Release date:	1:30pm on Wednesday 19 May, 2010
Due date:	11:00am on Wednesday 26 May, 2010

You are not discouraged from talking about assignment problems with other students, but every solution that you hand in must be your own work. Every page submitted should clearly indicate your name, student number, the course number, and the assignment number. Late assignments will not be accepted, unless under particularly extreme circumstances.

Problems

1. (a) Show that at a party with five people, it is not necessarily true that there exist three people who all know each other or three people who all don't know each other.
   (b) Note that if you take the dual of the tetrahedron, then you just get the tetrahedron again. Describe another polyhedron whose dual is itself.
   (c) Does there exist a map on the torus consisting of fifty triangles?
   (d) For which values of n and d is there a graph — without loops or multiple edges — with n vertices of degree d?
   (e) Draw a map on a genus two surface which consists of one polygon.
   (f) Divide the letters of the alphabet into the smallest possible number of groups so that, within each group, all letters are homeomorphic to each other. You should assume that the letters look like the following and consist of infinitely thin line segments and curves.

   A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

   [*3 marks*]

2. Prove that there does not exist a graph — without loops or multiple edges — with eight vertices which

   (a) has vertices of degrees 1, 1, 1, 1, 2, 2, 2, 8;
   (b) has vertices of degrees 1, 1, 2, 3, 3, 4, 4, 5;
   (c) has vertices of degrees 0, 1, 1, 2, 3, 4, 4, 7;
   (d) has vertices of degrees 1, 1, 1, 2, 2, 3, 4, 6.

   [*2 marks*]

3. (a) A domino consists of two squares, each of which is labelled with a number from 0 to 6. Verify that there are 28 different dominoes. Is it possible to arrange them all in a loop so that adjacent halves of neighbouring dominoes are labelled by the same number?
   (b) An n-domino consists of two squares, each of which is labelled with a number from 0 to n. For which values of n is it possible to arrange them all in a loop so that adjacent halves of neighbouring n-dominoes are labelled by the same number?
   Hint: You should use the theorem which tells you when a graph is Eulerian.

   [*2 marks*]

4. Consider a graph G — without multiple edges or loops — whose vertices all have degree greater than or equal to six.
   (a) If G has V vertices and E edges, use the handshaking lemma to prove that $V \leq \frac{E}{3}$.
   (b) Suppose that G is planar and can be drawn in the plane in such a way that there are F faces. Use the handshaking lemma on the dual graph to prove that $F \leq \frac{2E}{3}$.

(c) Explain how these two inequalities imply that there does not exist a planar graph — without loops or multiple edges — whose vertices all have degree greater than or equal to six.

[3 *marks*]

5. Suppose that you have a polyhedron and you are told that each face is a quadrilateral or a hexagon and that three faces meet at every vertex. Furthermore, every quadrilateral face shares an edge with four hexagonal faces, while every hexagonal face shares an edge with three quadrilateral faces and three hexagonal faces.
   (a) Deduce the number of quadrilateral faces and the number of hexagonal faces of the polyhedron and draw an example of such a polyhedron.
   (b) Cut off each vertex of the polyhedron using a plane which passes through the midpoints of the three edges which meet at that vertex. This process produces a convex polyhedron from a convex polyhedron. For example, applying this process to a tetrahedron produces an octahedron. How many vertices, edges and faces does this new polyhedron have? How many triangular faces, quadrilateral faces, pentagonal faces, and hexagonal faces does this new polyhedron have?

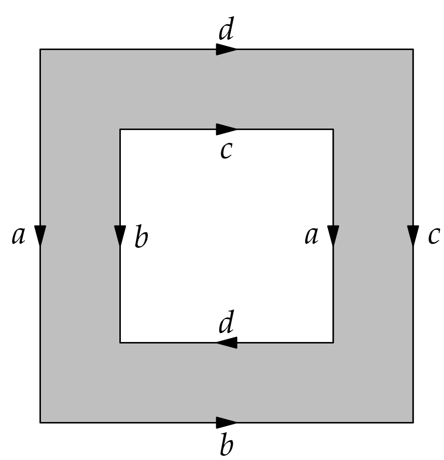
[3 *marks*]

6. For each of the following edge words, determine whether or not the corresponding surface is orientable and calculate its Euler characteristic. Also, for each surface, identify it as the sphere, a connect sum of tori or a connect sum of projective planes.
   (a) $abc^{-1}b^{-1}da^{-1}d^{-1}c$
   (b) $abacb^{-1}ded^{-1}e^{-1}c$
   (c) $a_1a_2a_3 \cdots a_na_1^{-1}a_2^{-1}a_3^{-1} \cdots a_n^{-1}$
   Hint: Try this problem for several values of n until you find a pattern.

[3 *marks*]

7. The shaded region, pictured below, is a shape which gives rise to a surface after gluing the edges together as indicated.
   (a) Show that it can be described by the edge word $abc^{-1}d^{-1}ecadb^{-1}e^{-1}$.
   (b) Determine whether or not the surface is orientable, calculate its Euler characteristic and identity the surface using the classification of surfaces.
   (c) What is the shortest possible edge word for this surface?
   (d) If the surface is homeomorphic to $T \# X$ where T represents a torus and X represents a surface, then what must X be?

[2 *marks*]



8. Given an edge word W, let $\sigma(W)$ denote the surface corresponding to W. Prove the following facts, where a and b represent letters, W, X, Y, Z represent words, and P and T represent the projective plane and the torus, respectively.
   (a) $\sigma(aXa^{-1}Y) \cong \sigma(b^{-1}XbY)$
   (b) $\sigma(XY) \cong \sigma(YX)$
   (c) $\sigma(X) \cong \sigma(X^{-1})$
   (d) $\sigma(aa^{-1}X) \cong \sigma(X)$
   (e) $\sigma(aXaY) \cong P \# \sigma(XY^{-1})$
   (f) $\sigma(bWxXb^{-1}Yx^{-1}Z) \cong T \# \sigma(WZYX)$
   (g) $T \# P \cong P \# P \# P$

   [*2 marks*]

9. Use the results from question 8 to prove the classification of surfaces — in other words, prove that every surface is either the sphere, a connect sum of tori or a connect sum of projective planes.

   [*BONUS*]

10. Consider the pair (S, p) where S is a surface and p is a point on the surface. To the pair (S, p) we associate a set G consisting of the paths in S which start at p and end at p. However, we consider two paths a and b to be the same if you can slide the path a along the surface S, while keeping its start and end points at p, until you get the path b. If a and b are two loops on S which start and end at p, then we can compose them to form the loop $a \cdot b$ which traverses a and then traverses b.
    (a) Prove that the set G is a group under this composition.
    (b) Prove that the group G does not depend on the choice of p — in other words, if you form the group G_1 using the pair (S, p) and the group G_2 using the pair (S, q), then $G_1 \cong G_2$.
    (c) Let a and b be two paths in a torus which start and end at some point p such that a and b intersect only once. Show that $a \cdot b = b \cdot a$.
    (d) Describe the group G when the surface is a torus, a projective plane and a genus two surface.

    [*CHALLENGE*]