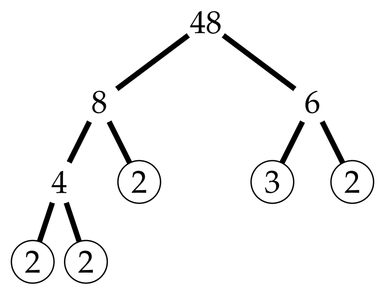
Reductionism for Dummies

Today, Wikipedia told me that *reductionism* is "an approach to understand the nature of complex things by reducing them to the interactions of their parts, or to simpler or more fundamental things". Here are two examples of reductionism at work which you may or may not be familiar with.

- *Prime factorisation*

  The Fundamental Theorem of Arithmetic says that every positive integer can be constructed by multiplying prime numbers together. In fact, it even tells us that, for every positive integer, there is really only one way to do this. So we can reduce every positive integer to its prime factorisation, which acts like a fingerprint for that number. You probably know that one way to find the prime factorisation of a number is to write down its factor tree, like I've done below for $48 = 2 \times 2 \times 2 \times 2 \times 3$.

  So in this case, the "fundamental things" are the prime numbers and it turns out that there are infinitely many of them. The oldest known proof of this fact is a beautiful piece of thinking by a guy named Euclid who lived a really really long time ago.

- *Physics*

  To understand the universe, it makes sense to explore the "stuff" that we observe — this is called matter. Quite a while back, we discovered that matter is made up of molecules and, a little bit later, that molecules are made up of atoms. These atoms are, in turn, made up of smaller particles like electrons, neutrons and protons. And more recently, physicists have found that these particles are made up of tiny tiny things called quarks. There are six different flavours of quark known as up, down, charm, strange, top, and bottom.

  In this overly simplistic view of physics, the "fundamental things" are the quarks and it turns out that there are only finitely many of them. I should mention that, even more recently still, some physicists believe that quarks are actually just incredibly tiny pieces of vibrating string.

Where's the Geometry?

So what does all this have to do with geometry? Well, suppose that you're trying to convince me of the simple geometric fact that all three angles in an equilateral triangle are 60°. Your argument would probably look something like this…

> First, you might use the fact that the base angles in an isosceles triangle are equal. But the equilateral triangle is isosceles "all three ways", so this means that all three angles are equal to each other. And then you might invoke the well-known fact that the angles in a triangle always add up to 180°. Since you now know that the three angles in an equilateral triangle are equal and add up to 180°, every single one of them must be 60°, and you're done.

Now imagine that I've been listening really carefully to your argument, but that I'm not a particularly intelligent nor knowledgeable individual.[1] In this purely hypothetical world, I probably don't know that the base angles in an isosceles triangle are equal. Nor would I believe the well-known fact that the angles in a triangle always add up to 180°. So you would have to prove these facts to me as well. And then in the process of proving these, you would probably use even more basic geometric facts and I would complain that I have never heard of them before, and on and on it goes. But where does it all end?

Of course, the game ends once you've broken the proof down into geometric facts which are so simple that I must know them to be true. What are these "fundamental things" in geometry? Do you need infinitely many of them or only finitely many of them? And if the answer is finitely many, then just how many do you need exactly?

Euclid's Elements

The "fundamental things", the LEGO blocks, the basic truths from which we build up all of geometry — or any mathematical theory, in fact — are called *axioms*. The interesting thing about axioms is that you simply cannot prove them, so you just have to assume that they're true.

The first person to successfully apply the reductionist approach to mathematics was a rather clever guy by the name of Euclid, whom I've already mentioned. He was a Greek mathematician who lived around 300 BC and is most famous for his series of thirteen books known as the *Elements*. Even though you've probably never heard of them, they're some of the most successful and influential books ever written, even more so than the Harry Potter or Twilight series. For example, Euclid's Elements is supposed to be second only to the Bible in the number of editions published. In fact, it was used as the basic text on geometry throughout the Western world for over two millennia, up until around one hundred years ago. In those days, if you didn't know Euclid's Elements, then people didn't think that you were well-educated. So to stop people from thinking that we're ignorant, let's have a look at Book 1 of Euclid's Elements, the most famous of them all.

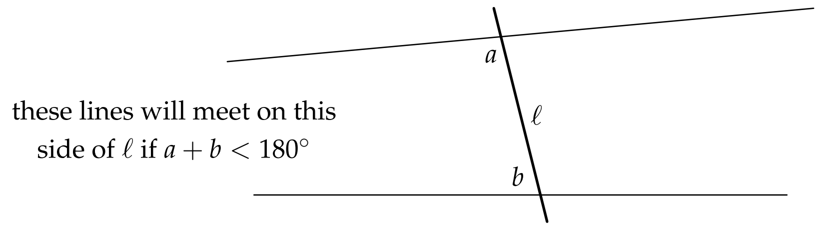
Axioms and Common Notions

In Book 1, Euclid states ten axioms from which he deduces everything that he knows about geometry. So thankfully, unlike the reductionist approach applied to prime factorisation, there are only finitely many "fundamental things". If there had been infinitely many, then Euclid would never have been able to write them all down.

The five main axioms of Euclidean geometry — which are often referred to as *postulates* — are stated below. I've taken some licence in paraphrasing Euclid's old-fashioned Ancient Greek into more modern English.

- A1. You can draw a unique line segment between any two given points.
- A2. You can extend a line segment to produce a unique line.
- A3. You can draw a unique circle with a given centre and a given radius.
- A4. Any two right angles are equal to each other.
- A5. Suppose that a line ℓ meets two other lines, making two interior angles on one side of ℓ which sum to less than 180°. Then the two lines, when extended, will meet on that side of ℓ.

[1] Of course, this is going to be a very difficult thing for you to imagine, but please try.



Euclid supplemented these with another set of five axioms of a slightly different nature, which we refer to as *common notions*.

C1. If $A = C$ and $B = C$, then $A = B$.

C2. If $A = X$ and $B = Y$, then $A + B = X + Y$.

C3. If $A = X$ and $B = Y$, then $A - B = X - Y$.

C4. If A and B coincide, then $A = B$.

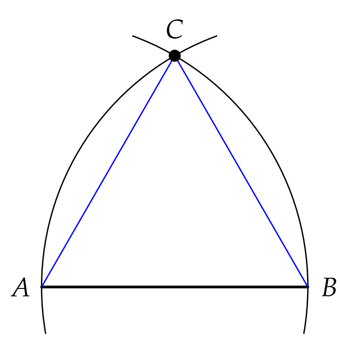
C5. The whole of something is greater than a part of something.

Surely you agree that each of these ten axioms is so obvious that it's self-evident. Essentially, Euclid was declaring his ten axioms to be the "rules of the game" from which he would build the world of Euclidean geometry. To really appreciate Euclid's work, you have to remember that before he entered the picture, geometry consisted of a bunch of useful rules, like the fact that a triangle with side lengths 3, 4, 5 has a right angle. You knew a fact like this was true because if you made a triangle with these side lengths out of rope and used it as a protractor, then the home that you were building for your family would appear to be vertical. Euclid came along and said that if you believe my ten axioms — and I'm pretty sure that you all do — then I can show by logic alone that you have to believe the more complicated things that I'm going to talk about. In this way, Euclid shows us what it means exactly for a theorem in mathematics to be true. In the remainder of Book 1, Euclid proceeds to deduce, one by one, forty-eight propositions, the proof of each one depending only on the axioms and on previously proven propositions.

The First Few Propositions

In the timeless words of Maria von Trapp from *The Sound of Music*, "Let's start at the very beginning, a very good place to start". To make digesting the proofs a little easier, I'll write which axiom or proposition I'm using to deduce each statement. It's generally a good habit when you're constructing geometry proofs — or any proofs, as a matter of fact — to provide reasoning for every single statement that you write down.

Proposition (Proposition I). *Given a line segment, you can draw an equilateral triangle on it.*



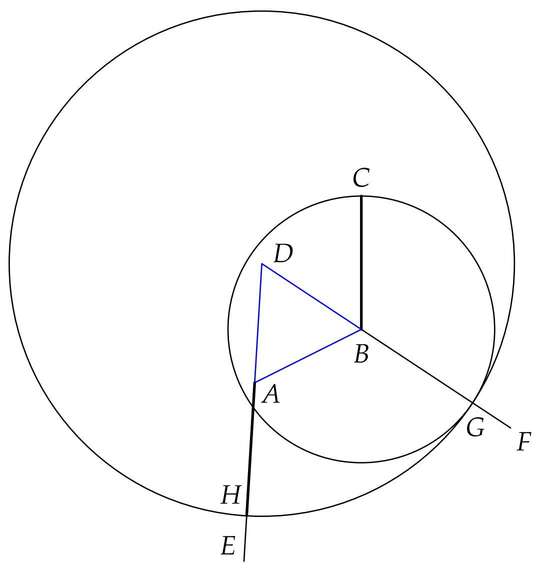
Proof. Let AB be the given line segment.

- Draw the circle with centre A and radius AB. [A3]
- Now draw the circle with centre B and radius BA. [A3]
- If the circles meet at a point C, then draw the line segments CA and CB. [A1]
- Since A is the centre of one circle, $AC = AB$. And since B is the centre of the other circle, $BC = BA$. But these two statements together imply that $AC = BC$. [C1]

So the line segments AB, BC, CA are all equal which simply means that triangle ABC is equilateral. □

Hopefully, you can appreciate that this is a fully rigorous proof and that it's pretty hard to find fault with any single part of it. Because for every statement written down, Euclid can assert its truth, merely by pointing to one of his axioms. In some sense, Euclid's axioms are like the ten commandments, and thou shalt not argue with them. One down and forty-seven to go...

Proposition (Proposition II)**.** *Given a line segment and a point, you can draw a line segment from the given point, equal in length to the given line segment.*

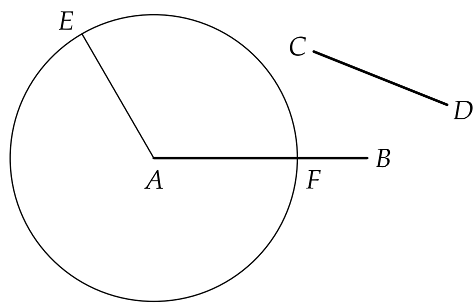


Proof. Let A be the given point and BC the given line segment.

- Draw the line segment AB. [A1]
- Now draw the equilateral triangle DAB. [P1]
- Extend the line segments DA and DB to obtain lines DE and DF. [A2]
- Draw the circle with centre B and radius BC and let it meet DF at G. [A3]
- Now draw the circle with centre D and radius DG and let it meet DE at H. [A3]
- Since D is the centre of a circle, we know that $DH = DG$. And because we constructed $DA = DB$, we may now subtract to obtain $DH - DA = DG - DB$, or equivalently, $AH = BG$. [C3]
- Since B is the centre of a circle, we know that $BC = BG$. So we can deduce that $AH = BC$. [C1] □

One thing you've probably noticed is that these propositions are ridiculously simple and seem pretty obvious. You've hopefully also noticed that it takes some ingenuity to prove them using only the axioms.

Proposition (Proposition III). *Given two line segments of unequal lengths, you can divide the longer one into two parts, one of which is equal in length to the shorter one.*



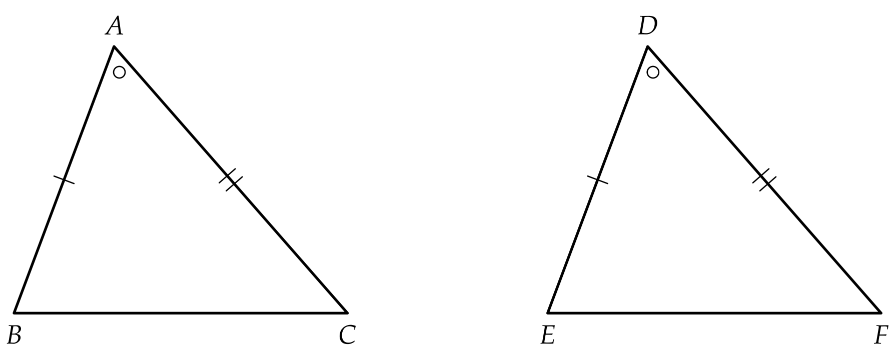
Proof. Let AB and CD be the two given line segments, where the longer one is AB.

- Draw the line segment AE equal to the line segment CD. [P2]
- Draw the circle with centre A and radius AE and let it meet AB at F. [A3]
- This means that $AE = AF$, but we also know that $AE = CD$, so we can deduce that $AF = CD$. [C1]

Therefore, we have divided AB into two parts using the point F, one of which is equal to CD. □

Hopefully, you're getting the hang of things now...

Proposition (Proposition IV). *If two triangles have two pairs of equal sides and the angles formed by these sides are equal, then the two triangles are congruent — in other words, they have the same side lengths and the same angles.*



Proof. Let the two triangles be ABC and DEF, where $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$. Place triangle ABC so that A coincides with D and the line AB coincides with the line DE. Since $AB = DE$, it must be the case that B coincides with E. The equal angles $\angle BAC = \angle EDF$ guarantee that the line AC will coincide with the line DF while the equal lengths $AC = DF$ guarantee that C will coincide with F.

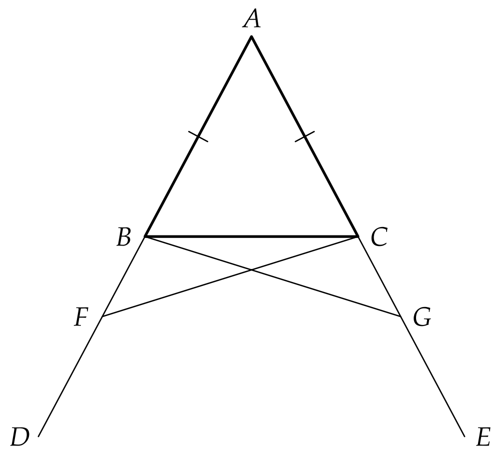
- The line segment BC must now coincide with the line segment EF. [A1]
- Therefore, we know that $BC = EF$. [C4]
- Furthermore, the angles of triangle ABC coincide with the angles of triangle DEF, so $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$. [C4]

We now know that triangles ABC and DEF are congruent. □

Pons Asinorum

We now come to Euclid's fifth proposition from Book 1 of his Elements, a proposition which has historically been given the nickname *Pons Asinorum*. If you're well-versed in Latin, you'll know that this means the Bridge of the Asses. Why is it called this? One reason that has been proposed is that the diagram used in the proof looks like a steep bridge which can be crossed by an ass but not by the fuller-figured horse. The more commonly accepted reason is that this proposition is the first real test of intelligence. At this point, the intelligent people are able to cross over to the harder propositions while the unintelligent asses get left behind. So that we don't feel like unintelligent asses, let's now try to understand Euclid's fifth proposition.

Proposition (Proposition V — Pons Asinorum)**.** *In an isosceles triangle, the base angles are equal and the angles under the base angles are equal.*



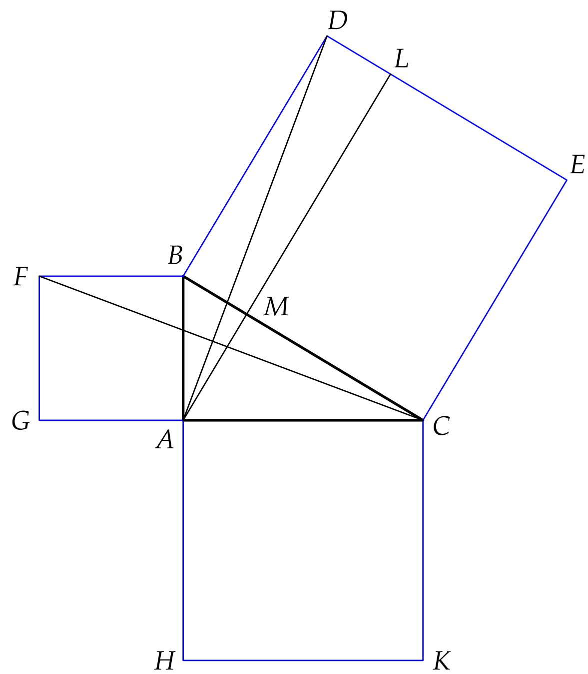
Proof. Let the isosceles triangle be ABC, where $AB = AC$.

- Extend the sides AB and AC to produce the lines AD and AE, respectively. [A2]
- Now choose a random point F on BD and let the point G divide the segment AE in such a way that $AF = AG$. [P3]
- Draw the line segments FC and GB. [A1]
- Since $AF = AG$ and $AB = AC$, the two line segments FA and AC are equal to the two line segments GA and AB, respectively. They also make a common angle $\angle FAC = \angle GAB$. So the two triangles AFC and AGB are congruent. [P4]
- We know that $AB = AC$, but also that $AF = AG$, so we can subtract to obtain $AF - AB = AG - AC$. Of course, this is the same thing as $BF = CG$. [C3]
- Remember that we already have $FC = GB$. So the two line segments BF and FC are equal to the two line segments CG and GB, respectively. We've also shown that $\angle CGB = \angle BFC$, so triangle BFC is congruent to triangle CGB. [P4]
- Hence, we have the equal angles $\angle ABG = \angle ACF$ as well as the equal angles $\angle CBG = \angle BCF$. After subtracting, we obtain $\angle ABG - \angle CBG = \angle ACF - \angle BCF$ or, equivalently, $\angle ABC = \angle ACB$, which are the base angles. [C3]
- Furthermore, we've already shown that $\angle FBC = \angle GCB$, which are precisely the angles under the base angles. □

Pythagoras' Theorem

Now that I've led you over the Pons Asinorum, it seems like a reasonably safe assumption that if you had enough time and energy, you could understand the rest of Euclid's Book 1. Rather than spend our time working through the remaining propositions, let's fast forward to the second last proposition in Euclid's Book 1, a theorem which you should already know and love.

Proposition (Proposition XLVII — Pythagoras' Theorem)**.** *In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*



Proof. Let ABC be the right-angled triangle with $\angle BAC = 90°$.

- Draw the square $BDEC$ on BC, the square $AGFB$ on AB, and the square $CKHA$ on CA. [P46]
- Draw the line segment AL parallel to BD, where L lies on DE. [P31]
- Draw the line segments AD and FC. [A1]
- Since $\angle BAC = \angle BAG = 90°$, the line segment CG passes through A and, similarly, the line segment BH passes through A. [P14]
- Note that $\angle CBD = \angle FBA$, so adding $\angle ABC$ to both sides, we obtain $\angle ABD = \angle FBC$. [C2]
- We also have $BD = BC$ and $AB = FB$, so triangles ABD and FBC are congruent. [P4]
- The parallelogram $BDLM$ is twice the area of triangle ABD because they have the same base BD and lie between the same parallel lines BD and AL. [P41]
- The square $AGFB$ is twice the area of triangle FBC because they have the same base FB and are between the same parallel lines FB and GC. [P41]

Thus, the parallelogram $BDLM$ is equal in area to the square $AGFB$. In a similar fashion, we can deduce that the parallelogram $CELM$ is equal in area to the square $CKHA$. Thus, the area of the square $BDEC$ is equal to the sum of the areas of the two squares $AGFB$ and $CKHA$ — exactly what we set out to prove. □

Of course, Euclid's proof of Pythagoras' Theorem relies on all sorts of propositions that we haven't seen yet. So, for the sake of begin complete, here is a list of all forty-eight propositions from Book 1 of Euclid's Elements.

Euclid's Propositions

P1. Given a line segment, you can draw an equilateral triangle on it.

P2. Given a line segment and a point, you can draw a line segment from the given point, equal in length to the given line segment.

P3. Given two line segments of unequal lengths, you can divide the longer one into two parts, one of which is equal in length to the shorter one.

P4. If two triangles have two pairs of equal sides and the angles formed by these sides are equal, then the two triangles are congruent — in other words, they have the same side lengths and the same angles.

P5. In an isosceles triangle, the base angles are equal and the angles under the base angles are equal.

P6. If a triangle has two equal angles, then the two sides opposite these angles are equal.

P7. Given a triangle ABC, there is no other point P on the same side of AB as C such that $AC = AP$ and $BC = BP$.

P8. If two triangles have three pairs of equal sides, then they also have three pairs of equal angles.

P9. Given a right angle, you can draw a line which bisects it.

P10. Given a line segment, you can draw its midpoint.

P11. Given a line and a point on that line, you can draw another line perpendicular to the given line and passing through the given point.

P12. Given a line and a point not on the line, you can draw another line perpendicular to the given line and passing through the given point.

P13. If a line intersects another line, then it creates two angles which sum to $180°$.

P14. Given a point on a line, if two line segments drawn from the point lie on opposite sides of the line and form adjacent angles which sum to $180°$, then the line segments lie on a line.

P15. If two lines meet, then the two vertical angles are equal.

P16. In any triangle, if one of the sides is extended, then the exterior angle formed is greater than either of the two interior opposite angles.

P17. In any triangle, the sum of any two angles is less than $180°$.

P18. In any triangle, the angle opposite the longest side is the largest.

P19. In any triangle, the side opposite the largest angle is the longest.

P20. In any triangle, the sum of any two sides is longer than the remaining side.

P21. If P is a point inside triangle ABC, then $AP + BP < AC + BC$ and $\angle APB > \angle ACB$.

P22. Given three line segments, it is possible to draw a triangle whose sides are equal in length to the given line segments whenever the sum of any two is longer than the remaining one.

P23. Given an angle and a line with a point on it, you can draw a line passing through the given point which creates an angle with the given line equal to the given angle.

P24. If two triangles have two pairs of equal sides and the angle formed by these two sides is larger in one triangle, then the third side is longer in that triangle.

P25. If two triangles have two pairs of equal sides and the third side is longer in one triangle, then the angle formed by these two sides is larger in that triangle.

P26. If two triangles have two pairs of equal angles and one pair of corresponding equal sides, then the two triangles are congruent.

P27. If a line meets two lines and forms equal alternate angles, then the two lines are parallel.

P28. If a line meets two lines and forms an exterior angle equal to the interior opposite angle on the same side, or the sum of the interior angles on the same side is equal to $180°$, then the lines are parallel.

P29. If a line meets two parallel lines, then alternate angles are equal, the exterior angle is equal to the interior opposite angle, and the interior angles on the same side sum to $180°$.

P30. Lines parallel to the same line are also parallel to each other.

P31. Given a line and a point, you can draw a line through the given point parallel to the given line.

P32. In any triangle, if one of the sides is extended, then the exterior angle formed equals the sum of the two interior opposite angles, and the sum of the three interior angles of the triangle equals $180°$.

P33. Line segments which join the ends of equal parallel line segments are equal and parallel.

P34. In a parallelogram, opposite sides and angles are equal to each other, and the diagonals bisect the area.

P35. Parallelograms with the same base and equal heights have equal areas.

P36. Parallelograms with equal bases and equal heights have equal areas.

P37. Triangles with the same base and equal heights have equal areas.

P38. Triangles with equal bases and equal heights have equal areas.

P39. Triangles with equal areas and the same base have equal heights.

P40. Triangles with equal areas and equal bases have equal heights.

P41. If a parallelogram has the same base and equal height with a triangle, then the parallelogram is twice the area of the triangle.

P42. Given an angle and a triangle, you can draw a parallelogram with area equal to the area of the given triangle and with one angle equal to the given angle.

P43. Consider a point on the diagonal of a parallelogram. If lines are drawn through this point parallel to the sides of the parallelogram, then four triangles and two parallelograms are formed. The two parallelograms have equal area.

P44. Given a triangle, a line segment and an angle, you can draw a parallelogram on the line segment with one angle equal to the given angle and with area equal to the area of the given triangle.

P45. Given a polygon and an angle, you can draw a parallelogram with one angle equal to the given angle and with area equal to the area of the given polygon.

P46. You can draw a square on a given line segment.

P47. In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

P48. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is right-angled.

The Problem with Reductionism

At this point, we're going to leave Euclid's Elements behind and make our own way through the world of geometry. We do this for three main reasons.

- So far, we have followed Euclid's reductionist approach and carefully proved the first five of his propositions. At this rate, it would take us more than half the course to finish Book 1 of the Elements and we'd still have twelve books left to go. As Willy Wonka from *Willy Wonka and the Chocolate Factory* once said, "So much time and so little to do! Strike that, reverse it."
- To be honest, I think that working through the remainder of Book 1 of Euclid's Elements would not only be incredibly laborious, but also excruciatingly boring.
- Whenever you get too involved in the reductionist approach, you run the risk of missing the bigger picture. For example, it is far from true that a physicist who unlocks the mysteries of the tiny tiny quarks and strings has a grasp of the universe around them. They would still have no further knowledge about topics such as beauty, music, linguistics, hockey and personal hygiene, to name a few. I think we should take a broader overview of geometry, whereas if we continued with Euclid, *we would not see the forest for the trees*.

Still, there is great beauty in Euclid's work, but it doesn't lie in the individual propositions themselves. It lies in the fact that the reductionist approach lets us truly understand what it means for a mathematical theorem to be true, the fact that the Elements was the paradigm of mathematical rigour for centuries upon centuries, and the fact that Euclid has done a lot of the dirty work for us and we can use his results as a stepping stone to look at some much more interesting facets of geometry.

Problems

At the end of each lecture, I'll usually state a couple of example problems and solve them, to help you learn how to approach problems in geometry. However, I'm going to end today's lecture with the are-you-smarter-than-Euclid challenge.

Problem. *As I mentioned earlier, Euclid's fifth proposition — otherwise known as Pons Asinorum — is the first real test of intelligence in Euclid's Elements. This is because Euclid's proof is relatively long and intricate. However, if you look at the first four propositions very carefully, you might notice that these can be used to give a much slicker proof to the Pons Asinorum. Try to find the proof that Euclid missed and write it out carefully.*

Euclid

As I mentioned earlier, Euclid was a Greek mathematician who lived a really really long time ago — around 300 BC, to be precise. But we can't be any more precise than that, because there's very little known about Euclid's life. In fact, it's been stated that any of the following three possibilities may be the actual truth about Euclid.

- Euclid was a person who wrote the Elements as well as various other works which are attributed to him.
- Euclid led a team of mathematicians in Alexandria, who all contributed to writing the complete works of Euclid, even continuing after Euclid died.
- Euclid was entirely fictional and his complete works were written by a team of mathematicians in Alexandria who borrowed the name of Euclid from an actual person — Euclid of Megara — who had lived about 100 years earlier.

Most people who aren't conspiracy theorists tend to believe the first possibility, in which case Euclid was a very clever guy indeed. He has quite rightly been referred to as the "Father of Geometry". You might be wondering, if we don't even know whether or not Euclid existed, then why is there a picture of him studying geometry below? Well, this particular portrait is just the product of an artist's imagination.