Release date: 1:30pm on Tuesday 4 May, 2010
Due date: 11:00am on Tuesday 11 May, 2010

You are not discouraged from talking about assignment problems with other students, but every solution that you hand in must be your own work. Every page submitted should clearly indicate your name, student number, the course number, and the assignment number. Late assignments will not be accepted, unless under particularly extreme circumstances.

Problems

1. Let ABC be a triangle with circumcentre O and incentre I. If O and I are the same point, prove that the triangle must be equilateral.

   [*1 mark*]

2. Let ABC be a triangle with circumcentre O and orthocentre H. Prove that $\angle ABH = \angle CBO$.

   [*2 marks*]

3. Let ABC be an acute triangle with altitudes AD, BE, CF.
   (a) Prove that triangle AEF is similar to triangle ABC.
   (b) Extend AD until it meets the circumcircle of triangle ABC at X and extend BE until it meets the circumcircle of triangle ABC at Y. Prove that $CX = CY$.

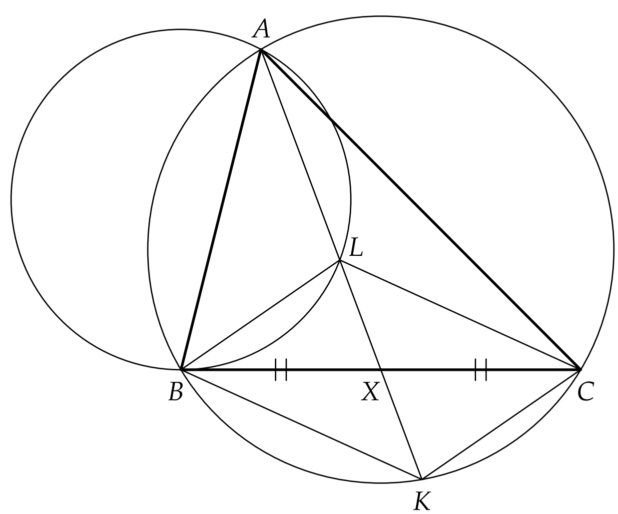
   [*2 marks*]

4. In the lectures, we tried to prove the following fact. If AB is a chord of a circle with centre O and C is a point on the circle on the same side of AB as O, then $\angle AOB = 2\angle ACB$. Unfortunately, our proof was incomplete because we only considered the case when O lies inside triangle ABC. Prove that the statement remains true in the case that O lies on one of the sides of triangle ABC and in the case that O lies outside triangle ABC.

   [*3 marks*]

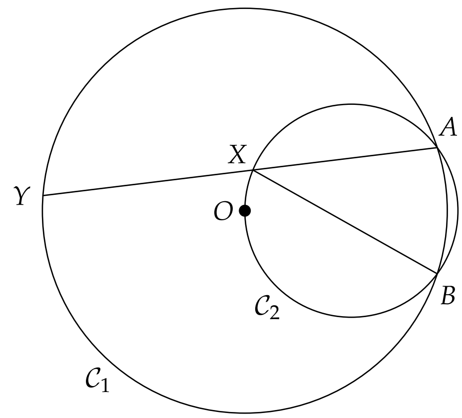
5. Extend the median AX of triangle ABC until it meets the circumcircle of triangle ABC at K. There is a unique circle which passes through A and B and is tangent to the line BC. If this circle meets AK at L, prove that triangle LBX and triangle KCX are congruent to each other. Hence, deduce that $BKCL$ is a parallelogram.

   [*3 marks*]



6. Suppose that the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points A and B and that $\mathcal{C}_2$ passes through the centre of $\mathcal{C}_1$, which we denote by O. Let X be a point on the arc AB which contains O and extend AX until it intersects $\mathcal{C}_1$ at Y. Prove that $XY = XB$.

[*3 marks*]



7. (a) In the lectures, we proved that the diameter of a circle subtends an angle of 90°. Prove the converse of this statement — in other words, prove that if ABC is a triangle with $\angle ACB = 90°$, then the circle with diameter AB passes through C.

   (b) Let $ABCD$ be a cyclic quadrilateral such that the diagonals AC and BD are perpendicular and intersect at the point P. Let M be the midpoint of AB and let N be the point on CD such that PN is perpendicular to CD. Prove that the points M, P, N lie on a line.

[*4 marks*]

8. In the hyperbolic plane, there exist pentagons all of whose angles are equal to 90°. Sketch one example of such a pentagon in the hyperbolic plane using the Poincaré disk model. What is the area of this pentagon?

[*2 marks*]

9. Let D, E, F be points on sides AB, BC, CA of triangle ABC, respectively. If $DE = BE$ and $FE = CE$, prove that the circumcentre of triangle ADF lies on the angle bisector of $\angle DEF$.

[*BONUS*]

Hints

1. Draw in the line segments AI, BI, CI or equivalently, AO, BO, CO. What is special about the lengths in the diagram? What is special about the angles in the diagram?
2. We already know how to work out all the angles in the diagram with triangle ABC and circumcentre O. We also already know how to work out all the angles in the diagram with triangle ABC and orthocentre H.
3. (a) Altitudes lead to right angles and right angles often lead to cyclic quadrilaterals. You should be able to use a cyclic quadrilateral or two to obtain some useful relations between angles.
   (b) Equal chords in a circle subtend equal angles — but remember that equal angles are also subtended by equal chords.
4. Try to mimic the case that we covered in lectures as much as possible.
5. Since the problem involves a tangent, you should try to use the alternate segment theorem. Use this to prove that $\angle LBX = \angle KCX$.
6. How would you prove that triangle XYB is isosceles?
7. (a) Suppose that C lies outside the circle with diameter AB — can you find a contradiction?
   (b) You should use the result from part (a), even if you are unable to prove it.
8. Do you remember how we found the sum of the angles in a polygon with n sides?
9. Do you really expect hints for a bonus question?