THE GEOMETRY AND COMBINATORICS OF GROMOV-WITTEN THEORY

Melbourne University Algebra - Geometry - Topology Seminar (03/10/11)

Norm Do

GW THEORY: count maps from algebraic curves to a fixed target space, satisfying prescribed conditions

PARADIGM: Schubert calculus

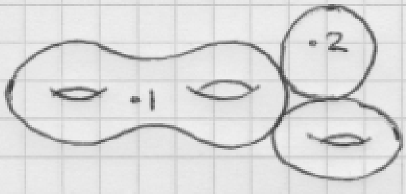
- count subspaces of $\mathbb{C}^n$, satisfying prescribed conditions
- geometry: construct a suitable moduli space $Gr_k(\mathbb{C}^n)$ and study $H^*(Gr)$ with respect to natural classes and maps
- combinatorics: symmetric functions

MODULI SPACES:

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable algebraic curves with} \\ n \text{ distinct labelled smooth points} \end{array} \right\}$$

"stable" = allow nodal singularities + finitely many automorphisms

e.g.



$\in \overline{\mathcal{M}}_{4,2}$

For X an algebraic variety and $\beta \in H_2(X)$, define the moduli space of maps

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \left\{ \begin{array}{l} f: C \to X \text{ a stable map from} \\ \text{a genus } g \text{ nodal curve with } n \\ \text{labelled points and } f_*([C]) = \beta \end{array} \right\}$$

WITTEN-KONTSEVICH:

Fact: $H^*(\overline{\mathcal{M}}_{g,n})$ is VERY complicated

Fix: study the tautological ring $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$ generated by certain natural cohomology classes and pushing / pulling via natural maps

Maps:

$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ forget a labelled point

$$\left.\begin{array}{l} gl_1 : \overline{\mathcal{M}}_{g,n+2} \longrightarrow \overline{\mathcal{M}}_{g+1,n} \\ gl_2 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2} \end{array}\right\} \text{glue two labelled points to create a node}$$

Classes:

$$\begin{array}{ccc} \mathcal{L}_i & & \text{cotangent line at } i^{\text{th}} \text{ labelled point} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & C \end{array}$$

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

Goal: Calculate $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}$, where

$$a_1 + a_2 + \cdots + a_n = 3g - 3 + n$$

Witten's conjecture: If you store these numbers in the right generating function, then it will satisfy the KdV hierarchy.

Proofs: Kontsevich, Mirzakhani, Okounkov - Pandharipande

GW THEORY OF $\mathbb{P}^1$:

For $\omega \in H^*(\mathbb{P}^1)$ dual to the point class, we want to calculate

$$GW_d(\tau_{a_1}(\omega), \ldots, \tau_{a_n}(\omega)) = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{\text{vir}}} \psi_1^{a_1} ev_1^*(\omega) \cdots \psi_n^{a_n} ev_n^*(\omega),$$

where $ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \to \mathbb{P}^1$

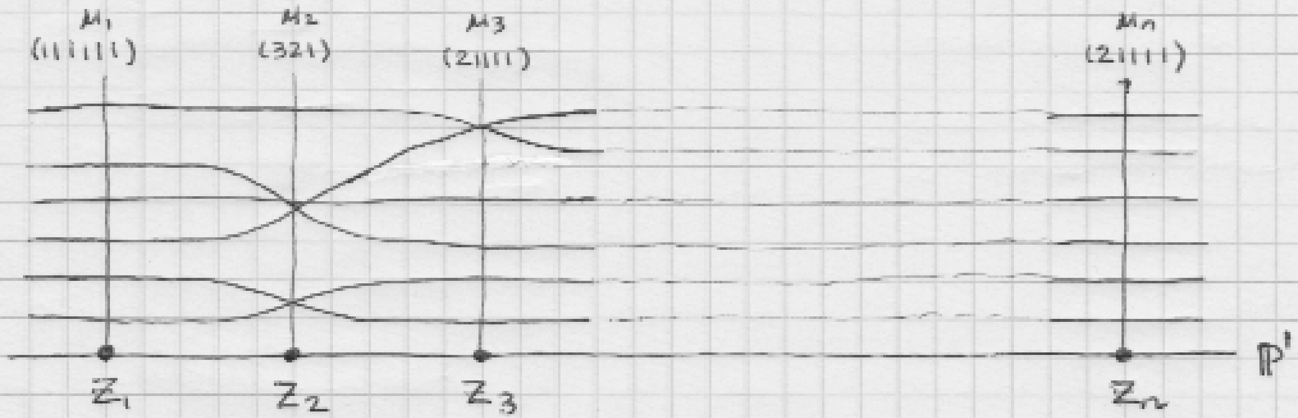
$$(f, C, p_1, p_2, \ldots, p_n) \longmapsto f(p_i)$$

and $a_1 + a_2 + \cdots + a_n = 2g - 2 + 2d$.

Geometrically, this counts degree d genus g branched covers of $\mathbb{P}^1$ with ramification order $a_i + 1$ at p_i.

HURWITZ THEORY:

Combinatorially counts branched covers of $\mathbb{P}^1$ via monodromy.



$$H^\bullet_d(\mu_1, \mu_2, \ldots, \mu_n) = \frac{1}{d!} \times \# \left\{ (\sigma_1, \sigma_2, \ldots, \sigma_n) \in (S_d)^n \;\middle|\; \begin{array}{l} \sigma_1\sigma_2 \cdots \sigma_n = id \\ \sigma_i \text{ has cycle type } \mu_i \end{array} \right\}$$

$$\overset{\text{(Burnside)}}{=} \sum_{|\lambda| = d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{i=1}^n f_{\mu_i}(\lambda)$$

where $f_\mu(\lambda) = |C_\mu| \frac{\chi^\lambda_\mu}{\dim \lambda}$

GW/H CORRESPONDENCE:

Morally, $GW_d(\tau_{a_1}(\omega), \ldots, \tau_{a_n}(\omega)) \sim H^\bullet_d((a_1+1), \ldots, (a_n+1))$

where (k) denotes the k-cycle in S_d which has cycle type $(k1\cdots1)$

- H overcounts by a combinatorial factor
- GW counts connected covers, H doesn't
- GW counts singular curves, H doesn't

Theorem (Okounkov - Pandharipande, 2006)

$$GW^\bullet_d(\tau_{a_1}(\omega), \ldots, \tau_{a_n}(\omega)) = \prod_{i=1}^n \frac{1}{a_i!} H^\bullet_d(\overline{(a_1+1)}, \ldots, \overline{(a_n+1)})$$

$$= \sum_{|\lambda| = d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{i=1}^n \frac{\mathbf{p}_{a_i+1}(\lambda)}{(a_i+1)!}$$

Remarks:

- The completed cycle $\overline{(k)}$ lies in $\mathbb{Z}\,\mathbb{C}[S_d]$ for all d.

  e.g. $\overline{(1)} = (1) - \frac{1}{24}(\cdot)$

  $\overline{(2)} = (2)$

  $\overline{(3)} = (3) + (1,1) + \frac{1}{12}(1) + \frac{7}{2880}(\cdot)$

- $p_k(\lambda) = \sum_{i=1}^{\infty} [(\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k] + (1 - 2^{-k})\,\zeta(-k)$

  is a power sum SHIFTED symmetric function on partitions

- Kerov and Olshanski showed that f_μ is also a shifted symmetric function

- Okounkov and Pandharipande found the coefficient of $\overline{(k)}$ using the Fock/infinite wedge space technology

- The GW/H correspondence holds when the target is a genus g curve — just change $\left(\frac{\dim \lambda}{d!}\right)^2$ to $\left(\frac{\dim \lambda}{d!}\right)^{2-2g}$.

- For $g=0$ targets: $\mathbb{P}^1 \rightsquigarrow$ integrability (Toda)

  For $g=1$ targets: $E \rightsquigarrow$ quasimodularity