DOMINOES, DIMERS AND DETERMINANTS

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If you take two squares and glue them together, then you will obtain a domino. If you take two atoms and join them by a bond, then you will obtain a dimer. And if you take first year linear algebra and attend a tutorial, then you will have to compute a determinant. In this seminar, we'll discuss some of the amazing mathematics connecting these three objects.

Dominoes on a checkerboard

Questions

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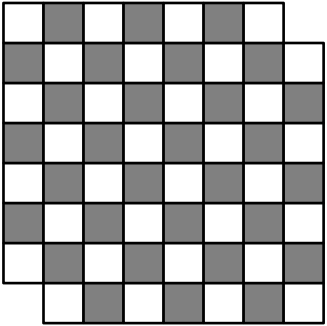

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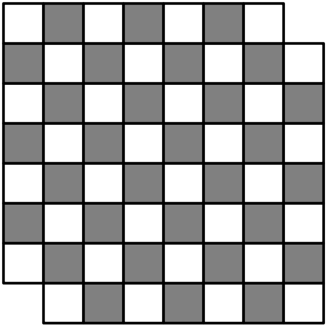

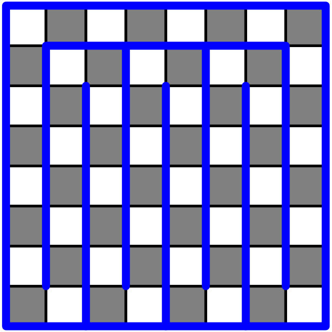

- Can you always tile the checkerboard if one white square and one black square are removed?

Dominoes and marriage

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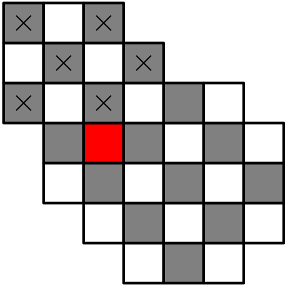
When can you tile a subset of a checkerboard with dominoes?



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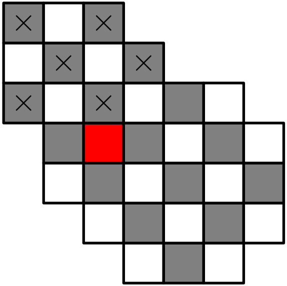

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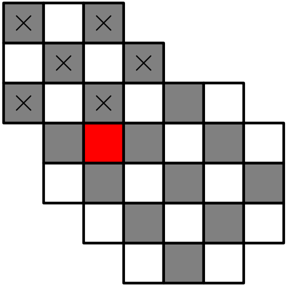

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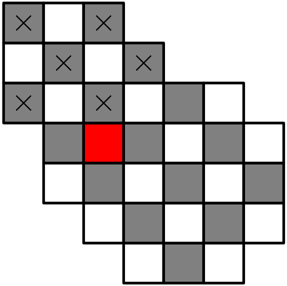

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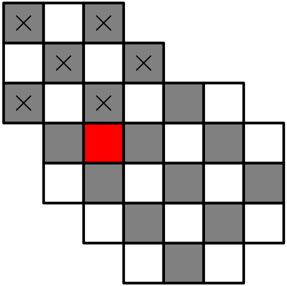

- Think of black squares as men and white squares as women.
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- We need gender balance... but we also need every group of men to have enough neighbours to marry.
- It turns out that these conditions are all you need.

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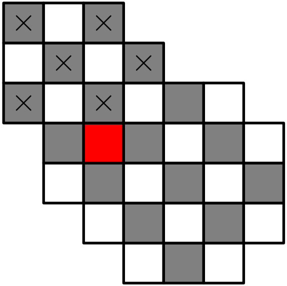

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- It turns out that these conditions are all you need.

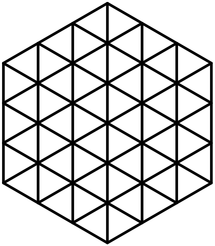
Hall's marriage theorem

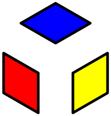
In any set of men and women, it is possible to marry off each man to a woman that he knows if and only if every group of men has enough acquaintances to marry.

The problem of the calissons

Question

A calisson is a parallelogram made by gluing together two equilateral triangles with side length 1. We want to tile a regular hexagon with side length n with calissons. Must there be an equal number of calissons in each orientation?





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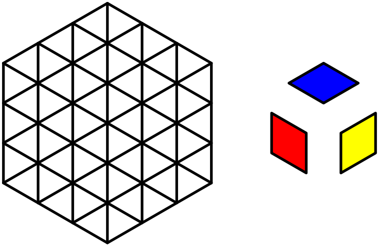

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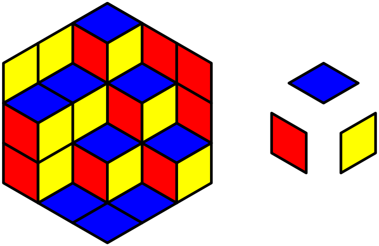

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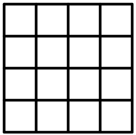
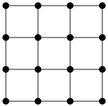
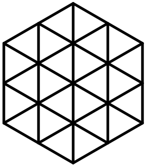
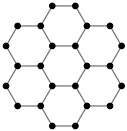
A primer on dimers

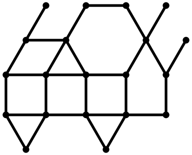
Tiling a checkerboard with dominoes or a hexagon with calissons are examples of dimer problems. A dimer covering of a graph is a collection of edges that covers every vertex once.











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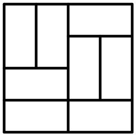

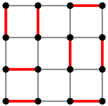

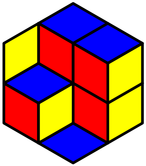

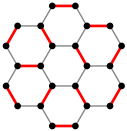

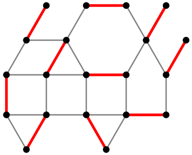

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n	# tilings of an $n \times n$ checkerboard	prime factorisation
0	1	1
2	2	2^1
4	36	$2^2.3^2$
6	6728	$2^3.29^2$
8	12988816	$2^4.17^2.53^2$
10	258584046368	$2^5.241^2.373^2$
12	53060477521960000	$2^6.5^4.31^2.53^2.701^2$
14	112202208776036178000000	$2^7.3^{10}.5^6.19^2.29^4.61^2$
16	24448887702508927958020791708 16	$2^8.101^2.5849^2.9929^2.16661^2$

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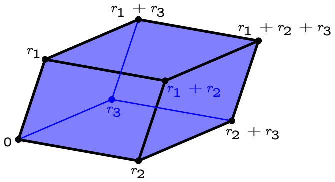
Surprisingly — or maybe unsurprisingly if you read the title of this seminar — you need to know what a determinant is to calculate the answer.

Discovering determinants

- The determinant of an $n \times n$ matrix M is the signed volume of the n-dimensional parallelepiped spanned by the row vectors of M.



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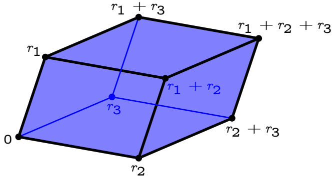

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$$\det M = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) M_{1,\sigma(1)} M_{2,\sigma(2)} \cdots M_{n,\sigma(n)}$$

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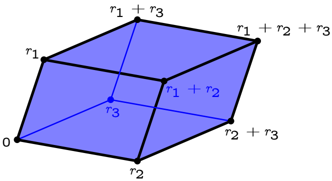

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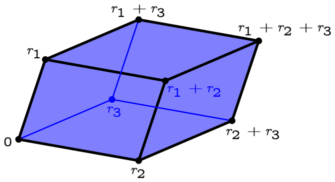

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Example

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \Rightarrow \det M = \begin{array}{l} +M_{11}M_{22}M_{33} - M_{11}M_{23}M_{32} - M_{12}M_{21}M_{33} \\ +M_{12}M_{23}M_{31} + M_{13}M_{21}M_{32} - M_{13}M_{22}M_{31} \end{array}$$

Kasteleyn's method

Let S be a subset of a checkerboard.

Construct the matrix K whose rows are labelled by the black squares and whose columns are labelled by the white squares. Let the entry corresponding to a black square and a white square be

- 1 if the squares are horizontally adjacent;
- i if the squares are vertically adjacent; and
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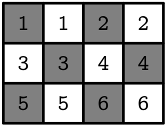

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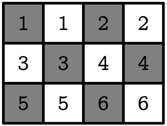

$$K = \begin{bmatrix} 1 & 0 & i & 0 & 0 & 0 \\ 1 & 1 & 0 & i & 0 & 0 \\ i & 0 & 1 & 1 & i & 0 \\ 0 & i & 0 & 1 & 0 & i \\ 0 & 0 & i & 0 & 1 & 0 \\ 0 & 0 & 0 & i & 1 & 1 \end{bmatrix}$$

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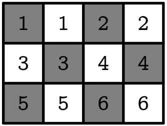

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Why does Kasteleyn's method work?

Example

- In the determinant calculation, the permutation σ is trying to "marry" black square 1 to white square $\sigma(1)$, black square 2 to white square $\sigma(2)$, and so on.

$$\det K = \sum_{\sigma \in S_6} \operatorname{sign}(\sigma)\ K_{1,\sigma(1)} K_{2,\sigma(2)} \cdots K_{6,\sigma(6)}$$

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Kasteleyn's method can count dimer coverings on any planar graph. You just need to be clever about the choice of "weight" attached to each edge.

Kasteleyn's method at work

Theorem (Fisher–Temperley and Kasteleyn, 1961)

The number of domino tilings of an $m \times n$ checkerboard is $\sqrt{|\triangle|}$, where

$$\triangle = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(2\cos\frac{j\pi}{m+1} + 2i\cos\frac{k\pi}{n+1} \right).$$

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Check that the eigenvalues of $M = \left[\begin{array}{c|c} 0 & K \\ \hline K^T & 0 \end{array}\right]$ are given by the equation

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Example

Consider the calculation for the regular 8×8 checkerboard.

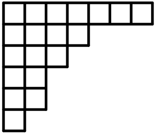
$$\begin{array}{cccc} \left(2\cos\frac{\pi}{9} + 2i\cos\frac{\pi}{9}\right) & \left(2\cos\frac{\pi}{9} + 2i\cos\frac{2\pi}{9}\right) & \cdots & \left(2\cos\frac{\pi}{9} + 2i\cos\frac{8\pi}{9}\right) \\ \left(2\cos\frac{2\pi}{9} + 2i\cos\frac{\pi}{9}\right) & \left(2\cos\frac{2\pi}{9} + 2i\cos\frac{2\pi}{9}\right) & \cdots & \left(2\cos\frac{2\pi}{9} + 2i\cos\frac{8\pi}{9}\right) \\ \vdots & \vdots & & \vdots \\ \left(2\cos\frac{8\pi}{9} + 2i\cos\frac{\pi}{9}\right) & \left(2\cos\frac{8\pi}{9} + 2i\cos\frac{2\pi}{9}\right) & \cdots & \left(2\cos\frac{8\pi}{9} + 2i\cos\frac{8\pi}{9}\right) \end{array} = 12988816^2$$

Partitions

A partition of n is a way to

- write it as a sum of positive integers where order doesn't matter; or
- a way to push n square boxes into the corner of a 2-D room.

$$19 = 6 + 5 + 3 + 2 + 1 + 1 + 1 \quad \longleftrightarrow$$



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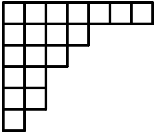
- a way to push n square boxes into the corner of a 2-D room.

$$19 = 6 + 5 + 3 + 2 + 1 + 1 + 1 \quad \longleftrightarrow$$



Theorem

The number of partitions of n is the coefficient of x^n in $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

Partitions

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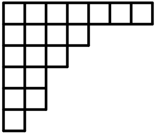

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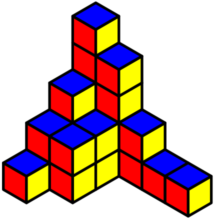
Proof

Write each term in the product as a geometric series and expand out the brackets.

$$\begin{array}{ccccccccccccc}
(1 & + & x^1 & + & x^2 & + & x^3 & + & x^4 & + & \cdots) & \times & \\
(1 & + & x^2 & + & x^4 & + & x^6 & + & x^8 & + & \cdots) & \times & \\
(1 & + & x^3 & + & x^6 & + & x^9 & + & x^{12} & + & \cdots) & \times & \\
(1 & + & x^4 & + & x^8 & + & x^{12} & + & x^{16} & + & \cdots) & \times & \cdots
\end{array}$$

Plane partitions

A plane partition of n is a way to push n cubic boxes into the corner of a 3-D room.



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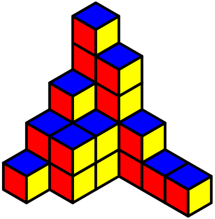

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Theorem (MacMahon, 1916)

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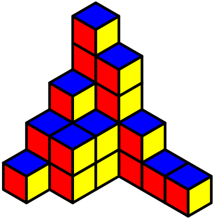

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Theorem (MacMahon, 1916)

- The number of plane partitions of n is the coefficient of x^n in $\prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}$.
- The number of plane partitions which fit inside an $a \times b \times c$ box is

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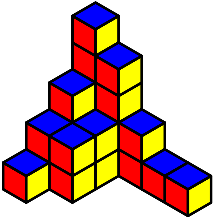

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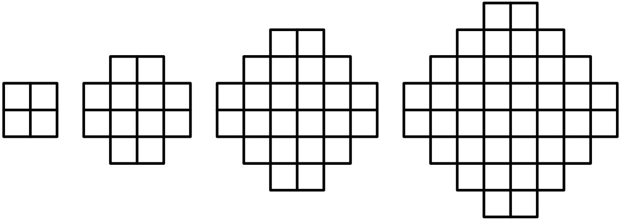
You can prove this using determinants via the "Gessel–Viennot trick".

Aztec diamonds

The Aztec diamond $AZ(n)$ is the nth term in the following sequence of shapes.



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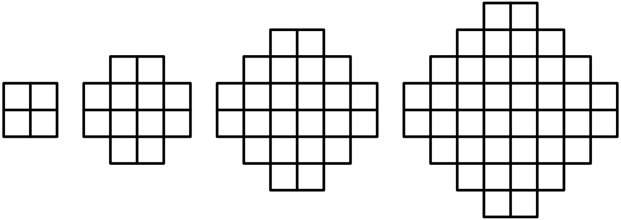

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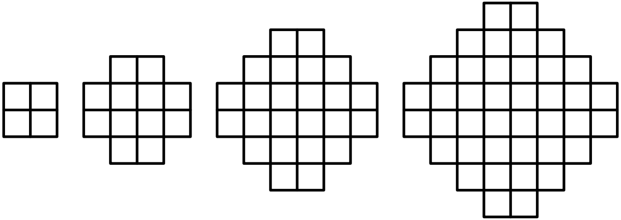

Question

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Answer (Elkies–Kuperberg–Larsen–Propp, 1992)

The number of ways to tile the Aztec diamond $AZ(n)$ is $2^{n(n+1)/2}$.

From Aztec diamonds to arctic circles

Colour the domino tiling of the Aztec diamond $AZ(n)$ according to

- whether the dominoes are horizontal or vertical; and
- whether the left/upper square of the domino lands on a black or white square.

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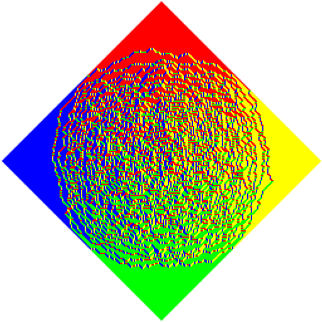

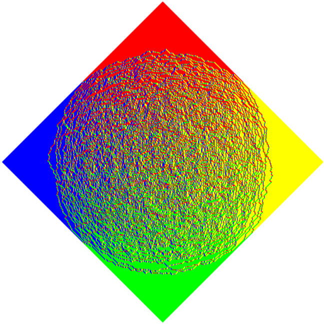

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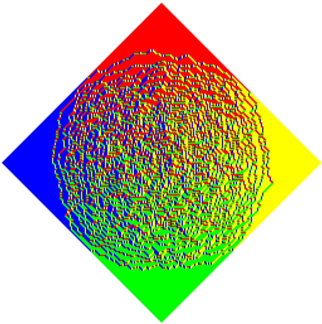

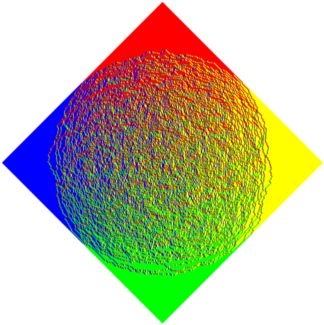

Arctic circle theorem (Jockusch–Propp–Shor, 1998)

In a tiling of the Aztec diamond, the ordered region is called frozen while the disordered region is called liquid. The frozen–liquid boundary of almost all domino tilings of the Aztec diamond $AZ(n)$ approaches a circle, as n approaches infinity.

Limit shapes

Theorem (Cohn–Kenyon–Propp, 2001)

Take a polygon which can be tiled with dominoes or calissons and consider tilings where the size of the tiles approaches zero. In the limit, a frozen–liquid boundary occurs and this limit shape is an algebraic curve — in other words, it can be described by a polynomial equation in two variables.

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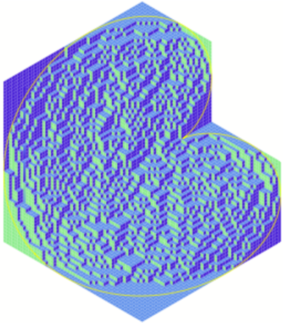

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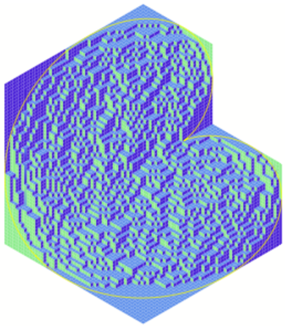

- One of the main ideas is to consider a tiling as a “random surface”.
- In 2007, Kenyon and Okounkov described the limit shape for these random surfaces using ideas from mathematical physics.

Thanks

If you would like more information, you can

- find the slides at `http://www.ms.unimelb.edu.au/~nndo`
- email me at `normdo@gmail.com`
- speak to me at `the front of the lecture theatre`