THE COMBINATORICS OF TETRAHEDRON INDEX RATIOS

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The tetrahedron index assigns a q-series to a pair of integers and is the building block for the physically motivated 3-manifold invariant known as the 3D index. Garoufalidis observed empirically that a particular ratio of tetrahedron indices yields a q-series whose coefficients are positive integers. We will prove this statement by showing that these integers enumerate combinatorial objects.

Tetrahedron index

Definition

$$\mathcal{I}_\Delta(m,e)=\sum_{n=0}^{\infty}(-1)^n\,\frac{q^{\frac{1}{2}n(n+1)-m(n+\frac{1}{2}e)}}{(q)_n\,(q)_{n+e}} \qquad \text{where } (q)_n=\prod_{i=1}^{n}(1-q^i)$$

Example

$$\mathcal{I}_\Delta(0,0)=1-q-2q^2-2q^3-2q^4+q^6+5q^7+7q^8+11q^9+13q^{10}+\cdots$$

$$\mathcal{I}_\Delta(0,1)=1-q^2-2q^3-3q^4-3q^5-3q^6-q^7+q^8+5q^9+9q^{10}+\cdots$$

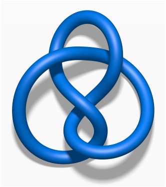
$$\mathcal{I}_\Delta(-1,0)=1-q^2-2q^3-3q^4-3q^5-3q^6-q^7+q^8+5q^9+9q^{10}+\cdots$$

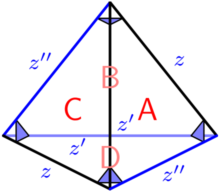
$$\mathcal{I}_\Delta(1,1)=-q^{\frac{3}{2}}(1+q+q^2-q^4-3q^5-5q^6-7q^7-9q^8-10q^9+\cdots)$$

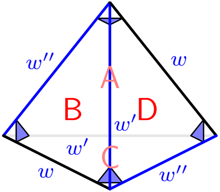
Properties

- Positivity. $\mathcal{I}_\Delta(m,e)\in\mathbb{Z}[[q^{\frac{1}{2}}]]$
- Linear recursion. $q^{\frac{e}{2}}\,\mathcal{I}_\Delta(m\pm 1,e)+q^{-\frac{m}{2}}\,\mathcal{I}_\Delta(m,e\pm 1)-\mathcal{I}_\Delta(m,e)=0$
- Duality. $\mathcal{I}_\Delta(m,e)=\mathcal{I}_\Delta(-e,-m)$
- Triality. $\mathcal{I}_\Delta(m,e)=(-q^{\frac{1}{2}})^{-e}\,\mathcal{I}_\Delta(e,-e-m)=(-q^{\frac{1}{2}})^{m}\,\mathcal{I}_\Delta(-e-m,m)$

Ideal triangulations of knot complements







- Let $\mathcal{T}$ be an ideal triangulation of $S^3 \setminus K$ with N tetrahedra (and hence N edges).
- Assign variables z_i, z_i', z_i'' to opposite edges of each tetrahedron.
- Neumann–Zagier matrices. Read off $N \times N$ matrices $\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}}$ with rows indexed by the N edges of $\mathcal{T}$ and columns indexed by the variables z_i, z_i', z_i''.
- Gluing equations. Use $\alpha + \beta + \gamma = (1, \dots, 1)^T$ to eliminate β from $\overline{\mathbf{A}}\alpha + \overline{\mathbf{B}}\beta + \overline{\mathbf{C}}\gamma = (2, \dots, 2)^T$ and obtain $\mathbf{A}\alpha + \mathbf{B}\gamma = \mathbf{v}$.
- Let the columns of $\mathbf{A}$ and $\mathbf{B}$, with the bottom entry removed, be $\mathbf{a}_i$ and $\mathbf{b}_i$.
- The "turning numbers" of the longitude and meridian give integers $p_1, \dots, p_N, q_1, \dots, q_N, r_1, \dots r_N, s_1, \dots, s_N$.

The 3D index

Definition

The 3D index $\mathcal{I}_{\mathcal{T}} : H_1(\partial\mathcal{T};\mathbb{Z}) \to \mathbb{Z}[[q^{\frac{1}{2}}]]$ is defined by

$$\mathcal{I}_{\mathcal{T}}(m,e) = \sum_{\mathbf{k}\in\mathbb{Z}^{N-1}} q^{\frac{1}{2}\mathbf{v}\cdot\mathbf{k}} \prod_{i=1}^{N} \mathcal{I}_{\Delta}(s_i m - q_i e - \mathbf{b}_i\cdot\mathbf{k}, p_i e - r_i m + \mathbf{a}_i\cdot\mathbf{k}).$$

- [DGG] If convergent, it "should depend" only on the underlying 3-manifold.
- [Garoufalidis] It converges if and only if the triangulation has an index structure.
- [Garoufalidis] It is invariant under 3-2 moves but not always under 2-3 moves.

Example

- $\mathcal{I}_{4_1}(m,e) = \sum_{k\in\mathbb{Z}} \mathcal{I}_{\Delta}(m-k, m+e-k)\mathcal{I}_{\Delta}(e-k,-k)$
- $\mathcal{I}_{4_1}(0,0) = 1 - 2q - 3q^2 + 2q^3 + 8q^4 + 18q^5 + 18q^6 + 14q^7 - 12q^8 - 52q^9 + \cdots$
- $\mathcal{I}_{3_1}(m,e) = \sum_{k\in\mathbb{Z}} (-q^{\frac{1}{2}})^{2k+m} \mathcal{I}_{\Delta}(e-2m,k)\mathcal{I}_{\Delta}(2m-e,k+m)$
- $\mathcal{I}_{3_1}(m,e) = \delta_{3m,e}$

Tetrahedron indices

Definition

A tetrahedron index is a function $f : \mathbb{Z}^2 \to \mathbb{Z}[[q^{\frac{1}{2}}]]$ that satisfies

- Positivity. $f(m,e) \in \mathbb{Z}[[q^{\frac{1}{2}}]]$;
- Linear recursion. $q^{\frac{e}{2}} f(m \pm 1, e) + q^{-\frac{m}{2}} f(m, e \pm 1) - f(m,e) = 0$.

Theorem (Garoufalidis)

The set of tetrahedron indices is a free q-holonomic $\mathbb{Z}[[q]]$-module of rank 1.

Proof.

- Use the linear recursion and `HolonomicFunctions.m` to show that

$$f(m, e+1) + (q^{e+\frac{m}{2}} - q^{-\frac{m}{2}} - q^{\frac{m}{2}}) f(m,e) + f(m, e-1) = 0.$$

  So $f(0,e)$ can be written in terms of $f(0,0)$ and $f(0,1)$.
- Use positivity to derive relations between coefficients of $f(0,0)$ and $f(0,1)$. □

Corollary

If f is a non-zero tetrahedron index, then $\dfrac{f(m,e)}{f(0,0)} = \dfrac{\mathcal{I}_\Delta(m,e)}{\mathcal{I}_\Delta(0,0)}$.

A remark of Garoufalidis

Remark 3.5. The proof of part (b) of Theorem 3.2 implies that if $f(m,e)$ is a tetrahedron index, then $f(m,e)$ is uniquely determined by $f(0,0) = \sum_{n=0}^{\infty} a_n q^n$. In particular, if $f(1,0) = \sum_{n=0}^{\infty} b_n q^n$, then b_n are $\mathbb{Z}$-linear combinations of a_k for $k \leq n$. For example, we have:

$$
\begin{aligned}
b_0 &= a_0 \\
b_1 &= a_0 + a_1 \\
b_2 &= 2a_0 + a_1 + a_2 \\
b_3 &= 4a_0 + 2a_1 + a_2 + a_3 \\
b_4 &= 9a_0 + 4a_1 + 2a_2 + a_3 + a_4 \\
b_5 &= 20a_0 + 9a_1 + 4a_2 + 2a_3 + a_4 + a_5 \\
b_6 &= 46a_0 + 20a_1 + 9a_2 + 4a_3 + 2a_4 + a_5 + a_6 \\
b_7 &= 105a_0 + 46a_1 + 20a_2 + 9a_3 + 4a_4 + 2a_5 + a_6 + a_7 \\
b_8 &= 242a_0 + 105a_1 + 46a_2 + 20a_3 + 9a_4 + 4a_5 + 2a_6 + a_7 + a_8 \\
b_9 &= 557a_0 + 242a_1 + 105a_2 + 46a_3 + 20a_4 + 9a_5 + 4a_6 + 2a_7 + a_8 + a_9 \\
b_{10} &= 1285a_0 + 557a_1 + 242a_2 + 105a_3 + 46a_4 + 20a_5 + 9a_6 + 4a_7 + 2a_8 + a_9 + a_{10} \\
b_{11} &= 2964a_0 + 1285a_1 + 557a_2 + 242a_3 + 105a_4 + 46a_5 + 20a_6 + 9a_7 + 4a_8 + 2a_9 + a_{10} + a_{11} \\
b_{12} &= 6842a_0 + 2964a_1 + 1285a_2 + 557a_3 + 242a_4 + 105a_5 + 46a_6 + 20a_7 + 9a_8 + 4a_9 + 2a_{10} \\
&\quad + a_{11} + a_{12}
\end{aligned}
$$

In fact, it appears that b_n is a $\mathbb{N}$-linear combination of a_k for $k \leq n$, although we do not know how to show this, nor do we know of a geometric significance of this experimental fact.

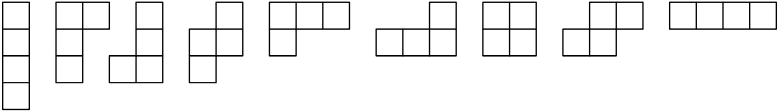
Skew Young diagrams

OEIS A006958: $1, 2, 4, 9, 20, 46, 105, 242, \ldots$

Number of connected skew Young diagrams with area n.

Definition

A connected skew Young diagram is a union of unit squares that lies between two N-E lattice paths that intersect only at their beginning and end.



Theorem (Bousquet-Mélou)

Let $\mathcal{SY}$ be the set of connected skew Young diagrams (excluding the empty one).

$$\frac{\mathcal{I}_\Delta(0,1)}{\mathcal{I}_\Delta(0,0)} = 1 + \sum_{D \in \mathcal{SY}} q^{area(D)}$$

Generatingfunctionology

Define $F(s) = 1 + \sum_{D \in \mathcal{SY}} q^{\text{area}(D)} s^{r(D)}$, where r is the height of the rightmost column.

Build diagrams by adding columns to the right.

Start with a connected skew Young diagram	$F(s)$
Add a square to the top-right	$sqF(1)$
Add squares above and below this square	$\frac{sq}{(1-sq)^2}F(1)$
Remove the offending diagrams	$\frac{sq}{(1-sq)^2}F(1) - \frac{sq}{(1-sq)^2}F(sq)$
RESULT: Skew Young diagrams with ≥ 2 columns	$F(s) - \frac{sq}{1-sq}$

Solving the equation

- We have

$$F(s) = a(s) + b(s)\,F(1) + c(s)\,F(sq),$$

where $a(s) = \frac{sq}{1-sq}$, $b(s) = \frac{sq}{(1-sq)^2}$, $c(s) = -\frac{sq}{(1-sq)^2}$.

$$F(s) = a(s) + b(s)\,F(1) + c(s)\,F(sq) \tag{1}$$

$$F(sq) = a(sq) + b(sq)\,F(1) + c(sq)\,F(sq^2) \tag{2}$$

$$F(sq^2) = a(sq^2) + b(sq^2)\,F(1) + c(sq^2)\,F(sq^3) \tag{3}$$

$$\vdots$$

- Substitute (2) into (1), then (3) into the result, then (4) into the result, etc.

$$F(s) = \sum_{n=0}^{\infty} c(s)\,c(sq)\cdots c(sq^{n-1})\,a(sq^n) + F(1)\sum_{n=0}^{\infty} c(s)\,c(sq)\cdots c(sq^{n-1})\,b(sq^n)$$

- Set $s = 1$ and rearrange.

$$F(1) = \frac{\sum_{n=0}^{\infty} c(1)\,c(q)\cdots c(q^{n-1})\,a(q^n)}{1 - \sum_{n=0}^{\infty} c(1)\,c(q)\cdots c(q^{n-1})\,b(q^n)} = \frac{\mathcal{I}_\Delta(0,1)}{\mathcal{I}_\Delta(0,0)} - 1$$

Final remarks

Theorem

$$\frac{\mathcal{I}_\Delta(0,e)}{\mathcal{I}_\Delta(0,0)} = 1 + \sum_{D\in\mathcal{SY}} \min(e, r(D)) \times q^{area(D)}$$

- In fact, the coefficients of $\dfrac{\mathcal{I}_\Delta(m_1,e_1)}{\mathcal{I}_\Delta(m_2,e_2)}$ are eventually positive, negative, or zero.
- Invariance under 2–3 Pachner moves relies on the pentagon identity.

$$\mathcal{I}_\Delta(m_1-e_2,e_1)\,\mathcal{I}_\Delta(m_2-e_1,e_2) = \sum_{e\in\mathbb{Z}} q^e\,\mathcal{I}_\Delta(m_1,e_1+e)\,\mathcal{I}_\Delta(m_2,e_2+e)\,\mathcal{I}_\Delta(m_1+m_2,e)$$

- Let $\hat{p}f(m,e) = q^{\frac{e}{2}} f(m+1,e)$ and $\hat{x}f(m,e) = q^{\frac{m}{2}} f(m,e-1)$, so $\hat{p}\hat{x} = q\hat{x}\hat{p}$.
  The linear recursion for $\mathcal{I}_\Delta$ can be written as $(\hat{p}+\hat{x}^{-1}-1)\,\mathcal{I}_\Delta = 0$.
- Recall that $\mathcal{I}_{\mathbf{3}_1}(m,e) = \delta_{3m,e}$, so it satisfies $(-\hat{p}+q^{\frac{3}{2}}\hat{x}^3)\,\mathcal{I}_{\mathbf{3}_1} = 0$.
  This is the Â-polynomial for the trefoil and this property "should hold" in general.

References

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- S. Garoufalidis. *The 3D index of an ideal triangulation and angle structures.* `arXiv:1208.1663`