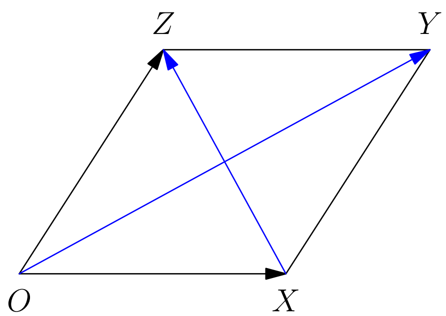
Today we covered Subsections 3.3.2 and 3.3.3 in the textbook.

Examples

- Show that the diagonals of a rhombus $OXYZ$ are perpendicular to each other.



Remember that a rhombus is a shape with four equal side lengths. A rhombus is always has opposite sides which are parallel. We want to show that XZ is perpendicular to OY and, of course, we're going to use the dot product to do it.

$$\vec{XZ} \cdot \vec{OY} = (Z - X) \cdot Y = (Z - X) \cdot (Z + X)$$
$$= Z \cdot Z - X \cdot X = \|Z\|^2 - \|X\|^2 = 0$$

- Determine when two lines $\ell, \ell' \subseteq \mathbb{R}^2$ given by the equations $\ell : y = Qx + b$ and $\ell' : y = Q'x + b'$ are parallel and when they are perpendicular.

  The corresponding parametric equations are $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} + t \begin{bmatrix} 1 \\ Q \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b' \end{bmatrix} + t \begin{bmatrix} 1 \\ Q' \end{bmatrix}$. This means that the direction vectors are $\begin{bmatrix} 1 \\ Q \end{bmatrix}$ and $\begin{bmatrix} 1 \\ Q' \end{bmatrix}$.

  So $\ell \| \ell' \Leftrightarrow \begin{bmatrix} 1 \\ Q \end{bmatrix} \| \begin{bmatrix} 1 \\ Q' \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 \\ Q \end{bmatrix} = s \begin{bmatrix} 1 \\ Q' \end{bmatrix}$ for some real number s. But this can only happen if $s = 1$, in which case $Q = Q'$.

  We also have $\ell \perp \ell' \Leftrightarrow \begin{bmatrix} 1 \\ Q \end{bmatrix} \perp \begin{bmatrix} 1 \\ Q' \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 \\ Q \end{bmatrix} \cdot \begin{bmatrix} 1 \\ Q' \end{bmatrix} = 0 \Leftrightarrow 1 + QQ' = 0 \Leftrightarrow QQ' = -1$.

Remark: In $\mathbb{R}^3$ (and also in $\mathbb{R}^n$ for $n > 3$) a line cannot be given by 1 equation. This is why we use the parametric equation to describe a line. Note that one linear equation in $\mathbb{R}^3$ produces a *plane*.

Planes

Example

Let $N = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and let $\pi = \{X \in \mathbb{R}^3 : X \cdot N = 0\}$. Prove that π is a linear subspace of $\mathbb{R}^3$.

To check that π is a subspace of $\mathbb{R}^3$, there are only three simply things to check.

- $O \cdot N = 0 \Rightarrow O \in \pi$
- $X \in \pi$ and $Y \in \pi \Rightarrow X \cdot N = 0$ and $Y \cdot N = 0 \Rightarrow (X + Y) \cdot N = X \cdot N + Y \cdot N = 0 \Rightarrow X + Y \in \pi$
- $X \in \pi$ and $t \in \mathbb{R} \Rightarrow (tX) \cdot N = t(X \cdot N) = tO = O \Rightarrow tX \in \pi$

Example

How do you find the equation of the plane π which is perpendicular to $N = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and passes through $X_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$? Note that π won't actually be a linear subspace of $\mathbb{R}^3$ any more.

Geometrically, $X \in \pi$ if and only if $X - X_0 \perp N$. So π is the set

$$\begin{aligned}&\{X \in \mathbb{R}^3 : (X - X_0) \cdot N = 0\}\\ =&\left\{ \begin{bmatrix} x\\y\\z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} x-1\\y-0\\z-2 \end{bmatrix} \cdot \begin{bmatrix} 1\\-2\\3 \end{bmatrix} = 0 \right\}\\ =&\{[x,y,z]^T \in \mathbb{R}^3 : 1(x-1) - 2y + 3(z-2) = 0\}.\end{aligned}$$

So in the end, we can write the equation for π as $x - 2y + 3z = 7$.

Definition

A non-zero vector perpendicular to a plane is called a *normal vector* for the plane.

Theorem

If the plane π in $\mathbb{R}^3$ is perpendicular to $N = \begin{bmatrix} a\\b\\c \end{bmatrix}$ and contains the point $X_0 = \begin{bmatrix} x_0\\y_0\\z_0 \end{bmatrix}$, then the equation which describes the points $X = \begin{bmatrix} x\\y\\z \end{bmatrix}$ lying on π is $(X - X_0) \cdot N = 0$. This can be written as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \text{or} \qquad ax + by + cz = k$$

for some real number k.

Example

Check whether or not the two planes

$$\begin{aligned}\pi_1 &: z = 2x - 3y + 1\\ \pi_2 &: z = 2x + y - 4\end{aligned}$$

are parallel.

Write the equations as $2x - 3y - z = -1$ and $2x + y - z = 4$. Then you can tell just by looking at the equations that the normal vectors are $\begin{bmatrix} 2\\-3\\-1 \end{bmatrix}$ and $\begin{bmatrix} 2\\1\\-1 \end{bmatrix}$. The fact that these two vectors aren't parallel to each other tells us that the two planes cannot be parallel.

Corollary

The two planes described by the equations

$$\begin{aligned}z &= ax + by + c\\ z &= a'x + b'y + c'\end{aligned}$$

are parallel if and only if $a = a'$ and $b = b'$.

Note that the parametric equation of a plane $\pi : 2x + 3y - 2z = 1$ can be easily obtained by taking $x = s$, $y = t$ and solving for $z = \frac{1}{2}(2x + 3y - 1) = \frac{1}{2}(2s + 3t - 1)$. This gives us

$$\begin{bmatrix} x\\y\\z \end{bmatrix} = \begin{bmatrix} s\\t\\s+\frac{3}{2}t-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0\\0\\-\frac{1}{2} \end{bmatrix} + s\begin{bmatrix} 1\\0\\1 \end{bmatrix} + t\begin{bmatrix} 0\\1\\\frac{3}{2} \end{bmatrix}.$$

There are two parameters s and t here and this is great, because a plane is a two dimensional object.

Examples

- Find a normal vector of $\pi : 7x - 4y + 8z = 14$.

  You can immediately see a normal vector by looking at the equation — the answer is $\begin{bmatrix} 7 \\ -4 \\ 8 \end{bmatrix}$.

- Find the equation of the plane which passes through $X_0 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$ and is parallel to the plane given by the equation $3x - 2y + z = 0$.

  The plane π which solves the question must have the equation $3x - 2y + z = k$ for some real number k. Since we need $X_0 \in \pi$, just plug in the coordinates of X_0 into this equation to determine what k must be.

  $$3 \cdot 4 - 2 \cdot 0 + 1 \cdot (-1) = k$$

  But this just means that $k = 11$, so the answer is $\pi : 3x - 2y + z = 11$.

- Is the line $\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 2t \\ 3+2t \end{bmatrix}$ contained in the plane $\pi : 4x + 2y = 5$? Are they parallel to each other?

  It's easy to see that ℓ isn't contained in π since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ lies on the line ℓ while it doesn't lie on the plane π.

  If ℓ is parallel to π, then a direction vector for ℓ will be perpendicular to a normal vector for π. (You should think carefully about why this is true — it should be obvious if you can visualize what is going on.) A direction vector for ℓ is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ while a normal vector for π is $\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. And these two are perpendicular to each other because

  $$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = (-1) \cdot 4 + 2 \cdot 2 + 2 \cdot 0 = 0.$$

  It follows that the line ℓ and the plane π are parallel to each other.