If we suppose that these two lines meet at a point O, then it must be the case that $AO = BO$ and also that $BO = CO$. These two equations together imply that $AO = CO$, in which case O lies on the perpendicular bisector of the side CA as well. □

In the previous proof, we noted that if the perpendicular bisectors of triangle ABC meet at O, then the distances from O to the vertices are all equal. Another way to say this is that there's a circle with centre O which passes through the vertices A, B, C. This circle is called the *circumcircle* of the triangle and the point O is called the *circumcentre*.[2] Furthermore, the radius of the circumcircle is known as the *circumradius* for obvious reasons. We now know that every triangle has exactly one circumcircle and that its centre lies on the perpendicular bisectors of the triangle.

Something interesting to note is that when triangle ABC is acute, O lies inside the triangle; when triangle ABC is right-angled, O lies on the hypotenuse of the triangle, at its midpoint; and when triangle ABC is obtuse, O lies outside the triangle.

Let's draw an acute triangle ABC and draw in the three perpendicular bisectors XO, YO, ZO, just like I've done below. There are three cyclic quadrilaterals lurking in the diagram — surely you can spot them.

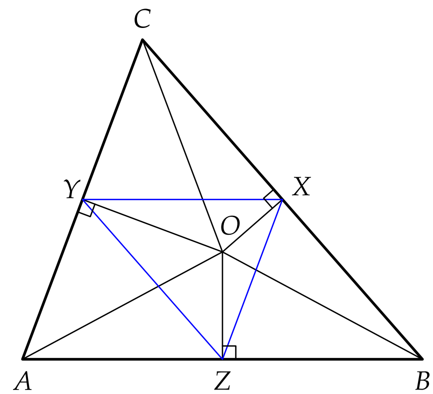


The cyclic quadrilaterals are

- $AZOY$ — the opposite angles $\angle AZO$ and $\angle OYA$ add to $180°$;
- $BXOZ$ — the opposite angles $\angle BXO$ and $\angle OZB$ add to $180°$; and
- $CYOX$ — the opposite angles $\angle CYO$ and $\angle OXC$ add to $180°$.

[2] In Latin, the word "circum" means around and this makes sense because the circumcentre goes around the triangle.

Let's now draw in the three line segments AO, BO, CO as well as the triangle XYZ. An extremely useful exercise is to label all of the thirty-six angles in the diagram in terms of $a = \angle CAB$, $b = \angle ABC$ and $c = \angle BCA$.



- The midpoint theorem states that the triangles AZY and ABC are similar, so we have $\angle YZA = b$. You can use this strategy to label six of the angles in the diagram.
- Since the quadrilateral $AZOY$ is cyclic, the hockey theorem tells us that $\angle YOA = \angle YZA = b$. You can use this strategy to label six more of the angles in the diagram.
- Since the sum of the angles in triangle YOA is 180°, we must have $\angle YAO = 90° - \angle YOA = 90° - b$. You can use this strategy to label six more of the angles in the diagram.
- Since the quadrilateral $AZOY$ is cyclic, the hockey theorem tells us that $\angle YZO = \angle YAO = 90° - b$. You can use this strategy to label six more of the angles in the diagram.
- The remaining twelve angles can be labelled by using the fact that the angles in a triangle add to 180°.

The Incentre

An *angle bisector* of a triangle is a line which passes through a vertex and bisects the angle at that vertex. Note that three randomly chosen lines will almost never ever meet at a point and yet, for any particular triangle we choose, we'll see that its three angle bisectors always do.

Proposition. *The three angle bisectors of a triangle meet at a point.*

Proof. Our proof relies crucially on the following lemma, which can be proven using congruent triangles.

> *Lemma*. A point P lies on the angle bisector of $\angle ABC$ if and only if the distance from P to AB is equal to the distance from P to CB.

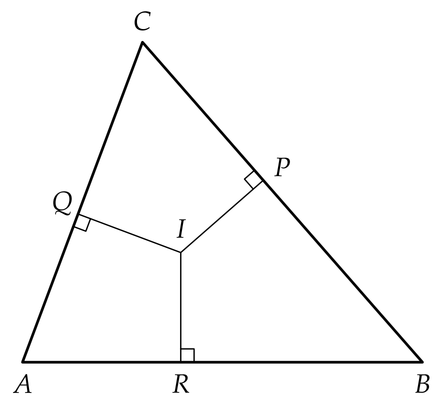
So take a triangle ABC and the angle bisectors at the vertices A and B. If we suppose that these two lines meet at a point I, then it must be the case that the distance from I to CA equals the distance from I to AB and the distance from I to AB equals the distance from I to BC. These two statements together imply that the distance from I to CA equals the distance from I to BC, in which case I lies on the angle bisector at the vertex C as well. □

In the previous prof, we noted that if the angle bisectors of triangle ABC meet at I, then the distances from I to the three sides are all equal. Another way to say this is that there's a circle with centre I which touches the sides AB, BC, CA. This circle is called the *incircle* of the triangle and the point I is called the *incentre*.[3]

[3] In Latin, the word "in" means inside and this makes sense because the incentre goes inside the triangle.

Furthermore, the radius of the incircle is known as the *inradius* for obvious reasons. We now know that every triangle has exactly one incircle and that its centre lies on the angle bisectors of the triangle.

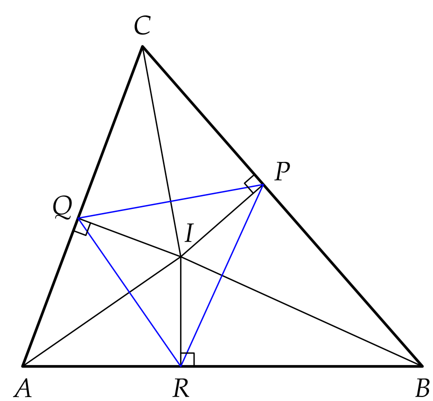
Let's draw a triangle ABC and draw in the three radii of the incircle PI, QI, RI, just like I've done below. There are three cyclic quadrilaterals lurking in the diagram — surely you can spot them.



The cyclic quadrilaterals are

- $ARIQ$ — the opposite angles $\angle ARI$ and $\angle IQA$ add to 180°;
- $BPIR$ — the opposite angles $\angle BPI$ and $\angle IRB$ add to 180°; and
- $CQIP$ — the opposite angles $\angle CQI$ and $\angle IPC$ add to 180°.

Let's now draw in the three line segments AI, BI, CI as well as the triangle PQR. An extremely useful exercise is to label all of the thirty-six angles in the diagram in terms of $a = \angle CAB$, $b = \angle ABC$ and $c = \angle BCA$.



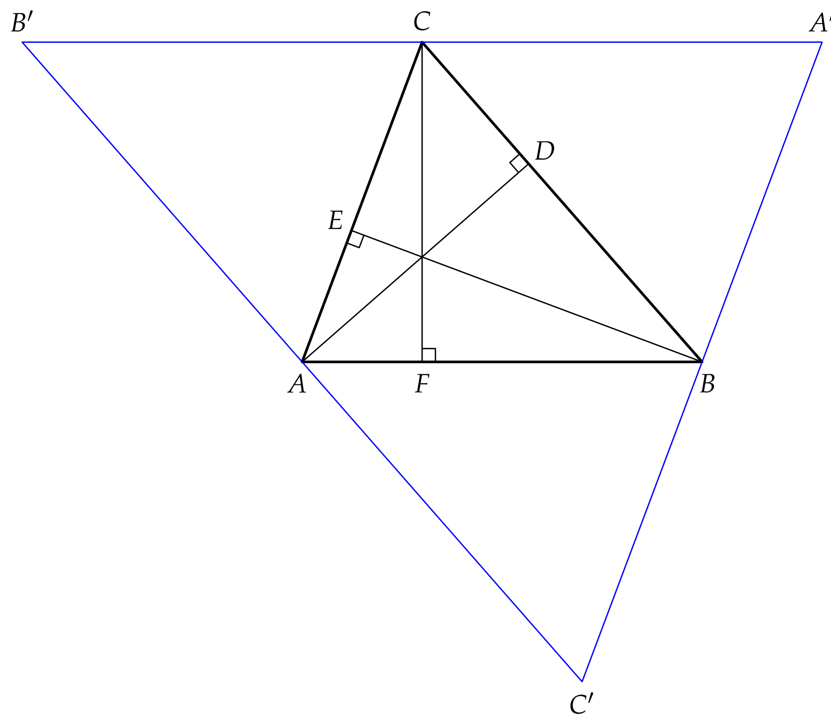
- The ice cream cone theorem states that the angles QAI and RAI are equal, so we have $\angle QAI = \angle RAI = \frac{a}{2}$. You can use this strategy to label six of the angles in the diagram.
- Since the quadrilateral $ARIQ$ is cyclic, the hockey theorem tells us that $\angle QRI = \angle QAI = \frac{a}{2}$. You can use this strategy to label six more of the angles in the diagram.
- Since the sum of the angles in triangle QAI is 180°, we must have $\angle QIA = 90° - \angle QAI = 90° - \frac{a}{2}$. You can use this strategy to label six more of the angles in the diagram.
- Since the quadrilateral $ARIQ$ is cyclic, the hockey theorem tells us that $\angle QRA = \angle QIA = 90° - \frac{a}{2}$. You can use this strategy to label six more of the angles in the diagram.
- The remaining twelve angles can be labelled by using the fact that the angles in a triangle add to 180° and, in fact, they are all equal to 90°.

The Orthocentre

An *altitude* of a triangle is a line which passes through a vertex and is perpendicular to the opposite side. If the triangle happens to have an angle greater than $90°$, then you will need to extend the sides in order to draw all three altitudes. Note that three randomly chosen lines will never ever meet at a point yet, for any particular triangle we choose, we'll see that its three altitudes always do.

Proposition. *The three altitudes of a triangle meet at a point.*

Proof. Draw the triangle $A'B'C'$ such that $A'B'$ is parallel to AB and C lies on $A'B'$, $B'A'$ is parallel to BC and A lies on $B'C'$, and $C'C'$ is parallel to CA and b lies on $C'A'$.



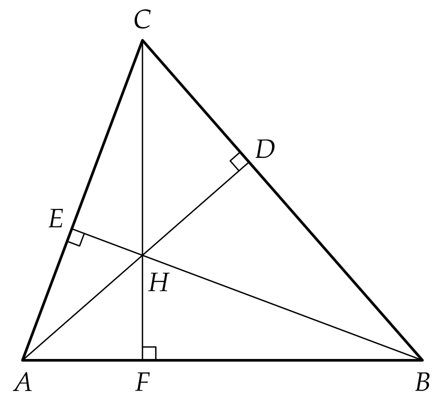
This creates the parallelograms $ABCB'$ and $ABA'C$, so that we have the equal lengths $B'C = AB = CA'$. Similarly, we have the equations $C'A = BC = AB'$ and $A'B = CA = BC'$. So the points A, B, C are simply the midpoints of the sides $B'C', C'A', A'B'$, respectively. This is precisely the setup for the midpoint theorem.

Since CF is perpendicular to AB, it's also parallel to $B'A'$ — in other words, CF is the perpendicular bisector of $A'B'$. Similarly, we know that AD is the perpendicular bisector of $B'C'$ and BE is the perpendicular bisector of $C'A'$. But we proved earlier that the three perpendicular bisectors of a triangle meet at a point. Therefore, the three altitudes AD, BE, CF of triangle ABC meet at a point. □

The three altitudes AD, BE, CF of triangle ABC meet at a single point H called the *orthocentre*.[4] Something interesting to note is that when triangle ABC is acute, H lies inside the triangle; when triangle ABC is right-angled, H lies at a vertex of the triangle; and when triangle ABC is obtuse, H lies outside the triangle.

Let's draw an acute triangle ABC and draw in the three altitudes AD, BE, CF, just like I've done below. Amazingly, there are six cyclic quadrilaterals lurking in the diagram — can you find them all?

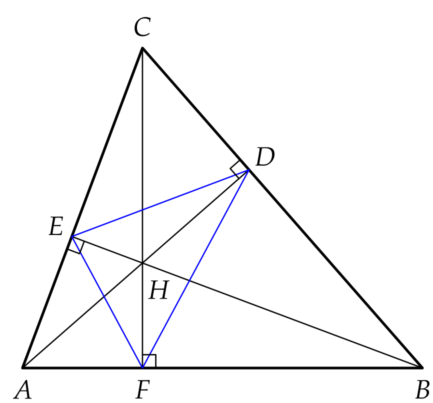
[4] In ancient Greek, the word "ortho" means vertical and this makes sense because the altitude of a triangle is vertical with respect to the base.



The cyclic quadrilaterals are

- $AFHE$ — the opposite angles $\angle AFH$ and $\angle HEA$ add to $180°$;
- $BDHF$ — the opposite angles $\angle BDH$ and $\angle HFB$ add to $180°$;
- $CEHD$ — the opposite angles $\angle CEH$ and $\angle HDC$ add to $180°$;
- $ABDE$ — the angles $\angle ADB$ and $\angle AEB$ are equal;
- $BCEF$ — the angles $\angle BEC$ and $\angle BFC$ are equal; and
- $CAFD$ — the angles $\angle CFA$ and $\angle CDA$ are equal.

Let's now draw in the triangle DEF. An extremely useful exercise is to label all of the thirty-six angles in the diagram in terms of $a = \angle CAB$, $b = \angle ABC$ and $c = \angle BCA$.



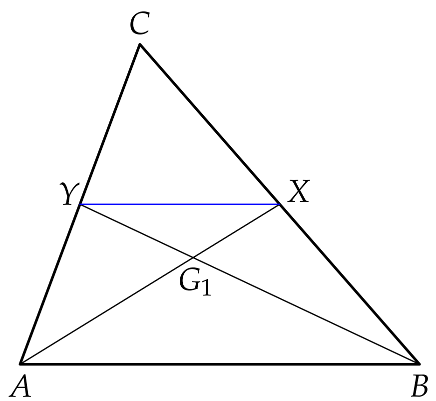
- Since the sum of the angles in triangle ABD is $180°$, we must have $\angle BAD = 90° - \angle DBA = 90° - b$. You can use this strategy to label six of the angles in the diagram.
- Since the quadrilateral $AFHE$ is cyclic, the hockey theorem tells us that $\angle FEH = \angle FAH = 90° - b$. You can use this strategy to label six more of the angles in the diagram.
- Since $\angle HEA$ is a right angle, we must have $\angle FEA = 90° - \angle FEH = b$. You can use this strategy to label six more of the angles in the diagram.
- Since the quadrilateral $AFHE$ is cyclic, the hockey theorem tells us that $\angle FHA = \angle FEA = b$. You can use this strategy to label six more of the angles in the diagram.
- The remaining twelve angles can be labelled by using the fact that the angles in a triangle add to $180°$.

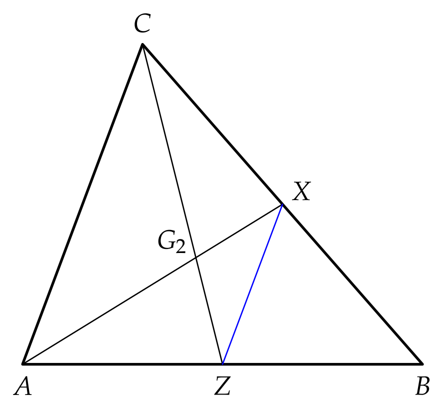
The Centroid

A *median* of a triangle is a line which passes through a vertex and the midpoint of the opposite side.

Proposition. *The three medians of a triangle meet at a point.*

Proof. Suppose that the medians AX and BY meet at G_1. By the midpoint theorem, we know that XY is parallel to AB which implies that the triangles ABG_1 and XYG_1 are similar by AAA. We also know by the midpoint theorem that $\frac{AB}{XY} = 2$, so the constant of proportionality is 2. This means that $\frac{AG_1}{G_1X} = 2$. So the median BY cuts the median AX at a point G_1 such that AG_1 is twice as long as G_1X.





Now suppose that the medians AX and CZ meet at G_2. By the midpoint theorem, we know that XZ is parallel to AC which implies that the triangles ACG_2 and XZG_2 are similar by AAA. We also know by the midpoint theorem that $\frac{AC}{XZ} = 2$, so the constant of proportionality is 2. This means that $\frac{AG_2}{G_2X} = 2$. So the median CZ cuts the median AX at a point G_2 such that AG_2 is twice as long as G_2X.

Putting these two pieces of information together, we deduce that the points G_1 and G_2 must actually be the same point. In other words, the medians BY and CZ meet the median AX at the same point. □

The three medians AX, BY, CZ of triangle ABC meet at a single point G called the *centroid*. It is the centre of gravity of ABC in the sense that if you cut the triangle out of cardboard, then it should theoretically balance on the tip of a pencil placed at the point G. One consequence of our proof above is the fact that we have the equal fractions

$$\frac{AG}{GX} = \frac{BG}{GY} = \frac{CG}{GZ} = 2.$$

More Fun with Triangle Centres

There is so much more to triangle centres than we have mentioned. Let's write down some interesting facts here, which you can try to prove on your own.

Proposition. *If the orthocentre of triangle ABC is H, then the orthocentre of triangle HBC is A, the orthocentre of triangle HCA is B and the orthocentre of triangle HAB is C.*

Proposition. *If you take a triangle ABC and draw in the three medians AX, BY, CZ, then the six resulting triangles all have equal area.*

The following proposition shows that the four triangle centres we have looked at are related in various peculiar ways — you should try to prove all of these statements.

Proposition. *Given a triangle ABC, let the midpoints of the sides be X, Y, Z, let the incircle touch the sides at P, Q, R, and let the feet of the altitudes be D, E, F.*

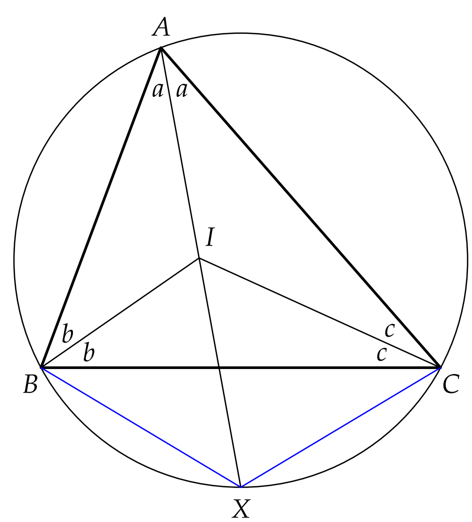
- *The circumcentre O of triangle ABC is the orthocentre of triangle XYZ.*
- *The incentre I of triangle ABC is the circumcentre of triangle PQR.*
- *The orthocentre H of triangle ABC is the incentre of triangle DEF.*
- *The centroid G of triangle ABC is the centroid of triangle XYZ.*

Problems

When solving difficult geometry problems, here are a few things to always keep a look out for. If you become well practised at spotting these objects in your geometry diagrams, then you are well on the way to becoming a geometry guru.

- cyclic quadrilaterals
- isosceles triangles
- equal angles and lengths
- similar or congruent triangles
- right angles
- ice cream cones

Problem. *Let ABC be a triangle with incentre I and extend AI until it meets the circumcircle of triangle ABC at X. Prove that X is the circumcentre of triangle BIC.*



Proof. When the incentre of triangle ABC is involved, I like to let the angles at A, B, C be $2a, 2b, 2c$, respectively. This is because I lies on the angle bisectors so I can label

$$\angle BAI = \angle CAI = a, \quad \angle CBI = \angle ABI = b, \quad \angle ACI = \angle BCI = c.$$

We want to prove that X is the circumcentre of triangle BIC or equivalently, that the lengths BX, IX, CX are all equal. Hopefully you can see that the problem is solved if we can prove that triangle BXI is isosceles with $BX = IX$. This is because the same reasoning will tell us that triangle CXI is isosceles with $CX = IX$.

So let's focus on proving that $BX = IX$. Since there are many relationships between angles in circles, it makes sense to try to instead prove that $\angle IBX = \angle BIX$. Using our notation, we obtain that

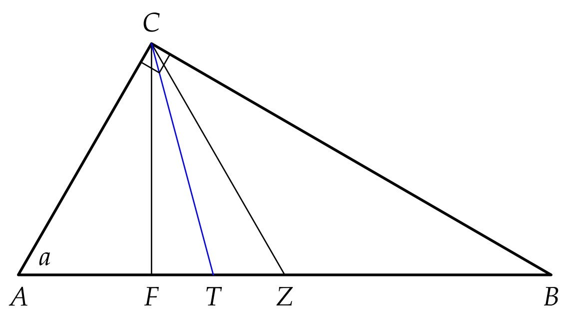
$$\angle IBX = \angle CBX + \angle IBC = \angle CAX + \angle IBC = a + b.$$

Here, we've used the hockey theorem on chord CX to deduce that $\angle CBX = \angle CAX = a$.

Now observe that $\angle BIX + \angle BIA = 180°$, since they form a straight line. And if we sum up the angles in triangle ABI, we obtain the equation $\angle BAI + \angle ABI + \angle BIA = 180°$ or equivalently, $a + b + \angle BIA = 180°$. These two facts imply that $\angle BIX = a + b$, so we have deduced that $\angle IBX = \angle BIX$. As we mentioned earlier, it follows that $BX = IX$ and similar reasoning will lead to $CX = IX$ as well. Hence, the three lengths BX, IX, CX are all equal and X is the circumcentre of triangle BIC. □

Problem. *If ABC is a triangle with a right angle at C, prove that the angle bisector from C bisects the angle formed by the altitude from C and the median from C.*

Proof. Let's call the altitude, the angle bisector and the median CF, CT and CZ, respectively. Then what we'd like to prove can be rephrased as $\angle FCT = \angle TCZ$. However, we already know that CT bisects the right angle, so that $\angle TCA = \angle TCB$. What this means is that we can rephrase the problem once again as $\angle FCA = \angle ZCB$. With this in mind, let's write $a = \angle CAB$ and try to determine the angles $\angle FCA$ and $\angle ZCB$ in terms of a.



The first of these angles is easy — since triangle CFA is right-angled, we can write $\angle FCA = 90° - a$. Now if we consider the sum of the angles in triangle ABC, we obtain $\angle ABC = 90° - a$ which is equivalent to $\angle ZBC = 90° - a$. Remember that our goal is to prove that $\angle ZCB = \angle FCA = 90° - a$. So what we should try to prove now is that triangle ZBC is isosceles, with $ZB = ZC$. But remember that the diameter of a circle subtends an angle of 90°, so a circle with diameter AB passes through C. As Z is the midpoint of AB, this means that Z is the circumcentre of triangle ABC and $ZB = ZC$, as desired. □

Fermat

Pierre de Fermat was actually a lawyer by day at the Parlement of Toulouse and an amateur mathematician by night. He lived in the early seventeenth century from 1601 to 1665 and is often credited with the development of a very early form of what we now call calculus. However, Fermat is probably most famous for his work in number theory. One theorem of his — Fermat's Little Theorem — says that if you pick your favourite positive integer a and your favourite prime number p, then the number $a^p - a$ will be divisible by p. The most famous story about Fermat tells of how he pencilled in the margin of a mathematics book the following problem.

> It is impossible to find two perfect cubes which sum to a perfect cube, two perfect fourth powers which sum to a perfect fourth power, or in general, two perfect nth powers which sum to a perfect nth power, if n is an integer greater than 2.

Rather tantalisingly, Fermat also wrote that he had a marvellous proof which was too small to fit into the margin. Of course, many people tried to recreate the supposedly unwritten proof, but to no avail. In fact, Fermat's Last Theorem — as the result is commonly called — was not proved until 1995, the proof being over one hundred pages long and using extremely technical mathematical tools which haven't been around for very long. Almost every mathematician believes that Fermat must have been either mistaken or lying.

In geometry, there is a triangle centre known as the Fermat point. Given a triangle ABC, it is the point P which makes the sum of the distances $PA + PB + PC$ as small as possible. If no angles of the triangle are greater than or equal to $120°$, the point P will be the unique point inside the triangle such that $\angle APB = \angle BPC = \angle CPA = 120°$.