Release date:	1:30pm on Tuesday 11 May, 2010
Due date:	11:00am on Tuesday 18 May, 2010

You are not discouraged from talking about assignment problems with other students, but every solution that you hand in must be your own work. Every page submitted should clearly indicate your name, student number, the course number, and the assignment number. Late assignments will not be accepted, unless under particularly extreme circumstances.

Problems

1. (a) Suppose that $ABCD$ is a square with the vertices labelled counterclockwise. Let
   - G_{AC} denote the glide reflection consisting of a reflection in the line AC followed by a translation which takes A to C;
   - T_{BD} denote the translation which takes B to D;
   - M_{AB} denote the reflection in the line AB; and
   - R_B denote a counterclockwise rotation by 90° about B.

   Identify the composition $G_{AC} \circ T_{BD} \circ M_{AB} \circ R_B$.

   (b) If X denotes the composition $G_{AC} \circ T_{BD} \circ M_{AB} \circ R_B$, let n be the minimum number of reflections whose composition is equal to X. Determine, with proof, the value of n and carefully describe n reflections whose composition is equal to X.

   [2 *marks*]

2. (a) Let M be a reflection and let R be a rotation by 90° counterclockwise. Is it possible for the compositions $M \circ R$ and $R \circ M$ to be the same?

   (b) Let M_k be a reflection through the line k and let M_ℓ be a reflection through the line ℓ. If someone tells you that $M_\ell \circ M_k = M_k \circ M_\ell$, what can you tell them about the lines k and ℓ?

   [3 *marks*]

3. (a) The following is a *groupoku* puzzle — a Cayley table for the group G, where some of the entries are missing. Use the properties which you know about Cayley tables to fill in all of the missing entries. Give full reasoning only for the first entry of the table that you manage to fill in.

·	a	b	c	d
a	*	*	*	*
b	*	c	*	*
c	b	*	d	*
d	*	*	*	*

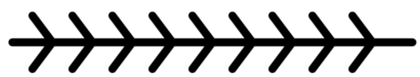
   (b) In the lectures, we have seen two groups with four elements — the cyclic group C_4 and the dihedral group D_2. Prove that G is isomorphic to one of these groups by writing down an explicit isomorphism. Furthermore, prove that what you have written down is in fact an isomorphism. How many isomorphisms are there?

   (c) Prove that the cyclic group C_4 and the dihedral group D_2 are not isomorphic to each other, even though they are both abelian groups with four elements.

(d) Give a subset of the Euclidean plane whose symmetry group is isomorphic to G. Does there exist a finite set of points in the Euclidean plane whose symmetry group is isomorphic to G? If so, determine the minimum number of points in such a set. If not, give a brief explanation of why such a set does not exist.

[*4 marks*]

4. (a) Write down the $HVRG$ symbol for the frieze pattern below.



(b) Write down the $HVRG$ symbol for the frieze pattern below.

LOΓOLOΓOLOΓOLOΓO

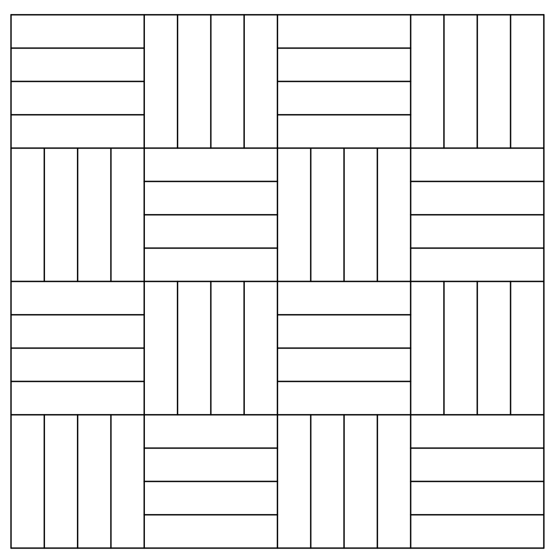
(c) For each of the seven types of frieze pattern, determine whether or not there exists a sequence of capital letters which can be written in a repeating pattern to form a frieze pattern of that type. You should assume that the capital letters look like the following.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

(d) Prove that if a symmetry group of a frieze pattern contains a glide reflection and a reflection in a vertical mirror, then it must also contain a rotation by 180°.

[*3 marks*]

5. (a) Identify the wallpaper pattern on the outside of Burnside Hall, using its RMG symbol.
   (b) Identify the wallpaper pattern on the walls of room 1B24, using its RMG symbol.
   (c) Write down the RMG symbol for the wallpaper pattern pictured below left.
   (d) Write down the RMG symbol for the wallpaper pattern pictured below right. Draw the wallpaper pattern and mark clearly on your diagram a point which is the centre of a rotational symmetry of order R, a point where M mirrors meet, and a point where G proper glide axes meet.



[*4 marks*]

6. Let ABC be a triangle with vertices labelled in a counterclockwise fashion. Three equilateral triangles BCX, CAY, ABZ are drawn on the sides of triangle ABC, outside of triangle ABC. Let the centres of BCX, CAY, ABZ be K, L, M, respectively. Let R_K be the rotation by 120° counterclockwise about K, let R_L be the rotation by 120° counterclockwise about L, and let R_M be the rotation by 120° counterclockwise about M.
   (a) Prove that $R_M \circ R_K \circ R_L$ is a translation.
   (b) Determine the isometry $R_M \circ R_K \circ R_L$ by considering the location of $R_M \circ R_K \circ R_L(A)$.
   (c) Describe the location of $R_M \circ R_K \circ R_L(L)$. If $P = R_K(L)$, what can you say about the quadrilateral $LKPM$?
   (d) Hence, what can you deduce about the triangle KLM?

   [*3 marks*]

7. Prove that Leonardo's theorem does not hold in three dimensions. In other words, prove rigorously that there exists a subset of Euclidean space whose symmetry group is neither cyclic nor dihedral.

   [*1 mark*]

8. Explain why it follows from S_3 and C_6 not being isomorphic that there is no shuffle of 3 cards that you can repeat which goes through all of the possible orderings of the deck. Prove that there is no shuffle of n cards that you can repeat which goes through all of the possible orderings of the deck for $n \geq 3$. Determine the largest number of possible orderings which you can go through if you repeat a shuffle of n cards for $n \leq 10$.

   [*BONUS*]

9. The groups we've been dealing with have all been examples of cyclic groups, dihedral groups or symmetric groups — however, there are many others around. Write down the Cayley table for the smallest non-abelian group which is not a cyclic group, not a dihedral group and not a symmetric group.

   [*CHALLENGE*]