LATTICE POINTS IN MODULI SPACES OF CURVES

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There appear to be only two essentially distinct ways to understand intersection numbers on moduli spaces of curves. The algebraic geometric techniques of localization and degeneration lead to relations with Hurwitz numbers while the hyperbolic geometric approach leads to relations with symplectic volumes. In this talk, we'll consider polynomials defined by Norbury which bridge the gap between these two pictures. These polynomials count lattice points in moduli spaces of curves and well see that some of their coefficients store interesting information.

6 March 2010

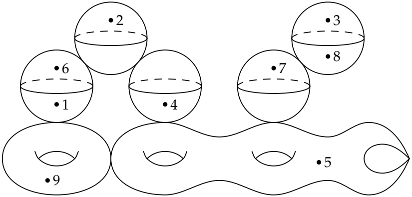
Moduli spaces of curves

- Moduli spaces of curves

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ smooth algebraic curves with distinct} \\ \text{points labelled from 1 up to } n \end{array} \right\}$$

- Deligne–Mumford compactification of moduli spaces of curves

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable algebraic curves with distinct} \\ \text{smooth points labelled from 1 up to } n \end{array} \right\}$$



- We need $2g - 2 + n > 0$, in which case $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

Ribbon graphs

Definition

A ribbon graph of type (g, n) is a graph such that

- every vertex has degree at least three;
- there is a cyclic ordering of the half-edges at every vertex; and
- the thickening of the graph is a genus g connected surface with boundary components labelled from 1 up to n.

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Example (A ribbon graph with genus 1 and 1 boundary)

A cell decomposition for moduli spaces of curves

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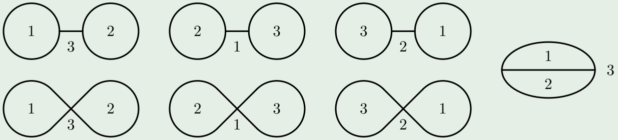

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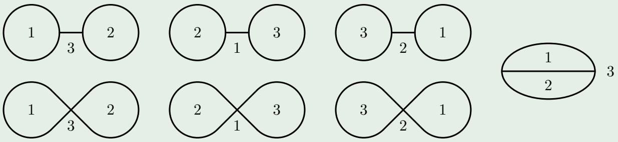


Theorem (Harer, Mumford, Penner, Thurston)

Let $MRG_{g,n}(\mathbf{b})$ be the space of metric ribbon graphs of type (g, n) with boundary lengths given by $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. Also, let $RG_{g,n}$ be the set of ribbon graphs of type (g, n). Then for all $\mathbf{b} \in \mathbb{R}_+^n$, we have

$$\mathcal{M}_{g,n} \cong MRG_{g,n}(\mathbf{b}) \cong \bigcup_{\Gamma \in RG_{g,n}} P_\Gamma(\mathbf{b}).$$

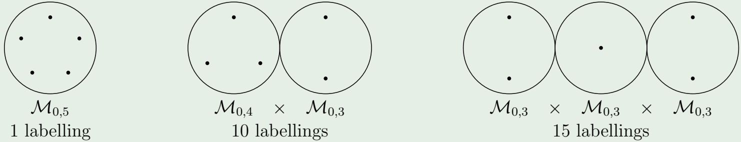
The set $P_\Gamma(\mathbf{b})$ consists of metric ribbon graphs in $MRG_{g,n}(\mathbf{b})$ whose underlying ribbon graph is Γ — this is a polytope of dimension $E(\Gamma) - n$.

A stratification for moduli spaces of curves

Points representing stable curves of given topology and labeling form strata.

Example (The stratification for $\overline{\mathcal{M}}_{0,5}$)

Points in $\overline{\mathcal{M}}_{0,5}$ represent curves of the following types.



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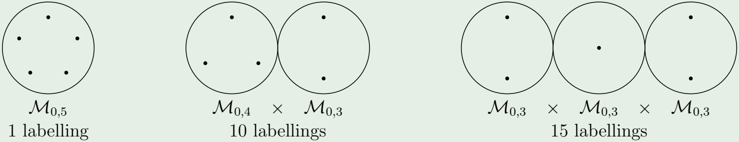

This lets us express $\overline{\mathcal{M}}_{0,5}$ as a disjoint union of strata, each one a product of uncompactified moduli spaces of curves.

$$\overline{\mathcal{M}}_{0,5} = \mathcal{M}_{0,5} \cup \left[\bigcup_{\text{10 copies}} \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} \right] \cup \left[\bigcup_{\text{15 copies}} \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \right]$$

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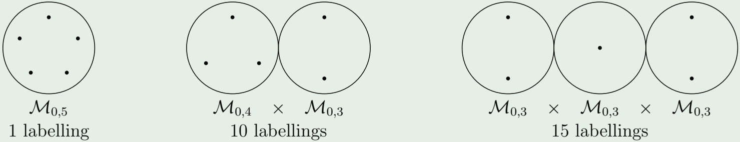

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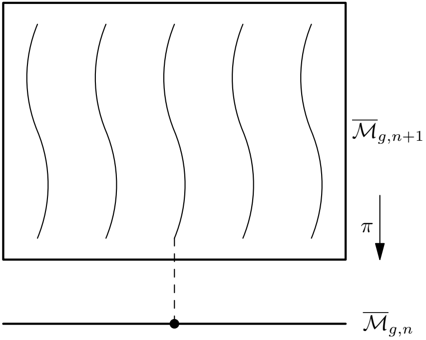
One consequence is the following Euler characteristic calculation.

$$\chi(\overline{\mathcal{M}}_{0,5}) = \chi(\mathcal{M}_{0,5}) + 10\chi(\mathcal{M}_{0,4}) \cdot \chi(\mathcal{M}_{0,3}) + 15\chi(\mathcal{M}_{0,3}) \cdot \chi(\mathcal{M}_{0,3}) \cdot \chi(\mathcal{M}_{0,3})$$



$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgets the point labelled $n+1$

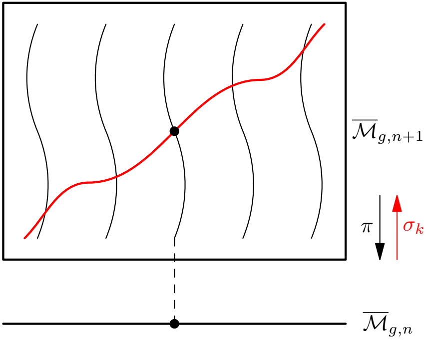
the fibre over a point in $\overline{\mathcal{M}}_{g,n}$ is the curve associated to that point

- For $k = 1, 2, \ldots, n$, define $\psi_k = c_1 [\sigma_k^* \mathcal{L}] \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$.
- For $|\boldsymbol{\alpha}| = 3g - 3 + n$, Witten considers the psi-class intersection number

$$\langle \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \in \mathbb{Q}.$$

- From these intersection numbers, you can conjecturally compute the tautological ring of $\overline{\mathcal{M}}_{g,n}$ — that is, *all classes you can easily think of.*

Intersection theory on moduli spaces of curves



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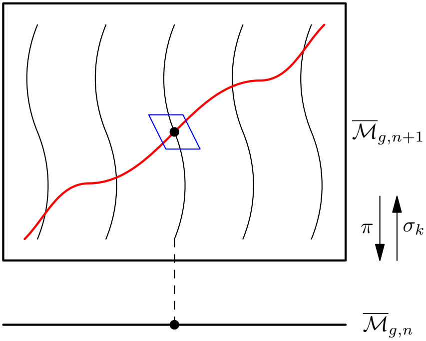

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$\mathcal{L}$ is the bundle consisting of cotangent lines to the fibres

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Witten's conjecture

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If we define the generating function

$$F(t_0, t_1, t_2, \dots) = \sum_{\mathbf{d}} \prod_{k=0}^{\infty} \frac{t_k^{d_k}}{d_k!} \langle \tau_0^{d_0} \tau_1^{d_1} \tau_2^{d_2} \cdots \rangle,$$

then $\frac{\partial^2 F}{\partial t_0^2}$ satisfies the KdV hierarchy. More explicitly, F satisfies the following partial differential equation for every non-negative integer n.

$$(2n+1)\frac{\partial^3 F}{\partial t_n \partial t_0^2} = \left(\frac{\partial^2 F}{\partial t_{n-1}\partial t_0}\right)\left(\frac{\partial^3 F}{\partial t_0^3}\right) + 2\left(\frac{\partial^3 F}{\partial t_{n-1}\partial t_0^2}\right)\left(\frac{\partial^2 F}{\partial t_0^2}\right) + \frac{1}{4}\frac{\partial^5 F}{\partial t_{n-1}\partial t_0^4}$$

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Using Witten's conjecture, you can calculate every psi-class intersection number from the base case $\langle \tau_0^3 \rangle = 1$ and the string equation.

$$\langle \tau_0 \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle = \sum_{k=1}^{n} \langle \tau_{\alpha_1} \cdots \tau_{\alpha_k - 1} \cdots \tau_{\alpha_n} \rangle$$

Kontsevich's proof

- Kontsevich used the combinatorial description of the moduli space and a volume calculation to prove the following result.

Theorem (Kontsevich's combinatorial formula)

If $TRG_{g,n}$ denotes the set of trivalent ribbon graphs of type (g,n), then

$$\sum_{|\boldsymbol{\alpha}|=3g-3+n} \langle \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle \prod_{k=1}^{n} \frac{(2\alpha_k - 1)!!}{s_k^{2\alpha_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|Aut\,\Gamma|} \prod_{e \in E(\Gamma)} \frac{1}{s_{\ell(e)} + s_{r(e)}}.$$

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- The ribbon graph enumeration on the RHS arises as a Hermitian matrix integral — this gives the desired connection to the KdV hierarchy.

Okounkov and Pandharipande's proof

- For $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, let $H_{g,\boldsymbol{\mu}}$ be the number of genus g branched covers of $\mathbb{P}^1$ with branching profile $\boldsymbol{\mu}$ over ∞ and simple branching over $r = 2g - 2 + |\boldsymbol{\mu}| + n$ given points — this is a Hurwitz number.

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- Using the Gromov–Witten theory of $\mathbb{P}^1$ and virtual localization, Okounkov and Pandharipande prove the ELSV formula.

$$H_{g,\boldsymbol{\mu}} = \frac{r!}{|\text{Aut }\boldsymbol{\mu}|} \prod_{k=1}^{n} \frac{\mu_k^{\mu_k}}{\mu_k!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{(1 - \mu_1\psi_1)(1 - \mu_2\psi_2)\cdots(1 - \mu_n\psi_n)}$$

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- Hurwitz numbers can also be formulated as the enumeration of certain maps on surfaces or certain factorizations in the symmetric group.

- The asymptotics of the map enumeration for $H_{g,\boldsymbol{\mu}}$ combined with the ELSV formula give Kontsevich's combinatorial formula.

Mirzakhani's proof

- Define $V_{g,n}(\mathbf{L})$ to be the Weil–Petersson volume of the moduli space of genus g hyperbolic surfaces with n geodesic boundaries of lengths $\mathbf{L} = (L_1, L_2, \ldots, L_n)$.

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- Mirzakhani used symplectic reduction to prove that $V_{g,n}(\mathbf{L})$ equals

$$\sum_{|\boldsymbol{\alpha}|+m=3g-3+n} \frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \kappa_1^m}{2^{|\boldsymbol{\alpha}|} \alpha_1! \alpha_2! \cdots \alpha_n! m!} L_1^{2\alpha_1} L_2^{2\alpha_2} \cdots L_n^{2\alpha_n}.$$

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- She then used hyperbolic geometry to prove the following recursion, where $S = \{2, 3, \ldots, n\}$.

$$\begin{aligned}
2\frac{\partial}{\partial L_1} L_1 V_{g,n}(L_1, L_S) &= \int_0^\infty \int_0^\infty xy\, H(x+y, L_1)\, V_{g-1,n+1}(x, y, L_S)\, dx\, dy \\
&+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \int_0^\infty \int_0^\infty xy\, H(x+y, L_1)\, V_{g_1,|I|+1}(x, L_I)\, V_{g_2,|J|+1}(y, L_J)\, dx\, dy \\
&+ \sum_{k=2}^{n} \int_0^\infty x[H(x, L_1 + L_k) + H(x, L_1 - L_k)]\, V_{g,n-1}(x, L_{S\setminus\{k\}})\, dx
\end{aligned}$$

More proofs of Witten's conjecture

There are now several proofs of Witten's conjecture — but they generally fall into one of two categories.

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- Kontsevich
- Mirzakhani

HURWITZ NUMBERS

- Okounkov and Pandharipande
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- Kazarian and Lando

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Idea

Count lattice points in moduli spaces of curves

- This gives a discrete version of Kontsevich and Mirzakhani's volume calculations.
- This gives a combinatorial problem which has a similar flavor to the enumeration of Hurwitz numbers.

Lattice points in moduli spaces of curves

- Recall the cell decomposition

$$\mathcal{M}_{g,n} \cong MRG_{g,n}(\mathbf{b}) \cong \bigcup_{\Gamma \in RG_{g,n}} P_\Gamma(\mathbf{b}).$$

- If $\mathbf{b} \in \mathbb{N}^n$, then $P_\Gamma(\mathbf{b}) \subseteq \mathbb{R}^{E(\Gamma)}$ is an integral polytope and we can count lattice points inside it.

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Definition

$$N_{g,n}(\mathbf{b}) = \sum_{\Gamma \in RG_{g,n}} \frac{\#\text{ lattice points in } P_\Gamma(\mathbf{b})}{|\text{Aut }\Gamma|}$$

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Definition

$$N_{g,n}(\mathbf{b}) = \sum_{\Gamma \in RG_{g,n}} \frac{\#\text{ lattice points in } P_\Gamma(\mathbf{b})}{|\text{Aut }\Gamma|}$$

- In other words, $N_{g,n}(\mathbf{b})$ counts metric ribbon graphs of type (g, n) with integer edge lengths and boundary lengths given by $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.
- Equivalently, $N_{g,n}(\mathbf{b})$ counts ribbon graphs of type (g, n) which are now allowed to have vertices of degree at least two.

A recursive formula for counting lattice points

Theorem

We have the following recursion for $N_{g,n}$, where $S = \{1, 2, \ldots, n\}$.

$$(b_1 + b_2 + \cdots + b_n)\, N_{g,n}(b_S) = \sum_{\substack{\{i,j\} \subseteq S \\ p+q=b_i+b_j}} pq\, N_{g,n-1}(p, b_{S\setminus\{i,j\}})$$

$$+ \sum_{\substack{i \in S \\ p+q+r=b_i}} pqr \left[N_{g-1,n+1}(p, q, b_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}} N_{g_1,|I|+1}(p, b_I) N_{g_2,|J|+1}(q, b_J) \right]$$

This is a symmetrized discrete version of Mirzakhani's recursion.

Lattice point polynomials

- We could use Ehrhart's theorem to deduce that $N_{g,n}$ is piecewise quasi-polynomial… but the recursive formula lets us say more.

Theorem

The lattice point count $N_{g,n}(b_1, b_2, \ldots, b_n)$ is a degree $6g - 6 + 2n$ even quasi-polynomial which depends on the parity of $b_1, b_2, \ldots, b_n$.

- There exist even polynomials $N_{g,n}^{(k)}$ for $k = 0, 1, 2, \ldots, n$ such that

$$N_{g,n}(b_1, b_2, \ldots, b_n) = N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$$

  whenever $b_1, b_2, \ldots, b_k$ are odd and $b_{k+1}, b_{k+2}, \ldots, b_n$ are even.

- By the handshaking lemma, we know that $N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n) = 0$ whenever $b_1 + b_2 + \cdots + b_n$ is odd.

Examples of lattice point polynomials

g	n	k	$N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$
0	3	0 or 2	1
1	1	0	$\frac{1}{48}(b_1^2 - 4)$
0	4	0 or 4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - 2)(b_1^2 + b_2^2 - 10)$
2	1	0	$\frac{1}{2^{16}.3^3.5}(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(5b_1^2 - 32)$
3	1	0	$\frac{1}{2^{25}.3^6.5^2.7}(5b_1^4 - 188b_1^2 + 1152)\prod_{k=1}^{5}(b_1^2 - 4k^2)$

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1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - 2)(b_1^2 + b_2^2 - 10)$
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Lemma

If $b_1 + b_2 + \cdots + b_n \leq 4g - 4 + 2n$, then $N_{g,n}(b_1, b_2, \ldots, b_n) = 0$.

Coefficients of lattice point polynomials

Theorem

- *If* $|\alpha| = 3g - 3 + n$*, then the coefficient of the top degree monomial* $b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}$ *in* $N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$ *for all even* k *is*

$$\frac{\langle \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle}{2^{5g-6+2n} \alpha_1! \alpha_2! \cdots \alpha_n!}.$$

- *The quasi-polynomial* $N_{g,n}$ *satisfies* $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$.

Coefficients of lattice point polynomials

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Proof.

- The lattice point count $N_{g,n}$ approximates the volume of the moduli space up to a constant factor. Kontsevich and Mirzakhani tell us that this volume stores psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- Consider the following meromorphic function and calculate its value at infinity in two distinct ways.

$$R_{g,n}(z) = \sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} N_{g,n}(b_1, b_2, \ldots, b_n)\, z^{b_1 + b_2 + \cdots + b_n}$$

□

Eynard–Orantin invariants are where matrix integrals meet enumerative geometry.

- INPUT: A Riemann surface C with two meromorphic functions x and y, where the ramification points of x have multiplicty two.
- OUTPUT: A meromorphic multilinear form $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ on C for each pair of non-negative integers (g, n).

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Eynard–Orantin invariants are where matrix integrals meet enumerative geometry.

- INPUT: A Riemann surface C with two meromorphic functions x and y, where the ramification points of x have multiplicty two.
- OUTPUT: A meromorphic multilinear form $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ on C for each pair of non-negative integers (g, n).
- RULE: Start with $\omega_{0,1} = 0$ and $\omega_{0,2} =$ Bergman kernel on C and then use the following recursion.

$$\omega_{g,n}(z_1, z_S) = \sum_m \operatorname*{Res}_{z \to a_m} K_m(z_1, z) \left[\omega_{g-1,n+1}(z, \bar{z}, z_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \omega_{g_1,|I|+1}(z, z_I)\omega_{g_2,|J|+1}(\bar{z}, z_J) \right]$$

$$\omega_{g,0} = \sum_m \operatorname*{Res}_{z \to a_m} \left(\int y\, dx \right) \frac{\omega_{g,1}(z)}{2g-2}$$

Lattice point polynomials and Eynard–Orantin invariants

Theorem

The Eynard–Orantin invariants of the spectral curve $xy - y^2 = 1$ *are given by*

$$\omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} N_{g,n}(b_1, b_2, \ldots, b_n) \prod_{k=1}^{n} b_k z_k^{b_k - 1} dz_k.$$

This shouldn't be surprising, since the recursions for the lattice point polynomials and the Eynard–Orantin invariants look similar.

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Corollary

The Eynard–Orantin invariants of the spectral curve $xy - y^2 = 1$ satisfy $\omega_{g,0} = \chi(\mathcal{M}_{g,0})$.

Idea

Try to count the number of lattice points $\overline{N}_{g,n}$ in the Deligne–Mumford compactification of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$.

Lattice points in compactified moduli spaces of curves

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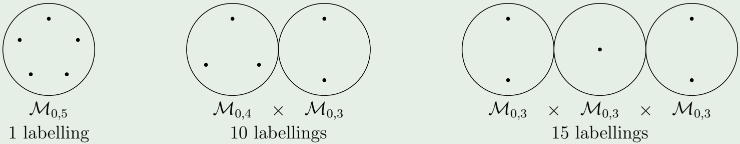
Example (Calculation of $\overline{N}_{0,5}$)



$$\begin{aligned}\overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) &= N_{0,5}(b_1, b_2, b_3, b_4, b_5) \\ &+ \sum_{\text{10 terms}} N_{0,4}(b_i, b_j, b_k, 0) \cdot N_{0,3}(b_\ell, b_m, 0) \\ &+ \sum_{\text{15 terms}} N_{0,3}(b_i, b_j, 0) \cdot N_{0,3}(b_k, 0, 0) \cdot N_{0,3}(b_\ell, b_m, 0)\end{aligned}$$

Compactified lattice point polynomials

Theorem

- *The lattice point count* $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ *is a degree* $6g - 6 + 2n$ *even quasi-polynomial which depends on the parity of* $b_1, b_2, \ldots, b_n$*.*
- *If* $|\alpha| = 3g - 3 + n$*, then the coefficient of the top degree monomial* $b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}$ *in* $\overline{N}^{(k)}_{g,n}(b_1, b_2, \ldots, b_n)$ *for all even k is*

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Proof.

- Use the algebra of quasi-polynomials.
- The only contribution to the top degree of $\overline{N}_{g,n}$ comes from $N_{g,n}$.
- This follows from the stratification of $\overline{\mathcal{M}}_{g,n}$, the definition of $\overline{N}_{g,n}$ and the fact that $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$. □

Examples of compactified lattice point polynomials

g	n	k	$\overline{N}^k_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	0 or 2	1
1	1	0	$\frac{1}{48}(b_1^2 + 20)$
0	4	0 or 4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2)$
1	2	0	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 48b_1^2 + 48b_2^2 + 192)$
1	2	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 48b_1^2 + 48b_2^2 + 84)$

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Claim

The compactified lattice point polynomials $\overline{N}_{g,n}$ seem to be the right things to look at (as opposed to $N_{g,n}$).

Questions

- *Are the coefficients of $\overline{N}_{g,n}$ always positive?*

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  We conjecture (and hope) that the answer is "yes".

- *The quasi-polynomials $N_{g,n}$ count certain factorizations in the symmetric group. What are the consequences of this viewpoint?*

  This should provide a link to characters of the symmetric group, τ-functions of integrable hierarchies and more.