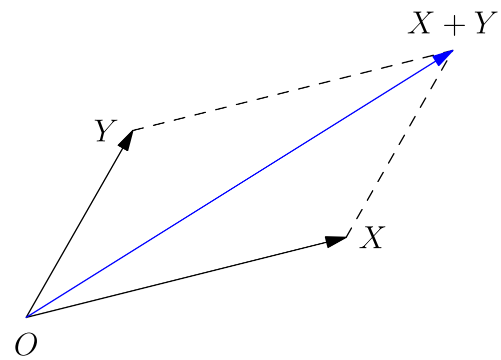
Today, we covered Section 3.1 in the textbook.

Vector geometry in $\mathbb{R}^n$

- Most of our examples will be in $\mathbb{R}^2$ or $\mathbb{R}^3$, where it's easier to visualize things.
- We will identify a point $X \in \mathbb{R}^n$ with the vector which starts at the origin O and ends at the point X. We say that such vectors with their tail at the origin are in *standard position*.
- Addition — Geometrically, you can visualize adding two vectors using the *tip-to-tail method* or the *parallelogram rule*. When adding the vectors X and Y using the tip-to-tail method, we move vector Y until its tail coincides with the tip of the vector X. The vector $X + Y$ now points from the tail of X to the tip of Y. The parallelogram rule keeps the two vectors in standard position and the vector $X + Y$ corresponds to the unique point Z such that $OXZY$ is a parallelogram.



- Scalar multiplication — Geometrically, you can visualize the result of multiplying a positive real number t by the vector X as the vector in the same direction as X, stretched by a factor of t. If we multiply the number -1 by the vector X, then the result is the vector in the opposite direction to X, with the same length. It follows that the result of multiplying a negative real number $-t$ by the vector X is the vector in the opposite direction to X, stretched by a factor of t.

  Note that the vectors O, X and tX all lie on the same line for any real number t.
- In the textbook, they often use notation like $\vec{v}, \vec{w}$, $\vec{A}B$, $\vec{B}C$, and so on. This is fine, but we won't use it so much since each vector can be moved so that its tail is at the origin.
- You should remember that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ — in other words, it's an object which satisfies the vector space axioms.

Dot product

We need a mathematical tool which can be used to do geometric things, like measure the length of a vector, measure the angle between two vectors, decide whether two vectors are perpendicular, and so on. It turns out that the dot product — sometimes called the scalar product — is a good choice.

Definition
Given two vectors $X = [x_1, x_2, \ldots, x_n]^T$ and $Y = [y_1, y_2, \ldots, y_n]^T$, we define the *dot product* to be the real number

$$X \cdot Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Note that the two vectors must both have the same number of coordinates — they are both from $\mathbb{R}^n$.

Example

Consider the following two examples of dot products.

$$\begin{bmatrix} 3 \\ -2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 7 \\ -1 \end{bmatrix} = 3 \times 2 + (-2) \times 1 + 0 \times 7 + 4 \times (-1) = 0 \qquad \begin{bmatrix} 3 \\ -2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 0 \\ 4 \end{bmatrix} = 3^2 + (-2)^2 + 0^2 + 4^2 = 29$$

It's useful to keep in mind that the dot product is a special case of matrix multiplication, as you can see in the following example. On the left, the two column vectors are multiplied using the dot product while on the right, a row vector and a column vector are multiplied using standard matrix multiplication.

$$\begin{bmatrix} 3 \\ -2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 7 \\ -1 \end{bmatrix} = [3, -2, 0, 4] \begin{bmatrix} 2 \\ 1 \\ 7 \\ -1 \end{bmatrix}$$

Fact

Here are some nice properties concerning the dot product. If X, Y and Z are vectors in $\mathbb{R}^n$ and t is a real number, then the following equations hold.

- $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$
- $X \cdot Y = Y \cdot X$
- $(tX) \cdot Z = t(X \cdot Z)$
- $X \cdot X = \|X\|^2$

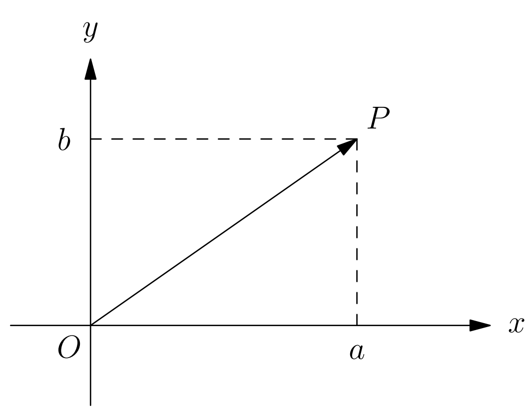
Here, $\|X\|$ denotes the length of the vector — in other words, the length of the line segment from O to X.

The first two properties are direct consequences of the fact that dot product is a special case of matrix multiplication. The third fact should be completely obvious if you think carefully about the definition of dot product. Let's now try and give a proof of the fourth fact in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$.

- The $\mathbb{R}^2$ case

  Using one of the right-angled triangles in the diagram below, Pythagoras' Theorem tells us that

$$\|P\|^2 = a^2 + b^2.$$

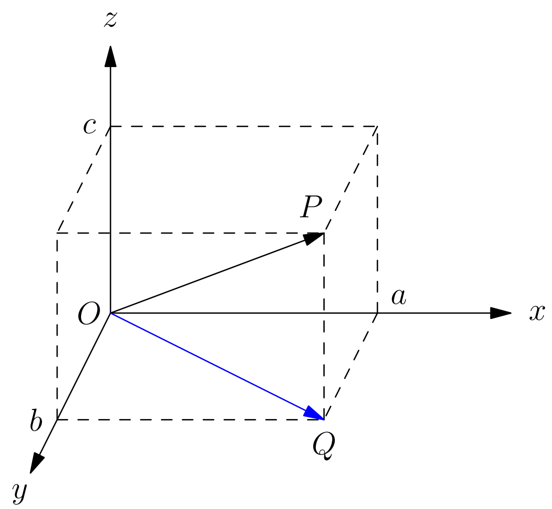


On the other hand, we can calculate the dot product of P with itself to obtain

$$P \cdot P = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = a^2 + b^2.$$

- The $\mathbb{R}^3$ case
  Using the right-angled triangle OQP in the diagram below, Pythagoras' Theorem tells us that

$$\|P\|^2 = \|Q\|^2 + \|\vec{QP}\|^2 = (a^2 + b^2) + \|\vec{QP}\|^2 = a^2 + b^2 + c^2.$$



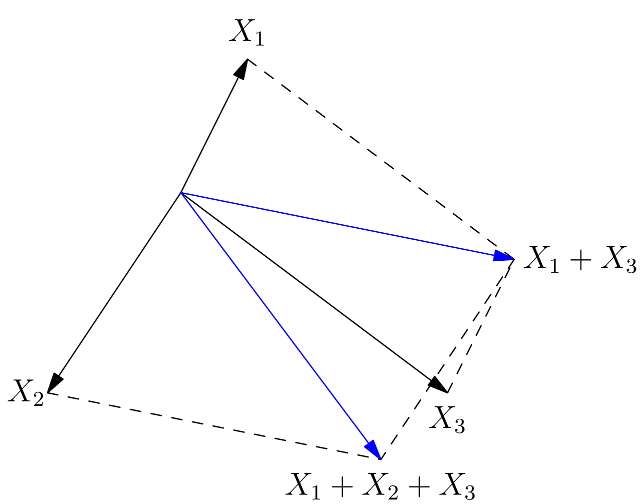
On the other hand, we can calculate the dot product of P with itself to obtain

$$P \cdot P = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a^2 + b^2 + c^2.$$

Examples

- Determine $X_1 + X_2 + X_3$ if $X_1 = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $X_2 = \left[\begin{smallmatrix} -2 \\ -3 \end{smallmatrix}\right]$ and $X_3 = \left[\begin{smallmatrix} 4 \\ -3 \end{smallmatrix}\right]$. Draw a picture of the vectors and add them using the tip-to-tail method or the parallelogram rule.

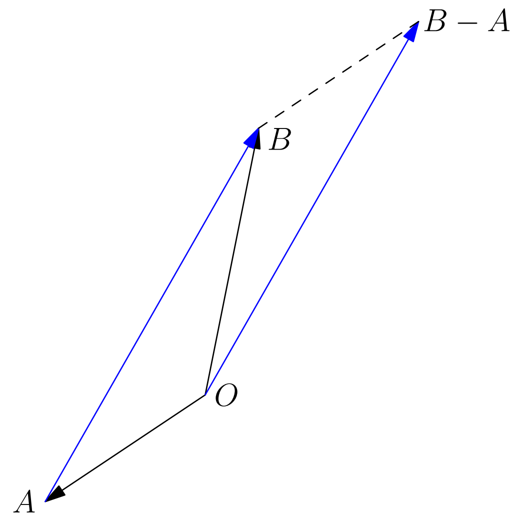
  We can easily calculate that $X_1 + X_2 + X_3 = \left[\begin{smallmatrix} 3 \\ -4 \end{smallmatrix}\right]$. The picture on the right shows what this means if you add the vectors using the parallelogram rule.



- Compute $B - A$ if $A = \left[\begin{smallmatrix}-3\\-2\end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix}1\\5\end{smallmatrix}\right]$.

  We can easily calculate that $B - A = \left[\begin{smallmatrix}3\\8\end{smallmatrix}\right]$. The most interesting thing to observe here is the fact that the vector which starts at A and ends at B is equal to the vector $B - A$.

  $$\vec{AB} = B - A$$



- Compute the distance between $X = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $Y = \begin{bmatrix}4\\-2\\1\end{bmatrix}$.

  If we write the distance between X and Y as $d(X, Y)$, then we have

  $$d(X, Y) = \|\vec{XY}\| = \|Y - X\|.$$

  Now we calculate $Y - X = [3, -4, -2]^T$ so $\|Y - X\|^2 = (Y - X) \cdot (Y - X) = 3^2 + (-4)^2 + (-2)^2 = 29$. Hence, we can conclude that $d(X, Y) = \sqrt{29}$.

- What is the distance $d(X, Y)$ between the vectors $X = [x_1, x_2, x_3, x_4]^T$ and $Y = [y_1, y_2, y_3, y_4]^T$ in $\mathbb{R}^4$?

  We're just going to do now with letters what we did in the previous exercise with numbers.

  $$d(X,Y)^2 = \|X - Y\|^2 = (X - Y) \cdot (X - Y) = \begin{bmatrix}x_1 - y_1\\x_2 - y_2\\x_3 - y_3\\x_4 - y_4\end{bmatrix} \cdot \begin{bmatrix}x_1 - y_1\\x_2 - y_2\\x_3 - y_3\\x_4 - y_4\end{bmatrix}$$
  $$= (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2$$

  Now just take the square root of both sides and you end up with

  $$d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2}.$$