Today we covered Subsections 3.2.1, 3.2.2, 3.3.1, 3.3.2 and some of Subsection 4.5.1 in the textbook.

Lines and spheres

Example

Find a parametric equation — also known as a vector equation — for the line ℓ in $\mathbb{R}^3$ which passes through $A = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

To find a point on this line, you can start at the point A and walk in the direction $\vec{AB} = B - A$ or in the opposite direction. Hence, every point on the line must be of the form $A + t(B - A)$ for some real number t. So the parametric equation that we are looking for is

$$X = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+t \\ -2+3t \\ 3+t \end{bmatrix}.$$

You can check that when $t = 0$ in the above equation, we get the point A and when $t = 1$ in the above equation, we get the point B.

Note that in general, the parametric equation of a line which passes through A and B takes the form

$$X = X_0 + t(B - A),$$

where X_0 can be any point which lies on the line. In the example above, we chose $X_0 = A$, but you could equally choose $X_0 = B$ or any other point which lies on the line.

Definition

A non-zero vector parallel to a line is called a *direction vector* for this line. For example, the vector $B - A$ is a direction vector for the line which passes through A and B.

Exercise

Find a parametric equation for the line ℓ, given that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \in \ell$.

Observe that the two lines given by the parametric equations

$$X = X_0 + tX_1$$
$$X = X_0' + tX_1'$$

are parallel if and only if their direction vectors are parallel. This happens if and only if $X_1 = sX_1'$ for some real number s.

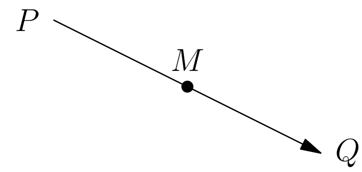
Exercise

Find a line through $X_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which is parallel to the line ℓ: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Which of the points $A_1 = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -7 \\ -1 \end{bmatrix}$ belong to ℓ?

Example

Find the midpoint between the points $P = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$.

To get to the point M, you can start at P and walk halfway along the vector $\vec{PQ}$. Hence, we have the equation



$$M = P + \frac{1}{2}\vec{PQ} = P + \frac{1}{2}(Q-P) = \frac{1}{2}(P+Q) = \begin{bmatrix} 1/2 \\ 2 \\ 1 \end{bmatrix}.$$

Example

Find the equation of a sphere in $\mathbb{R}^3$ with center at $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and radius 2.

A point X is on the sphere only if its distance from the centre is equal to 2. Therefore, we can describe the sphere as the set $S = \{X \in \mathbb{R}^3 : \left\| X - \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\| = 2\}$. So the equation for the sphere is $\left\| X - \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\| = 2$ which we can write as $\left\| \begin{bmatrix} x-1 \\ y+2 \\ z \end{bmatrix} \right\|^2 = 4$. Recall that $\|X\|^2 = X \cdot X$, so the equation for the sphere is

$$(x-1)^2 + (y+2)^2 + z^2 = 4.$$

Fact

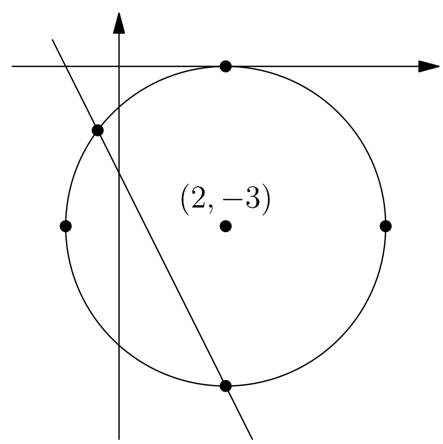
A sphere S in $\mathbb{R}^n$ with center $C = [c_1, c_2, \ldots, c_n]^T$ and radius r has the equation

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 + \cdots + (x_n - c_n)^2 = r^2.$$

Example

Draw the set $S = \{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 : (x-2)^2 + (y+3)^2 = 9\}$. Where does the line $y = -2x - 2$ intersect it?

The equation $(x-2)^2 + (y+3)^2 = 9$ looks exactly like the equation of a circle in $\mathbb{R}^2$ — in fact, it describes the circle with centre $(2, -3)$ and radius 3. This set is drawn in the coordinate plane on the right.



The line $y = -2x - 2$ intersects the circle at points $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2$ which simultaneously satisfy the following equations.

$$(x-2)^2 + (y+3)^2 = 9$$

$$y = -2x - 2$$

To solve this, we substitute the second equation into the first equation to obtain

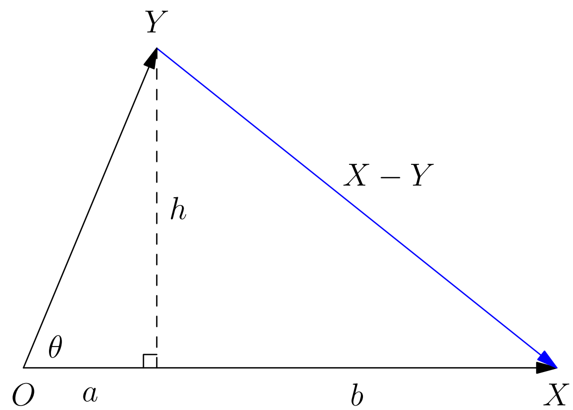
$$(x-2)^2 + (-2x-2+3)^2 = 9 \Leftrightarrow (x^2 - 4x + 4) + (4x^2 - 4x + 1) = 9 \Leftrightarrow 5x^2 - 8x - 4 = 0.$$

Using the quadratic equation, we obtain the solutions $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ -6 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} -2/5 \\ -6/5 \end{smallmatrix}\right]$.

Applications of the dot product

Consider two vectors $X, Y \in \mathbb{R}^n$. If you consider the diagram below, you should be able to see that

$$\cos\theta = \frac{a}{\|Y\|} \qquad \text{and} \qquad \sin\theta = \frac{h}{\|Y\|}.$$



Now consider the following calculations which start with Pythagoras' Theorem.

$$\begin{aligned}\|X-Y\|^2 = b^2+h^2 = (\|X\|-a)^2 + (\|Y\|\sin\theta)^2 &= (\|X\| - \|Y\|\cos\theta)^2 + (\|Y\|\sin\theta)^2 \\ &= \|X\|^2 - 2\|X\|\|Y\|\cos\theta + \|Y\|^2\cos^2\theta + \|Y\|^2\sin^2\theta \\ &= \|X\|^2 - 2\|X\|\|Y\|\cos\theta + \|Y\|^2\end{aligned}$$

We can calculate the same expression in a different way.

$$\|X-Y\|^2 = (X-Y)\cdot(X-Y) = X\cdot X - 2X\cdot Y + Y\cdot Y = \|X\|^2 - 2X\cdot Y + \|Y\|^2$$

Comparing these two expressions, we obtain the following result.

Theorem
For any two non-zero vectors $X, Y \in \mathbb{R}^n$, the angle θ between them satisfies

$$X \cdot Y = \|X\|\|Y\|\cos\theta.$$

Corollary
Two vectors $X, Y \in \mathbb{R}^n$ satisfy $X \cdot Y = 0$ if and only if X is perpendicular to Y. If X and Y are perpendicular, then we write $X \perp Y$.

Examples

- Find the angle between $X = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

If θ is the angle between the two vectors, then we have $X \cdot Y = \|X\|\|Y\| \cos\theta$. This rearranges to give us

$$\cos\theta = \frac{X \cdot Y}{\|X\|\|Y\|} = \frac{(-1)\cdot 2 + (-2)\cdot 1 + 1\cdot 1}{\sqrt{(-1)^2+(-2)^2+1^2}\sqrt{2^2+1^2+1^2}} = \frac{-3}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}.$$

So we can deduce that $\theta = 120°$. Actually, you could also say that $\theta = 240°$, but it's common to choose $0 \leq \theta < 180°$. Also, it's good to know that $120°$ is the same thing as $\frac{2\pi}{3}$ radians.

One useful fact to keep in mind is that $X \cdot Y \geq 0$ if and only if θ is acute — that is, $0 \leq \theta < \frac{\pi}{2}$.

- Show that the triangle XYZ with $X = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $Y = \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix}$ and $Z = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ is a right-angled triangle.

  We will have solved the problem if we can show that $\vec{XY} \perp \vec{XZ}$. To do this, we use the dot product as follows.

$$\vec{XY} \cdot \vec{XZ} = (Y - X) \cdot (Z - X) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 2 \cdot (-1) + (-1) \cdot 1 + 3 \cdot 1 = 0$$