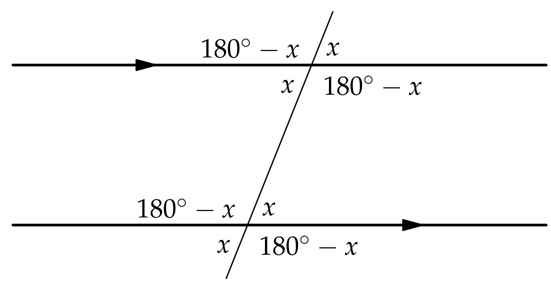
Warming Up with Parallel Lines

Now that we've decided to leave Euclid's Elements behind us, let's embark on a much less-detailed, though far more exciting, geometric journey. We'll warm up with a fact about parallel lines — Euclid proved it, but we'll assume it.

Proposition. *If a line meets two parallel lines and you label one angle x as shown in the diagram below, then you can label all eight angles as shown.*



Basic Facts about Triangles

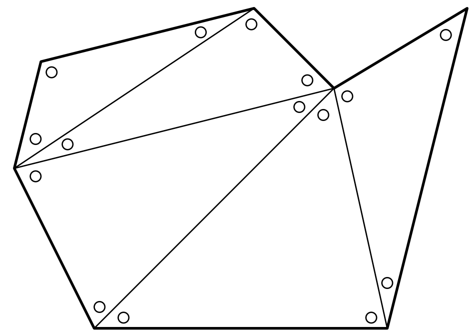
Now it's time for some basic facts about triangles, facts which you should know like the back of your hand and be able to use when solving geometry problems.

- The angles in a triangle add up to $180°$.
- If a triangle ABC satisfies $AB = AC$, then $\angle ABC = \angle ACB$. On the other hand, if a triangle ABC satisfies $\angle ABC = \angle ACB$, then $AB = AC$. If this is true, then we say that the triangle is *isosceles*, a word which comes from the Greek words "iso", meaning same, and "skelos", meaning leg.
- The area of a triangle is given by $\frac{1}{2} \times b \times h$, where b denotes the length of the base and h denotes the height of the triangle.

You might be wondering why we care so much about triangles. One reason is because a triangle is the simplest shape which encloses a region that you can draw using only straight line segments. Such shapes — of which triangles, quadrilaterals and pentagons are examples — are usually called *polygons*. In fact, we can consider triangles to be the "fundamental things" which we can glue together to build any polygon. For example, if you wanted to know what the sum of the angles are in a quadrilateral, then you could simply draw one of its diagonals to split it up into two smaller triangles. Each triangle on its own has angles which add to $180°$ so together, their angles add to $360°$. Since the angles of the individual triangles account for all of the angles in the quadrilateral, we have shown that every quadrilateral has angles which add to $360°$. This trick doesn't just work for quadrilaterals though, as we will now see.

Proposition. *In a polygon with n sides, the angles add to $(n-2) \times 180°$.*

Proof. We already saw that a quadrilateral can be cut along a diagonal into two triangles. If you draw a pentagon, you will notice that it can be cut along diagonals into three triangles. And if you draw a hexagon, you will notice that it can be cut along diagonals into four triangles. In general, a polygon with n sides can be cut along diagonals into $n-2$ triangles. This is not a particularly easy fact to prove, so we'll just take it for granted.



The figure above shows what you might get if you try this at home. Of course, each triangle on its own has angles which add to $180°$ so together, their angles add to $(n-2) \times 180°$. Since the angles of the individual triangles account for all of the angles in the polygon, we now know that every polygon with n sides has angles which add to $(n-2) \times 180°$. □

Congruence and Similarity

Congruence and similarity are two of the most important notions in geometry. We say that two shapes are *congruent* if it is possible to pick one of them up and place it precisely on top of the other one. On the other hand, we say that two shapes are *similar* if it is possible to pick one of them up, enlarge or shrink it by a certain factor, and then place it precisely on top of the other one. If the triangles ABC and XYZ are congruent, then we write this using the shorthand $ABC \cong XYZ$ and if they are similar, then we write this using the shorthand $ABC \sim XYZ$. When we use this notation, we always mean that the first vertex from one triangle corresponds to the first vertex from the other, the second vertex from one triangle corresponds to the second vertex from the other, and the third vertex from one triangle corresponds to the third vertex from the other.

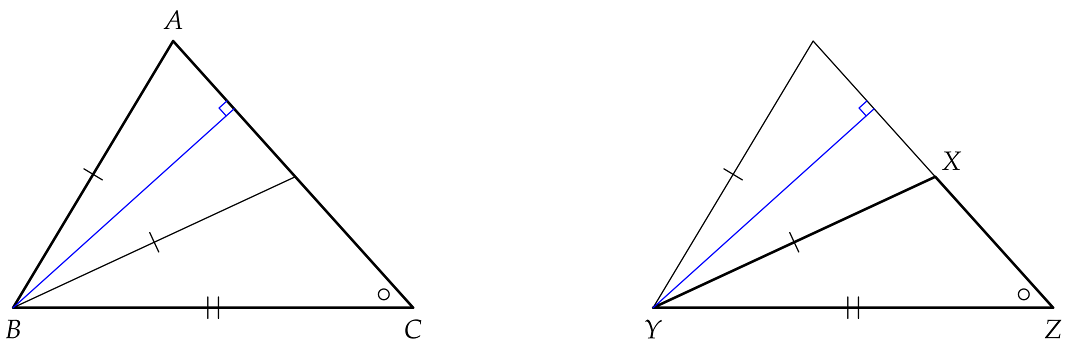
There are four simple rules to determine whether or not two triangles are congruent. Each one comes with a catchy TLA[1] which should hopefully be self-explanatory.

- *SSS (side-side-side)*
  If ABC and XYZ are two triangles such that $AB = XY$, $BC = YZ$ and $CA = ZX$, then the two triangles are congruent.
- *SAS (side-angle-side)*
  If ABC and XYZ are two triangles such that $AB = XY$, $BC = YZ$ and $\angle ABC = \angle XYZ$, then the two triangles are congruent.
- *ASA (angle-side-angle)*
  If ABC and XYZ are two triangles such that $\angle ABC = \angle XYZ$, $\angle ACB = \angle XZY$ and $BC = YZ$, then the two triangles are congruent.
- *RHS (right-hypotenuse-side)*
  If ABC and XYZ are two triangles such that $AB = XY$, $BC = YZ$ and $\angle BAC = \angle YXZ = 90°$, then the two triangles are congruent.

You have to be really careful with SAS, because the equal angles must be enclosed by the two pairs of equal sides for the rule to work. In other words, the fictitious rule SSA (side-side-angle) cannot be used to show that two triangles are congruent.

[1] Three Letter Acronym

To see this, we should be able to find two triangles ABC and XYZ such that $AB = XY$, $BC = YZ$ and $\angle BCA = \angle YZX$ such that the two triangles are not congruent. The following diagram should convince you that such pairs of triangles certainly do exist.

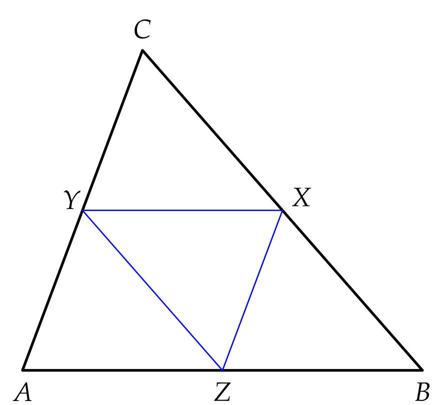


There are also three simple rules to determine whether or not two triangles are similar. Each one comes with a catchy TLA which once again should hopefully be self-explanatory.

- *AAA (angle-angle-angle)*
  If ABC and XYZ are two triangles such that $\angle ABC = \angle XYZ$, $\angle BCA = \angle YZX$ and $\angle CAB = \angle ZXY$, then the two triangles are similar. In fact, since we know that the angles in a triangle add to $180°$, we only need to know two of these equations and we get the third one for free.
- *PPP (proportion-proportion-proportion)*
  If ABC and XYZ are two triangles such that the fractions $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$ are equal, then the two triangles are similar.
- *PAP (proportion-angle-proportion)*
  If ABC and XYZ are two triangles such that the fractions $\frac{AB}{XY} = \frac{BC}{YZ}$ are equal and $\angle ABC = \angle XYZ$, then the two triangles are similar.

Once again, you have to be really careful with PAP, because the angle must be enclosed by the two proportional sides for the rule to work. In other words, the fictitious rule PPA (proportion-proportion-angle) cannot be used to show that two triangles are similar. Let's now apply our newfound knowledge about congruence and similarity to prove the following simple, but extremely useful, theorem.

Theorem (Midpoint Theorem)**.** *Let ABC be a triangle where the midpoints of the sides BC, CA, AB are X, Y, Z, respectively. Then the four triangles AZY, ZBX, YXC and XYZ are all congruent to each other and similar to triangle ABC.*



Proof. First, let's prove that triangle ABC is similar to triangle AZY. It's clear that $\angle CAB = \angle YAZ$ because they actually coincide. We also have the equal fractions $\frac{AB}{AZ} = \frac{AC}{AY} = 2$, so PAP tells us precisely that triangle ABC is similar to triangle AZY and is twice the size.

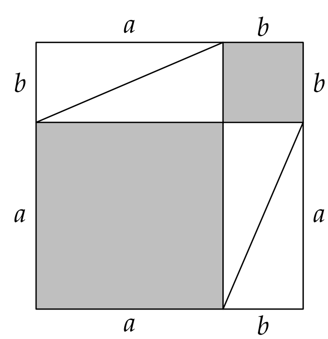
An entirely similar argument — no pun intended — can be used to prove that triangle ABC is similar to triangles ZBX and YXC and is twice the size. So what we have deduced is that the triangles AZY, ZBX and YXC are all congruent to each other. In particular, we have the three equations $XY = AZ$, $YZ = YZ$ and $ZX = AY$ which together imply that triangle XYZ is congruent to triangle AZY by SSS. □

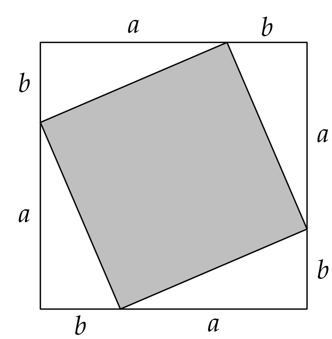
Pythagoras' Theorem (Reprise)

We've already seen Euclid's proof of Pythagoras' Theorem — let's reword the theorem a little differently now and consider a much slicker proof.

Theorem (Pythagoras' Theorem)**.** *Consider a right-angled triangle with side lengths a, b, c where c is the length of the hypotenuse. Then $a^2 + b^2 = c^2$.*

Proof. The entire proof is captured by the two diagrams below. On the left, we see a square of side length $a + b$. We've removed four right-angled triangles, each with side lengths a, b, c and the remaining shaded regions clearly have a combined area of $a^2 + b^2$.





On the right, we also see a square of side length $a + b$. We've once again removed four right-angled triangles, each with side lengths a, b, c, but in a slightly different way. The remaining shaded region clearly has area c^2 and it follows that $a^2 + b^2 = c^2$. □

Theorem (Converse of Pythagoras' Theorem)**.** *Consider a triangle with side lengths a, b, c where $a^2 + b^2 = c^2$. Then the triangle is right-angled and the hypotenuse has length c.*

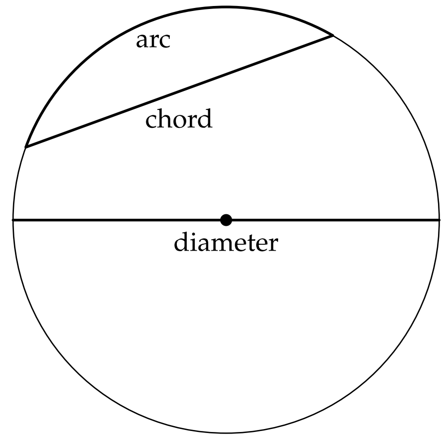
Proof. Construct a right-angled triangle whose legs have lengths a and b, and let its hypotenuse have length d. We can now invoke Pythagoras' theorem since we proved it just a little bit earlier. It tells us that $a^2 + b^2 = d^2$. But put this piece of information together with our assumption that $a^2 + b^2 = c^2$ and you have the equation $c^2 = d^2$. This implies that $c = d$.

Therefore, the triangle with side lengths a, b, c that we were given possesses exactly the same side lengths as the right-angled triangle that we have constructed. Because of SSS, this means that the two triangles are, in fact, congruent. So the given triangle with side lengths a, b, c is indeed right-angled, as we intended to prove. □

Basic Facts about Circles

We now know lots about triangles, so let's move on to circles — we start with some basic facts.

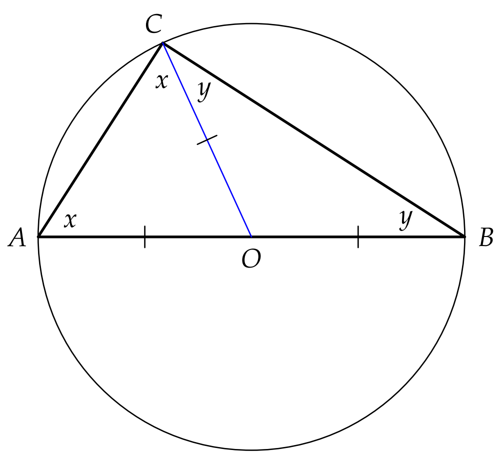
- A *circle* is the set of points which are the same distance r from some centre O. You should already know by now that r is called the *radius* of the circle and O is called the *centre* of the circle.
- A *chord* is a line segment which joins two points on a circle.
- An *arc* of a circle is the part of the circumference cut off by a chord.
- A *diameter* is a chord which passes through the centre of the circle. Note that its length is twice the radius.



This is all we need to know about circles in order to prove some pretty cool results, like the following.

Proposition. *The diameter of a circle subtends an angle of* $90°$. *In other words, if* AB *is the diameter of a circle and* C *is a point on the circle, then* $\angle ACB = 90°$.

Proof. Let O be the centre of the circle. The beauty of considering the centre is that we have the three equal radii $OA = OB = OC$. Equal lengths, for obvious reasons, often lead to isosceles triangles, and our diagram happens to have two of them. There is the isosceles triangle OAC which means that we can label the equal angles $\angle OAC = \angle OCA = x$. There's also the isosceles triangle OBC, which means that we can label the equal angles $\angle OBC = \angle OCB = y$. Labelling equal angles like this is an extremely common and extremely useful trick.



Now it's time for some angle chasing.[2] In particular, let's consider the sum of the angles in triangle ABC.

$$\angle BAC + \angle ACB + \angle CBA = 180°$$

[2] Angle chasing is the art of chasing angles. As Wikipedia puts it, the term is "used to describe a geometrical proof that involves finding relationships between the various angles in a diagram".

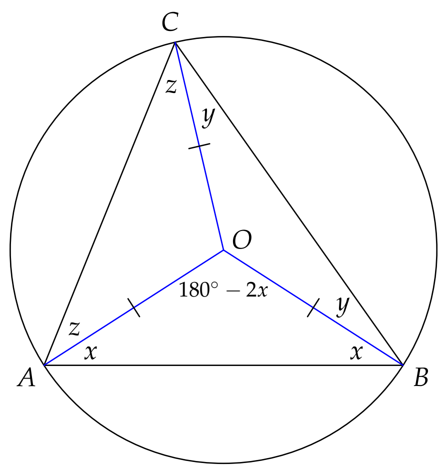
We can replace all of these confusing angles with x's and y's in the following way.

$$x + (x + y) + y = 180^\circ$$

This equation is, of course, the same thing as $2(x + y) = 180^\circ$ or, equivalently, $x + y = 90^\circ$. All we have to do now is recognise that $\angle ACB = x + y$ so we have proven that $\angle ACB = 90^\circ$. □

Proposition. *The angle subtended by a chord at the centre is twice the angle subtended at the circumference, on the same side. In other words, if AB is a chord of a circle with centre O and C is a point on the circle on the same side of AB as O, then $\angle AOB = 2\angle ACB$.*

Proof. The beauty of considering the centre is that we have the three equal radii $OA = OB = OC$. Equal lengths, for obvious reasons, often lead to isosceles triangles, and our diagram happens to have three of them. There is the isosceles triangle OAB which means that we can label the equal angles $\angle OAB = \angle OBA = x$. There's also the isosceles triangle OBC, which means that we can label the equal angles $\angle OBC = \angle OCB = y$. And there's also the isosceles triangle OCA, which means that we can label the equal angles $\angle OCA = \angle OAC = z$. Labelling equal angles like this is an extremely common and extremely useful trick.



Now it's time for some angle chasing. In particular, let's consider the sum of the angles in triangle ABC.

$$\angle BAC + \angle ACB + \angle CBA = 180^\circ$$

We can replace all of these confusing angles with x's, y's and z's in the following way.

$$(x + z) + (z + y) + (y + x) = 180^\circ$$

This equation is, of course, the same thing as $2(x + y + z) = 180^\circ$. Let's keep this equation in the back of our minds while we try and remember what it is exactly that we're trying to do. We want to prove that $\angle AOB = 2 \times \angle ACB$. Using the angle sum in triangle OAB, we can write $\angle AOB = 180^\circ - 2x$. We can also write $\angle ACB = y + z$. So what we're actually aiming for is the following equation.

$$180^\circ - 2x = 2(y + z)$$

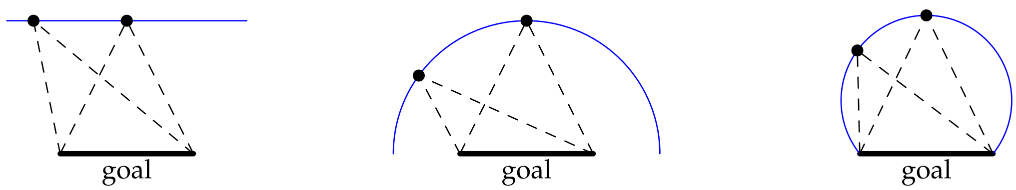
But after rearranging, this is just the same thing as $2(x + y + z) = 180^\circ$, which we already proved. □

Hopefully, you've managed to spot the similarity between these two proofs. They're both indicative of the general strategies that we'll be using to solve tougher geometry problems. More specifically, I guess what we've used here is the old "find isosceles triangles–label equal angles–sum up the angles in a triangle–work out what you're trying to prove–then prove it" trick.

I should probably mention that some of the proofs I've provided are a little incomplete. In particular, if you check the previous proof very carefully, you'll notice that it only works when O lies inside triangle ABC. It could be possible that O lies on or even outside triangle ABC. However, the proofs in these other cases are quite similar, so I'll leave it as a fun exercise for you to find them.

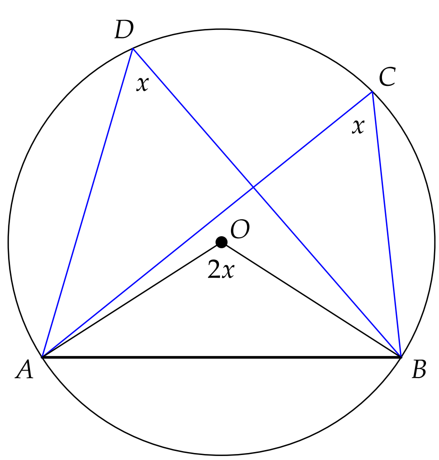
The Hockey Theorem

Suppose that a hockey coach wanted to see which player on their team had the best shot. They could line all the players parallel to the goal and ask them to shoot to see who scores and who misses. But this would certainly be unfair to the players on the ends who have to shoot further and have a smaller angle to aim for. The coach could try and fix the problem by placing everyone on a circle whose centre coincides with the middle of the goal. Of course, this means that everyone is the same distance from the goal now, but some players have a much greater angle to aim at than others. In fact, due to the frictionless nature of the sport, distance is no problem when shooting at a hockey goal. What we would rather test is the accuracy of each player. So it makes sense to place everyone somewhere where they all have the same angle to aim at. In that case, where should we put all the players? The following proposition tells us the answer — they should all stand on the arc of a circle whose end points are the goal posts.



Proposition (Hockey Theorem)**.** *Angles subtended by a chord on the same side are equal. In other words, if A, B, C, D are points on a circle with C and D lying on the same side of the chord AB, then $\angle ACB = \angle ADB$.*

Proof. The proof to this is delightfully simple — we start by letting O be the centre of the circle. We have already proved that $\angle AOB = 2\angle ACB$ and also that $\angle AOB = 2\angle ADB$. So it must be the case that $\angle ACB = \angle ADB$. □



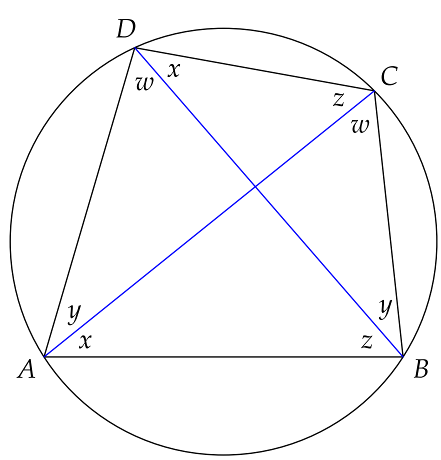
Cyclic Quadrilaterals

If I give you one point and ask you to draw a circle through it, then that's a pretty easy thing to do, right? I could make life slightly more difficult for you by giving you two points and asking you to draw a circle through both of them. And even if I give you three points and ask you to draw a circle through all of them, then you could almost always do it, as long as they don't lie on a line. To see this, consider a massive circle, so massive that all three points that I give you lie inside it. Now start shrinking the circle. Sooner or later, your circle has to hit one of the points. Now keep that point on the circle, but keep shrinking the circle. Sooner or later, your circle has to hit another of the points. We are now in the happy situation of having the circle pass through two of the given points. Now we either shrink or expand our circle, while keeping it in contact with these two points, until it finally passes through the third given point. Hopefully, this should convince you that there is actually only one circle which passes through any three points, as long as they don't lie on a line.

So now I'm going to make life particularly difficult for you by giving you four points and asking you to draw a circle through all of them. The shrink/expand trick we used above shows that there is a unique circle which passes through three of them. So to be able to accomplish the task, the fourth point must already lie on this circle. The probability that I was nice enough to actually give you four points where this is true is incredibly small. What I'm trying to get at here is that if I give you a random quadrilateral, then it is extremely rare for a circle to pass through all four of its vertices. So if a circle does pass through all four of its vertices, then the quadrilateral must be very special indeed — so special that we should give it a special name. In fact, we refer to such a quadrilateral as a *cyclic quadrilateral*. We're now going to prove a very important fact about cyclic quadrilaterals.

Proposition. *The opposite angles in a cyclic quadrilateral add up to* $180°$. *In other words, if* $ABCD$ *is a cyclic quadrilateral, then* $\angle ABC + \angle CDA = 180°$ *and* $\angle BCD + \angle DAB = 180°$.

Proof. If we draw in the diagonals AC and BD, the hockey theorem tells us that there are equal angles galore. For example, we can label $\angle ACB = \angle ADB = w$, $\angle BDC = \angle BAC = x$, $\angle CAD = \angle CBD = y$ and $\angle DBA = \angle DCA = z$. You can go crazy labelling equal angles like this whenever there's a cyclic quadrilateral somewhere in your diagram.



Now we're going to play a similar trick to one we played before — we're going to add up all of the angles in the quadrilateral and the answer should be $360°$.

$$\angle DAB + \angle ABC + \angle BCD + \angle CDA = 360°$$

We can replace all of these confusing angles with w's, x's, y's and z's in the following way.

$$(x+y)+(y+z)+(z+w)+(w+x)=360^\circ$$

This equation is, of course, the same thing as $2(w+x+y+z)=360^\circ$ or, equivalently, $w+x+y+z=180^\circ$. All we have to do now is recognise that $\angle ABC = y+z$ and $\angle CDA = w+x$, so that

$$\angle ABC + \angle CDA = (y+z)+(w+x) = 180^\circ.$$

You could also have chosen to recognise that $\angle BCD = z+w$ and $\angle DAB = x+y$, so that

$$\angle BCD + \angle DAB = (z+w)+(x+y) = 180^\circ.$$ □

How to Find a Cyclic Quadrilateral

Something we're going to learn is that it's incredibly useful to keep your eyes open for cyclic quadrilaterals when solving problems in Euclidean geometry. If the problem happens to mention a circle which has four points on it, then of course, those four points form a cyclic quadrilateral. But quite often, cyclic quadrilaterals can be hidden somewhere in your diagram, even when there are no circles involved. In those cases, you would probably use one of the two facts below to prove that the quadrilateral is cyclic.

We proved earlier that the opposite angles in a cyclic quadrilateral add up to 180°. One thing you might be wondering is whether the converse is true — in other words, if someone gives you a quadrilateral where the opposite angles add up to 180°, then is it necessarily true that the quadrilateral must have been cyclic? The following proposition tells you that the converse certainly is true.

Proposition. *If the opposite angles in a quadrilateral add up to* 180°*, then the quadrilateral is cyclic.*

Another thing we proved earlier was the hockey theorem — that if A, B, C, D are points on a circle with C and D lying on the same side of the chord AB, then $\angle ACB = \angle ADB$. And another thing you might be wondering is whether the converse is true — in other words, if someone gives you a quadrilateral $ABCD$ where $\angle ACB = \angle ADB$, then is it necessarily true that the quadrilateral must have been cyclic? The following proposition tells you that the converse certainly is true, once again.

Proposition. *If $ABCD$ is a convex quadrilateral such that $\angle ACB = \angle ADB$, then the quadrilateral is cyclic.*

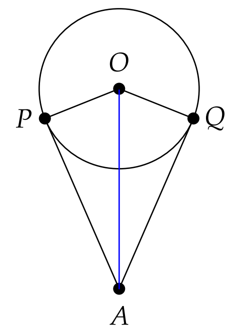
A particularly useful case occurs when you are given four points A, B, C, D such that $\angle ABC = \angle CDA = 90^\circ$. If $ABCD$ is a convex quadrilateral, then we can use the first proposition above to deduce that it must actually be a cyclic quadrilateral. On the other hand, if $ABCD$ is a convex quadrilateral, then we can use the second proposition above to deduce that it must actually be a cyclic quadrilateral. So either way, the four points A, B, C, D lie on a circle. This means that when right angles appear, you can often expect cyclic quadrilaterals to appear as well.

You can solve many many geometry problems by searching for cyclic quadrilaterals and using what you know about them. The beauty of cyclic quadrilaterals is that if you find one pair of equal angles, then you get three more for free. This is because if you find that $\angle ACB = \angle ADB$, then the quadrilateral $ABCD$ is cyclic, in which case we can apply the hockey theorem to also obtain the equal angles

$$\angle BDC = \angle BAC, \quad \angle CAD = \angle CBD, \quad \text{and} \quad \angle ABD = \angle ACD.$$

Tangents

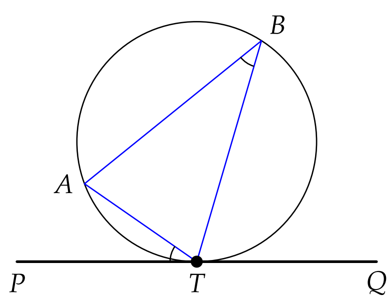
A *tangent* is a line which touches a circle at precisely one point. Given a point outside a circle, you can draw two tangents, thereby creating a diagram which looks very much like an ice cream cone.



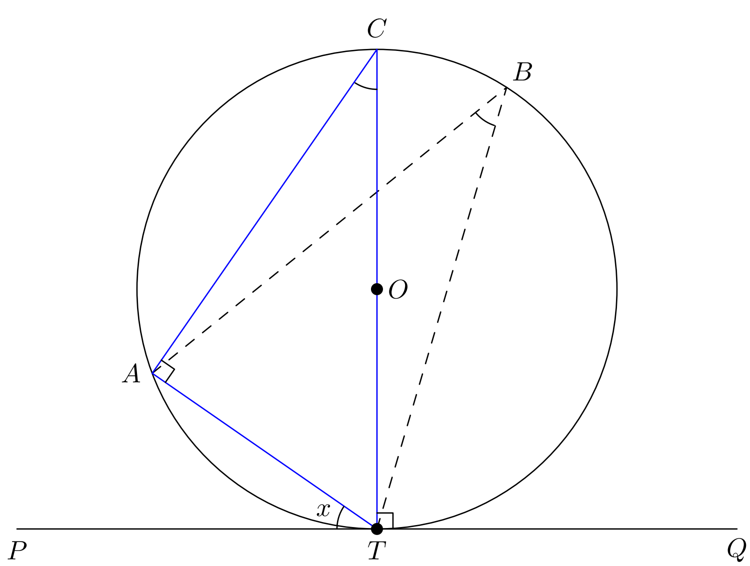
Theorem (Ice Cream Cone Theorem)**.** *The picture of the ice cream cone on the right is symmetric so that* $AP = AQ$ *and the line* OA *bisects the angle* PAQ*.*

The proof of the ice cream cone theorem is a very simple application of congruent triangles combined with the fact that the radius drawn from the centre of a circle to a point on its circumference is perpendicular to the tangent at that point. The following is another extremely useful fact about tangents.

Theorem (Alternate Segment Theorem)**.** *Suppose that* AT *is a chord of a circle and that* PQ *is a line tangent to the circle at* T*. If* B *lies on the circle, on the opposite side of the chord* AT *from* P*, then* $\angle ABT = \angle ATP$*.*



Proof. The hockey theorem guarantees that, as long as C lies on the circle, on the same side of AT as B, then $\angle ACT = \angle ABT$. So the trick here is to choose a particular point C on the circle and prove that $\angle ACT = \angle ATP$. One nice way to choose the location of the point C is so that TC is a diameter of the circle.



Choosing C in this way is great because we've introduced two right angles into the diagram. We have $\angle CAT = 90^\circ$, because it's an angle subtended by the diameter TC and we also have $\angle PTO = 90^\circ$, because it's an angle created by a radius and a tangent at the point T.

So if we label $\angle ATP = x$, then we have $\angle OTA = 90^\circ - x$. Now use the fact that the angles in triangle CAT add to 180° and you find that $\angle ACT = x$. However, we already mentioned that $\angle ABT = \angle ACT = x$ by the hockey theorem, so we can now conclude that $\angle ABT = \angle ATP$. □

It turns out that the converse of the alternate segment theorem is also true — but I'll let you try to write down exactly what it states.

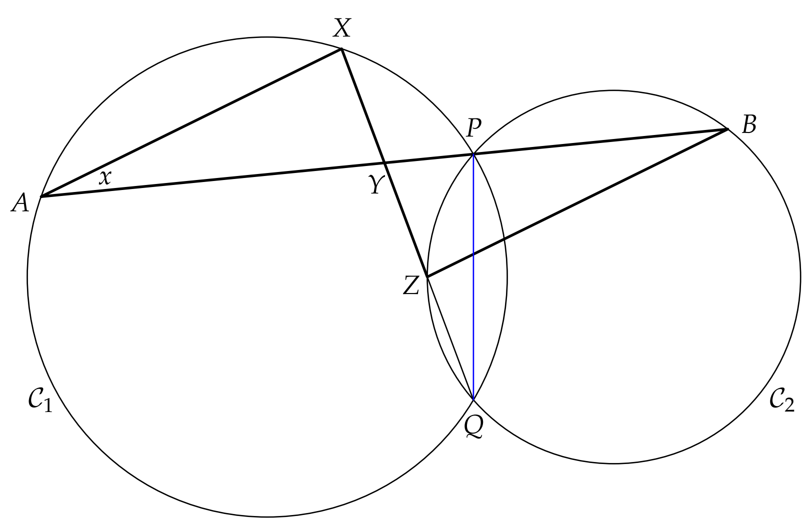
Problems

Many geometry problems can be solved by angle chasing. This means labelling some — but not too many — well-chosen angles in your diagram and using what you know to determine other ones. Here are some very common ways to relate different angles in your diagram.

- If two angles are next door to each other, then they add up to 180°.
- Parallel lines give you equal angles and angles which add up to 180°.
- The angles in a triangle add up to 180°.
- Congruent or similar triangles give you equal corresponding angles.
- In any cyclic quadrilateral, the opposite angles add up to 180°.
- In any cyclic quadrilateral, you can apply the hockey theorem to give you four pairs of equal angles.

Apart from these tips, possibly one of the most useful things I can tell you is to always draw a very accurate, very large diagram, preferably in multiple colours. Below are two example problems and their solutions. You should try to read the solutions carefully to make sure that you understand them and then reread them until you think you can reconstruct the proofs on your own.

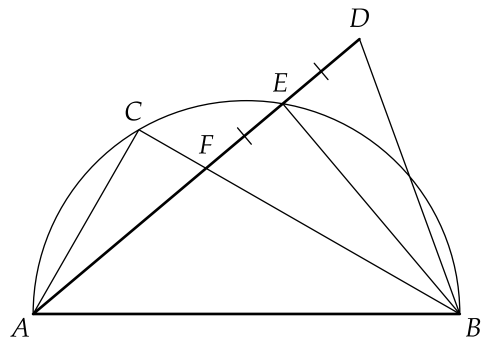
Problem. *Two circles intersect at P and Q. A line through P meets the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at A and B, respectively. Let Y be the midpoint of AB and suppose that the line QY meets the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at X and Z, respectively. Prove that triangle XYA is congruent to triangle ZYB.*



Proof. The first thing to notice is that the two triangles have the angles $\angle XYA = \angle ZYB$ as well as the sides $AY = BY$ in common. So to prove that they're congruent, we could try to use either ASA or SAS. It turns out that the first choice is better for this problem, since the circles in the diagram will give us equal angles.

So let's label $\angle YAX = x$ and note that the problem is completely solved if we can only show that $\angle YBZ = x$ as well. Since the angle $\angle YAX = \angle PAX$ is subtended by the chord PX, we can use the hockey theorem to deduce that $\angle PQX = \angle PAX = x$. And since the angle $PQZ = \angle PQX$ is subtended by the chord PZ, we can use the hockey theorem again to deduce that $\angle PBZ = \angle PQZ = x$. However, $\angle PBZ = \angle YBZ$ so we've proved that $\angle YBZ = x$ and the problem is solved. □

Problem. *Consider a semicircle with diameter AB. Suppose that D is a point such that $AB = AD$ and AD intersects the semicircle at the point E. Let F be the point on the chord AE such that $DE = EF$ and extend BF to meet the semicircle at the point C. Prove that $\angle BAE = 2\angle EAC$.*



Proof. A good start would be to label $\angle EAC = x$ and aim to prove that $\angle BAE = 2x$. Given that there is a circle — or at least half of one — there is probably a cyclic quadrilateral lurking about. Hopefully, you can see that $ABEC$ is a cyclic quadrilateral, so we can apply the hockey theorem to deduce that $\angle EBC = \angle EAC = x$ as well.

Note that AB is a diameter and there is a certain fact that we know about diameters — namely, they subtend angles of 90°. This means that $\angle AEB = \angle ACB = 90°$. Since $\angle EBC = x$, considering the angle sum in triangle EFB yields the fact that $\angle EFB = 90° - x$.

Now if you've drawn a nice accurate picture, then you'll see that the two triangles FEB and DEB look suspiciously congruent. These suspicions can be confirmed by observing that that $FE = ED$, $EB = EB$, $\angle FEB = \angle DEB = 90°$ and using SAS. One consequence of this is the fact that $\angle EDB = \angle EFB = 90° - x$.

One piece of information we haven't used yet is the fact that $AB = AD$ or, in other words, that triangle ABD is isosceles. The equal side lengths imply that there are equal angles in the diagram — namely, $\angle ABD = \angle ADB = \angle EDB = 90° - x$.

Now we note that we have labelled two of the angles in triangle ABD, so we can definitely determine what the third angle is.

$$\angle BAD = 180° - \angle ABD - \angle ADB = 180° - (90° - x) - (90° - x) = 2x$$

However, this tells us that $\angle BAE = 2x$ as well, exactly what we set out to prove. □

Pythagoras

Pythagoras was a Greek mathematician who lived even before Euclid was born — from around 570 BC to around 475 BC. Even though we know and love him for his theorem about right-angled triangles, it was essentially known before Pythagoras arrived on the scene and he never even managed to prove it. Actually, Pythagoras was not a great mathematician at all, but was the first person to call himself a philosopher — a word which in ancient Greek literally means a lover of wisdom.

Pythagoras founded a religious cult whose members believed that everything was related to mathematics and that numbers were the ultimate reality. Pythagoras himself was pretty serious about this belief, as you can tell from the following story. When his fellow cult member Hippasus of Metapontum managed to prove that the number $\sqrt{2}$ was irrational — a rather ingenious and important mathematical feat which didn't sit well with the Pythagorean philosophy — Pythagoras supposedly could not accept the result and sentenced Hippasus to death by drowning.

The Pythagorean cult also adhered to various rules, some mildly practical but most simply bizarre — the following are examples.

- Never eat beans.
- Don't pick up something that has fallen.
- Don't walk on highways.
- When you get out of bed, take the sheets and roll them all together.

Apart from these facts, there isn't a great deal which is known about Pythagoras because no written work of his has survived to this day. It's commonly believed that many of the accomplishments attributed to him were simply accomplishments of people who were in his cult.