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Weil–Petersson volumes and cone surfaces

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Abstract The moduli spaces of hyperbolic surfaces of genus g with n geodesic boundary components are naturally symplectic manifolds. Mirzakhani proved that their volumes are polynomials in the lengths of the boundaries by computing the volumes recursively. In this paper, we give new recursion relations between the volume polynomials.

Keywords Moduli space · Hyperbolic surface · Intersection number

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1 Introduction

For $\mathbf{L} = (L_1, L_2, \ldots, L_n)$, a sequence of non-negative numbers, let $\mathcal{M}_{g,n}(\mathbf{L})$ be the moduli space of connected oriented genus g hyperbolic surfaces with n labeled boundary components of lengths $L_1, \ldots, L_n$. A cusp at a point naturally corresponds to a zero length boundary component. When $\mathbf{L} = 0$, that is there are n cusps, the moduli space $\mathcal{M}_{g,n}(\mathbf{0})$ is naturally identified with the moduli space of conformal structures on a genus g oriented surface with n labeled points, also known as the moduli space of curves with n labeled points. The identification uses the fact that in any conformal class of metrics there is a unique complete hyperbolic metric, and for every conformal automorphism there is a corresponding isometry.

On the moduli space, $\mathcal{M}_{g,n}(\mathbf{L})$ lives a natural symplectic form ω, defined precisely in Sect. 2. The volume of the moduli space is

$$V_{g,n}(\mathbf{L}) = \int_{\mathcal{M}_{g,n}(\mathbf{L})} \frac{\omega^{3g-3+n}}{(3g-3+n)!}, \quad (g, n) \neq (1, 1).$$

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When $(g, n) = (1, 1)$ we instead take half of the integral of ω, an orbifold volume,

$$V_{1,1}(L_1) = \frac{1}{2} \cdot \int_{\mathcal{M}_{1,1}(L_1)} \omega = \frac{1}{48}(L_1^2 + 4\pi^2),$$

which fits well with recursion relations between volumes, and relations with intersection numbers on the moduli space. Mirzakhani uses the true volume of $\mathcal{M}_{1,1}(L_1)$ in [1,2] and includes an extra factor of a half in her formulae.

Theorem 1 (Mirzakhani [1]) *$V_{g,n}(\mathbf{L})$ is a polynomial in $\mathbf{L} = (L_1, \dots, L_n)$. The coefficient of $L^\alpha = L^{\alpha_1}, \dots, L^{\alpha_n}$ lies in $\pi^{6g-6+2n-|\alpha|}\mathbb{Q}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.*

Mirzakhani proved this using a recursion relation between volumes of moduli spaces:

$$\frac{\partial}{\partial L_1}(L_1 V_{g,n}(\mathbf{L})) = \mathcal{A}_{g,n}(\mathbf{L}) + \mathcal{B}_{g,n}(\mathbf{L}), \tag{1}$$

where $\mathcal{A}_{g,n}(\mathbf{L})$ consists of integral transforms of $V_{g-1,n+1}$ and $V_{g_1,n_1} \times V_{g_2,n_2}$ for $g_1+g_2 = g$ and $n_1 + n_2 = n + 1$, and $\mathcal{B}_{g,n}(\mathbf{L})$ consists of integral transforms of $V_{g,n-1}$. We have omitted the $\mathbf{L}$ dependence in $V_{g-1,n+1}$, $V_{g_1,n_1} \times V_{g_2,n_2}$ and $V_{g,n-1}$ because it requires further explanation. See Sect. 2.2 for precise definitions of $\mathcal{A}_{g,n}(\mathbf{L})$ and $\mathcal{B}_{g,n}(\mathbf{L})$.

The main idea of this paper is to use intermediary moduli spaces to give new recursion relations between volumes of moduli spaces. The intermediary moduli spaces consist of hyperbolic surfaces with a cone point of a specified angle. Hyperbolic geometry is an ideal setting for studying cone points, although a cone point does make sense more generally in terms of a conformal structure on a Riemann surface. A cone angle of 0 corresponds to a cusp marked point and as the cone angle goes from 0 to 2π this corresponds, in a sense, to removing the marked point. This leads to interesting relations between the moduli spaces. These intermediary moduli spaces are reminiscent of the moduli spaces of anti self dual connections with cone singularities around an embedded surface in a four-manifold, used by Kronheimer and Mrowka [3] to get relationships between intersection numbers on instanton moduli spaces.

In [4], it is shown that one can interpret a point with cone angle in terms of an imaginary length boundary component. Explicitly, a cone angle ϕ appears by substituting the length $i\phi$ in the volume polynomial. Mirzakhani's results, Theorem 1 and (1) use a generalized McShane formula [5] on hyperbolic surfaces, which was adapted in [4] to allow a cone angle ϕ that ends up appearing as a length $i\phi$ in such a formula, and hence in the volume polynomial. We do not describe the generalized McShane formula in this paper, although in Sect. 2.2 we give the underlying idea in terms of coordinates on the hyperbolic surface.

Theorem 2 *For $\mathbf{L} = (L_1, \dots, L_n)$*

$$V_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} \int_0^{L_k} L_k V_{g,n}(\mathbf{L}) dL_k \tag{2}$$

and

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\mathbf{L}, 2\pi i) = 2\pi i(2g - 2 + n)V_{g,n}(\mathbf{L}). \tag{3}$$

We think of the theorem as describing the limit of the volume and its derivative when a cone angle tends to 2π, and hence is removable, although the statement of the theorem is independent of this interpretation. One can make sense of $n = 0$ in the proof of Theorem 2 yielding the following results.

Theorem 3 *When there is exactly one marked point the volume factorises*

$$V_{g,1}(L) = (L^2 + 4\pi^2)P_g(L). \tag{4}$$

The classical volumes of moduli spaces are encoded in Mirzakhani's volume polynomials

$$V_{g,0} = \frac{V'_{g,1}(2\pi i)}{2\pi i(2g-2)} = \frac{P_g(2\pi i)}{g-1}, \tag{5}$$

where the polynomial $P_g(L)$ is defined by (4).

The recursion relations (2) and (3) give information about the volume of the moduli space $\mathcal{M}_{g,n+1}(L_1, \ldots, L_{n+1})$ from the volume of the single lower dimensional moduli space $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$. This contrasts with Mirzakhani's relation (1) which uses many lower dimensional moduli spaces as described above. In particular, when $g = 0$ or $g = 1$ this give a simpler algorithm to determine $V_{g,n}(\mathbf{L})$.

Theorem 4 *The relation* (2) *uniquely determines $V_{0,n+1}(L_1, \ldots, L_{n+1})$ from the single volume $V_{0,n}(L_1, \ldots, L_n)$. Similarly, relations* (2) *and* (3) *uniquely determine $V_{1,n+1}(L_1, \ldots, L_{n+1})$ from $V_{1,n}(L_1, \ldots, L_n)$.*

There are three approaches to the proof of Theorem 2. First, the theorem necessarily follows from Mirzakhani's recursion relation (1) since that relation uniquely determines the polynomials. This approach, which is nontrivial and not so transparent, is treated in [6]. Second, the statement of the theorem is equivalent to relations between the coefficients of the volume polynomials which are intersection numbers of ψ classes and κ classes (see Sect. 3 for definitions), so relations between the latter can be used to deduce the theorem. The recursion relations generalize the string and dilaton equations proven by Witten in [7]. It is the approach that we present in this paper which also allows us to prove Theorem 3. The third approach is the most interesting. The theorem should follow from an analysis of the intermediary cone angle moduli spaces, and although we can only do this in simple cases, it is still a useful approach because it sheds light on the recursion relations between intersection numbers. Furthermore, it predicts that there are other recursion relations between intersection numbers on the moduli space.

2 Volume of the moduli space

2.1 Teichmüller space and Fenchel–Nielsen coordinates

Fix a smooth oriented surface $S_{g,n}$ of genus g and n boundary components labeled from 1 to n. Define a marked hyperbolic surface of type (g, n) and lengths $(L_1, \ldots, L_n)$ to be a pair (Σ, f), where Σ is an oriented hyperbolic surface with n geodesic boundaries of lengths $L_1, L_2, \ldots, L_n$ and $f : S_{g,n} \to \Sigma$ is an orientation preserving diffeomorphism. We call f the marking of the hyperbolic surface and define the Teichmüller space

$$\mathcal{T}_{g,n}(\mathbf{L}) = \{(\Sigma, f)\}/\sim,$$

where $(\Sigma_1, f_1) \sim (\Sigma_2, f_2)$ if there exists an isometry $\phi : \Sigma_1 \to \Sigma_2$ such that $\phi \circ f_1$ is isotopic to f_2.

Fenchel–Nielsen coordinates are global coordinates for Teichmüller space defined as follows. Choose a maximal set of disjoint embedded simple closed curves on the topological surface $S_{g,n}$. Their complement is a collection of genus zero surfaces each with three boundary components, i.e., pairs of pants. We call such a decomposition a *pants decomposition* of the surface $S_{g,n}$. Each pair of pants contributes Euler characteristic -1, so there are $2g-2+n=-\chi(\Sigma)$ pairs of pants in the decomposition, and hence $3g-3+n$ closed geodesics (not counting the boundary curves.) Also on $S_{g,n}$, choose a further disjoint collection of g embedded closed curves and n embedded arcs between boundary components equal to the union of $6g-6+3n$ arcs which give the *seams* of each pair of pants, i.e., each pair of pants contains three embedded arcs joining its boundary components pairwise.

A marking $f: S_{g,n} \to \Sigma$ of a hyperbolic surface with n geodesic boundary components Σ induces a pants decomposition on Σ from $S_{g,n}$. The isotopy class of embedded closed curves contains a collection $\{\gamma_1,\ldots,\gamma_{3g-3+n}\}$ of disjoint embedded simple closed geodesics which cuts Σ into hyperbolic pairs of pants with geodesic boundary components. Their lengths $l_1,\ldots,l_{3g-3+n}$ give half the Fenchel–Nielsen coordinates, and the other half are the twist coordinates $\theta_1,\ldots,\theta_{3g-3+n}$ which we now define. Any hyperbolic pair of pants contains three geodesic arcs giving the shortest paths between boundary components. An arc of a seam passing through γ_j is isotopic to the non-embedded piecewise geodesic arc given by the union of two shortest path geodesic arcs between boundary components of the two pairs of pants meeting along γ_j together with a (generally non-integral) multiple of γ_j. The length of this multiple of γ_j is denoted by $\theta_j \in \mathbb{R}$. (If $\theta_j \in [0,l_j)$ then the piecewise geodesic arc is embedded.)

The coordinates (l_j,θ_j) for $j=1,2,\ldots,3g-3+n$ give rise to an isomorphism

$$\mathcal{T}_{g,n}(\mathbf{L}) \cong (\mathbb{R}^+ \times \mathbb{R})^{3g-3+n}$$

and are canonical coordinates for a symplectic form

$$\omega = \sum_i dl_i \wedge d\theta_i. \tag{6}$$

It is a quite deep fact that the symplectic form is invariant under the action of the mapping class group $\mathrm{Mod}_{g,n}$, of isotopy classes of orientation preserving diffeomorphisms of the surface that preserve boundary components. The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{T}_{g,n}(\mathbf{L})$ is induced by its action on markings. There is a finite number of pants decomposition up to the action of the mapping class group, each class consisting of infinitely many geometrically different types. Thus once a topological pants decomposition of the surface is chosen a given hyperbolic surface has infinitely many geometrically different pants decompositions equivalent under $\mathrm{Mod}_{g,n}$. Each different decomposition gives different lengths and twist coordinates to the same hyperbolic surface, and hence different coordinates, whereas the symplectic form (6) depends only on the hyperbolic surface. Hence the symplectic form descends to the moduli space which is a quotient of Teichmüller space

$$\mathcal{M}_{g,n}(\mathbf{L}) \cong \mathcal{T}_{g,n}(\mathbf{L})/\mathrm{Mod}_{g,n}.$$

The volume of the moduli space $V_{g,n}(\mathbf{L})$ is defined to be the integral of the top-dimensional form $\omega^{3g-3+n}/(3g-3+n)!$ over $\mathcal{M}_{g,n}(\mathbf{L})$, or equivalently over a fundamental domain for $\mathrm{Mod}_{g,n}$ in $\mathcal{T}_{g,n}(\mathbf{L})$.

When the moduli space describes hyperbolic surfaces with a specified cone angle and geodesic boundary components then the above description of Teichmüller space via pants decompositions goes through if the cone angle is less than π. Mirzakhani's proof [1] that the volume is a polynomial generalizes using the results of [4] to show that the volume of the moduli space with cone angle ϕ is also a polynomial with $i\phi$ in place of length. If the cone angle is greater than π—and we are interested in the cone angle tending to 2π—then the pants decomposition does not always exist. The failure of this coordinate system suggests that one might instead use something like Penner coordinates [8] to define the volume and recapture the volume polynomial. We explicitly calculated the volume of $\mathcal{M}_{0,4}$ with three cusp points and one cone angle tending to 2π which indeed resulted in the volume polynomial obtained by analytically continuing the case of cone angle less than π.

2.2 Coordinates on a hyperbolic surface

It is useful to view a hyperbolic surface from one geodesic boundary component, say ∂_1, chosen from the n boundary components. The boundary component ∂_1 gives a coordinate system on the surface—to any point on the surface assign its distance from ∂_1 and the point on ∂_1 where the shortest geodesic meets. More generally, take any geodesic beginning at a given point on the surface and meeting ∂_1 perpendicularly, and assign to the point its length and the point it meets ∂_1. This makes the coordinate system locally smooth, at the cost of losing uniqueness for the coordinates of a point.

Mirzakhani uses this coordinate system in the following way. Project points onto the second coordinate, which takes its values in ∂_1. Now suppose that there is another boundary component, ∂_i say. The projection of ∂_i is an interval $I_i^0 \subset \partial_1$. More precisely, the projection is a collection of infinitely many disjoint intervals $\{I_i^j \mid j = 0, \ldots, \infty\}$ since we take any perpendicular geodesic, not just the shortest one, resulting in non-unique coordinates.

The sum of the lengths $f_i = \sum_j l(I_i^j)$ is a well-defined function on the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$. The length of a single interval $l(I_i^j)$ is well defined on Teichmüller space $\mathcal{T}_{g,n}(\mathbf{L})$, and although it does not descend to the moduli space, $l(I_i^j)$ descends to an intermediate moduli space:

$$\begin{array}{c} \mathcal{T}_{g,n}(\mathbf{L}) \\ \downarrow \\ \widehat{\mathcal{M}}_{g,n}(\mathbf{L}) \\ \downarrow \\ \mathcal{M}_{g,n}(\mathbf{L}) \end{array}$$

and Mirzakhani shows that this enables one to integrate the function $f_i = \sum_j l(I_j^i)$ over $\mathcal{M}_{g,n}(\mathbf{L})$ yielding a polynomial, calculable from $V_{g,n-1}$. The $n-1$ collections of intervals $\{I_i^j \mid j = 0, \ldots, \infty\}$, $i = 2, \ldots, n$, are disjoint from each other and Mirzakhani similarly shows that the complementary region (up to a measure zero set) gives a well-defined function f^c on the moduli space which can be integrated in terms of lower volumes. Since $f^c + \sum f_i = L_1$, the sum of all of the integrals gives

$$\int_{\mathcal{M}_{g,n}(\mathbf{L})} L_1 d\mathrm{vol} = L_1 V_{g,n}(\mathbf{L})$$

the derivative of which can be calculated and leads to Mirzakhani's recursion relation:

$$\frac{\partial}{\partial L_1}(L_1 V_{g,n}(\mathbf{L})) = \mathcal{A}_{g,n}(\mathbf{L}) + \mathcal{B}_{g,n}(\mathbf{L}).$$

For completeness we will define the right-hand side although this will not be used further in the paper. Put $\hat{\mathbf{L}} = (L_2, \ldots, L_n)$ and let $(L_2, \ldots, \hat{L}_j, \ldots, L_n)$ mean we remove L_j. Then

$$\mathcal{A}_{g,n}(\mathbf{L}) = \int K_{L_1}(x, y) V'_{g-1,n+1}(x, y, \hat{\mathbf{L}}) dx dy$$

where

$$V'_{g-1,n+1}(x, y, \hat{\mathbf{L}}) = V_{g-1,n+1}(x, y, \hat{\mathbf{L}}) + \sum_{g_i, n_i, \mathbf{L_i}} V_{g_1,n_1}(x, \mathbf{L_1}) \times V_{g_2,n_2}(y, \mathbf{L_2})$$

and the sum is over all $g_1 + g_2 = g$, $n_1 + n_2 = n + 1$ and $\mathbf{L_1} \sqcup \mathbf{L_2} = \hat{\mathbf{L}}$. And

$$\mathcal{B}_{g,n}(\mathbf{L}) = \sum_{j=2}^{n} \int K_{L_1, L_j}(x) V_{g,n-1}(x, L_2, \ldots, \hat{L}_j, \ldots, L_n) dx.$$

The kernels are defined by

$$K_{L_1}(x, y) = H(x + y, L_1), \quad K_{L_1, L_j}(x) = H(x, L_1 + L_j) + H(x, L_1 - L_j)$$

for

$$H(x, y) = \frac{1}{2}\left(\frac{1}{1 + e^{\frac{x+y}{2}}} + \frac{1}{1 + e^{\frac{x-y}{2}}}\right).$$

The derivation of these kernels comes from a detailed study of a hyperbolic pair of pants—the simplest hyperbolic surface. We refer the reader to [1,2] for full details.

3 Characteristic classes of surface bundles

3.1 Surface bundles

To any oriented topological surface bundle

$$\begin{array}{ccc} \Sigma_g & \hookrightarrow & X \\ & & \pi \downarrow \uparrow s_i \qquad i = 1, \ldots, n \\ & & B \end{array}$$

with n sections having disjoint images we can associate characteristic classes in $H^*(B)$, [9]. On X there is a complex line bundle $\gamma \to X$ with fibre at $b \in B$ the vertical cotangent bundle $T^*\pi^{-1}(b)$. A local trivialization is obtained from a local trivialization of the fibre bundle X. For each $i = 1, \ldots, n$ pull back the line bundle γ to $s_i^*\gamma = \gamma_i \to B$. Define

$$\psi_i = c_1(\gamma_i) \in H^2(B).$$

Let $e = c_1(\gamma) \in H^2(X)$. (We use the terminology e because it is naturally the Euler class of γ. We have put a complex structure on γ for convenience.) Define the Mumford–Morita–Miller classes

$$\tilde{\kappa}_m = \pi_! e^{m+1} \in H^{2m}(B),$$

where $\pi_! : H^k(X) \to H^{k-2}(B)$ is the umkehr map, or Gysin homomorphism, obtained by integrating along the (oriented) fibres. Alternatively, the umkehr map is obtained from the composition

$$\pi_! : H^k(X) \stackrel{\text{PD}}{\to} H_{d-k}(X) \stackrel{\pi_*}{\to} H_{d-k}(B) \stackrel{\text{PD}}{\to} H^{k-2}(B),$$

where $d = \dim X$ and PD denotes Poincare duality. The Mumford–Morita–Miller classes ignore the n sections s_i. Use instead the sequence

$$\pi_! : H_c^k(X - \cup s_i(B)) \stackrel{\text{PD}}{\to} H_{d-k}(X - \cup s_i(B)) \stackrel{\pi_*}{\to} H_{d-k}(B) \stackrel{\text{PD}}{\to} H^{k-2}(B),$$

where $H_c^k(X - \cup s_i(B))$ denotes cohomology with compact supports. Define the kappa classes

$$\kappa_m = \pi_! e_c^{m+1} \in H^{2m}(B),$$

where $e_c = e(\gamma) \in H_c^2(X - \cup_i s_i(B))$ is the Euler class with compact support. It has the property that on any fibre Σ

$$\langle e_c, \Sigma - \cup_i s_i(B)\rangle = -\chi(\Sigma - \cup_i s_i(B))$$

which generalizes $\langle e, \Sigma\rangle = -\chi(\Sigma)$. It is convenient to work with the compact manifold X and in place of e_c use its image in $H^2(X)$

$$H_c^2(X - \cup_i s_i(B)) \to H^2(X)$$
$$e_c \mapsto e_n = e + \sum_{i=1}^n \text{PD}[s_i(B)].$$

The expression for e_n is deduced from its two properties

$$\langle e_n, \Sigma\rangle = -\chi(\Sigma - \cup_i s_i(B)), \quad e_n \cdot \text{PD}[s_j(B)] = 0, \quad j = 1, \dots, n, \tag{7}$$

the first because it is defined by restriction, and the second because it lies in the kernel of the map $H^2(X) \to H^2(\cup_i s_i(B))$.

We will need relations between classes obtained by simply forgetting a section. Now

$$e_{n+1} = e_n + \text{PD}[s_{n+1}(B)]$$

so from $e_{n+1} \cdot \text{PD}[s_{n+1}(B)] = 0$ and $c = a + b \Rightarrow c^{m+1} = a^{m+1} + b\sum_{j=0}^m c^j a^{m-j}$

$$e_{n+1}^{m+1} = e_n^{m+1} + \text{PD}[s_{n+1}(B)] \cdot e_n^m$$

thus the forgetful map π_{n+1} induces $\pi_{n+1}^* : H^*(B) \to H^*(B)$ satisfying

$$\kappa_m = \pi_{n+1}^* \kappa_m + \psi_{n+1}^m \tag{8}$$

and the straightforward relation

$$\psi_j = \pi_{n+1}^* \psi_j, \quad j = 1, \dots, n. \tag{9}$$

To any Σ bundle $\pi : X \to B$ with n sections s_i, there corresponds the pull-back Σ bundle $\pi^* X \to X$ with n sections $\pi^* s_i$ and a further tautological section s_{n+1}. In some sense, the section s_{n+1} gives all possible ways to add an $(n+1)$st section to the bundle over B. In this context, the forgetful map has two interpretations. As the map $\pi_{n+1}^* : H^*(B) \to H^*(B)$ discussed above, and also as $\pi^* : H^*(B) \to H^*(X)$. The two are related by

$$s_{n+1}^* \circ \pi^* = \pi_{n+1}^*.$$

The pull-back relation (8) looks the same for π^*

$$\kappa_m = \pi^* \kappa_m + \psi_{n+1}^m \tag{8a}$$

whereas the relation (9) needs to be adjusted to

$$\psi_j = \pi^*\psi_j + \mathrm{PD}[s_j(B)], \quad j = 1, \ldots, n. \tag{9a}$$

We have yet to mention that the tautological section $s_{n+1} : X \to \pi^*X$ does not have disjoint image from the other sections π^*s_i. After blowing up to separate the images of the sections, we are naturally led to consider surface bundles $\pi : X \to B$ that allow fibres with mild singularities. More precisely, the singular fibres may be *stable curves*—they consist of a collection of smooth components meeting at nodal singularities with the property that each component has multiplicity 1, and negative Euler characteristic after we subtract all labeled points and common points with other components. We call X a bundle of stable curves or simply a bundle with singular fibres, although strictly it is no longer a fibre bundle. The cohomology classes ψ_i and κ_m extend to this situation. Their definitions are best understood when we put a continuous family of conformal structures on the fibres, or we assume the stronger property that X and B are complex analytic varieties. Define $\gamma = K_X \otimes \pi^*K_B^{-1}$, essentially the vertical canonical bundle (relative dualising sheaf.) This coincides with the definition above on smooth fibres and generalizes the definition to singular fibres. One can make sense of sections of this bundle along singular fibres in terms of meromorphic 1-forms with simple poles and conditions on residues [10] but we will not explain this here. The definitions of ψ_i and κ_m are as above. Relations (8a) and (9a) generalize to bundles of stable curves. Proofs can be found in [11] and [7].

A simple example will demonstrate the definitions and relations. Let X be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points $(0, 0)$, $(1, 1)$ and (∞, ∞). The map from X to the first $\mathbb{P}^1$ factor realizes X as a surface bundle

$$\begin{array}{ccc} \mathbb{P}^1 \hookrightarrow & X & \\ & \pi \downarrow \uparrow s_i & \quad i = 1, \ldots, 4 \\ & \mathbb{P}^1 & \end{array}$$

which we equip with four sections $s_1(z) = (z, 0)$, $s_2(z) = (z, 1)$, $s_3(z) = (z, \infty)$ and $s_4(z) = (z, z)$. The general fibre is genus 0 with four labeled points, and the singular fibres, at 0, 1 and ∞, are stable curves with two irreducible components each with two labeled points (and a common intersection point). We can generate $H_2(X)$ by H, F, E_1, E_2 and E_3, where E_i are the exceptional divisors of the blow-up and $H = \mathbb{P}^1 \times \{w\}$ and $F = \{z\} \times \mathbb{P}^1$ for any w and z different from 0, 1 and ∞. We use these curves to represent their divisor class, homology class and their Poincare dual cohomology class. Then

$$c_1(\gamma) = -2H + E_1 + E_2 + E_3 \quad \Rightarrow \quad \tilde{\kappa}_1 = c_1(\gamma)^2 = -3,$$
$$c_1\left(\gamma\left[\sum s_i(B)\right]\right) = 2H + F - E_1 - E_2 - E_3 \quad \Rightarrow \quad \kappa_1 = c_1\left(\gamma\left[\sum s_i(B)\right]\right)^2 = 1,$$
$$\psi_1 = c_1(\gamma) \cdot s_1(B) = c_1(\gamma) \cdot (H - E_1) = 1 = \psi_i, \quad i = 2, 3, 4.$$

Since X is the blow-up of the pull-back of the $\mathbb{P}^1$ bundle over a point with three sections, (8a) and (9a) are also evident.

3.2 Intersection numbers

Let us use $\mathcal{M}_{g,n}$ to notate the moduli space of genus g curves with n labeled points, which is isomorphic to the moduli space of genus g hyperbolic surfaces with n labeled cusps, $\mathcal{M}_{g,n}(\mathbf{L})$ with $\mathbf{L} = \mathbf{0}$, and $\overline{\mathcal{M}}_{g,n}$ the Deligne–Mumford compactification which adds stable curves to $\mathcal{M}_{g,n}$. Wolpert [12] showed that the symplectic structure ω on $\mathcal{M}_{g,n}$ extends to

$\overline{\mathcal{M}}_{g,n}$. The ψ_i and κ_m classes naturally live in $H^*(\overline{\mathcal{M}}_{g,n})$. They are associated to a universal surface bundle over $\overline{\mathcal{M}}_{g,n}$, essentially given by $\overline{\mathcal{M}}_{g,n+1}$ with map forgetting the last labeled point, and any bundle X equipped with conformal structures on fibres is the pull-back of the universal bundle under a map $B \to \overline{\mathcal{M}}_{g,n}$.

Theorem 5 (Mirzakhani) *The coefficient* C_α *of* $L_1^{2\alpha_1}, \ldots, L_n^{2\alpha_n}$ *in* $V_{g,n}(\mathbf{L})$ *is*

$$C_\alpha = \frac{1}{2^{|\alpha|}\alpha!(3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}, \ldots, \psi_n^{\alpha_n} \omega^{3g-3+n-|\alpha|}. \tag{10}$$

This is proven in [2] by showing that $\mathcal{M}_{g,n}(\mathbf{L})$ is the symplectic quotient of a larger symplectic manifold by a Hamiltonian T^n action, where a fixed value of the moment map corresponds to fixing the lengths $L_1, \ldots, L_n$ of the geodesic boundary components. Any such quotient is equipped with n line bundles coming from the T^n action, and their Chern classes are related to the coefficients of the volume polynomial. In [2] Mirzakhani used this together with her recursion relation for the volume polynomials to give a new proof of Witten's conjecture [7] regarding intersections of ψ classes on $\overline{\mathcal{M}}_{g,n}$. In the original proof of Witten's conjecture, Kontsevich [13] calculated the Laplace transform of the top degree terms of $V_{g,n}(\mathbf{L})$. It would be interesting to understand the Laplace transform of the whole polynomial $V_{g,n}(\mathbf{L})$.

In the following, write ψ^α for $\psi_1^{\alpha_1}, \ldots, \psi_n^{\alpha_n}$ and ignore the term if there is an $\alpha_j < 0$. For ease of reading, note that in all formulae the variable j sums from 0 to m while the variable k sums from 1 to n.

Lemma 1 *The equation*

$$V_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} \int_0^{L_k} L_k V_{g,n}(\mathbf{L}) dL_k$$

is equivalent to

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^j \kappa_1^{m-j} = \sum_{k=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}, \ldots, \psi_k^{\alpha_k - 1}, \ldots, \psi_n^{\alpha_n} \kappa_1^m \tag{11}$$

for all α *and* m.

Proof Assume that $|\alpha| + m = 3g - 2 + n$ since otherwise (11) is zero on both sides. By (10) and substitution of L_{n+1}^{2j} with $(2\pi i)^{2j}$, the coefficient of $L_1^{2\alpha_1}, \ldots, L_n^{2\alpha_n}$ in $V_{g,n+1}(\mathbf{L}, 2\pi i)$ is

$$\begin{aligned} &\sum_{j=0}^{m} \frac{(2\pi i)^{2j}}{2^{|\alpha|+j}\alpha! j! (m-j)!} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^j \omega^{m-j} \\ &= \sum_{j=0}^{m} \frac{(2\pi i)^{2j}}{2^{|\alpha|+j}\alpha! j! (m-j)!} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^j (2\pi^2\kappa_1)^{m-j} \\ &= \frac{2^{m-|\alpha|}\pi^{2m}}{\alpha!\, m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^j \kappa_1^{m-j} \end{aligned}$$

where we have used the identity $\omega = 2\pi^2\kappa_1$ proven in [14].

The coefficient of $L_1^{2\alpha_1}, \ldots, L_n^{2\alpha_n}$ in $\int_0^{L_k} L_k V_{g,n}(\mathbf{L}) dL_k$ is

$$\frac{\alpha_k}{2^{|\alpha|-1}(2\alpha_k)\alpha! m!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}, \ldots, \psi_k^{\alpha_k - 1}, \ldots, \psi_n^{\alpha_n} \omega^m$$
$$= \frac{2^{m-|\alpha|}\pi^{2m}}{\alpha!\, m!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}, \ldots, \psi_k^{\alpha_k - 1}, \ldots, \psi_n^{\alpha_n} \kappa_1^m.$$

Add this expression over $k = 1, \ldots, n$ and divide both sides by the factor $2^{m-|\alpha|}\pi^{2m}/\alpha! m!$ to prove the lemma. □

Lemma 2 *The equation*

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\mathbf{L}, 2\pi i) = 2\pi i(2g - 2 + n)V_{g,n}(\mathbf{L})$$

is equivalent to

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^{j+1} \kappa_1^{m-j} = (2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} \psi^\alpha \kappa_1^m. \tag{12}$$

Proof The proof is much like the proof of the previous lemma. The coefficient of $L_1^{2\alpha_1}, \ldots, L_n^{2\alpha_n}$ in $\partial V_{g,n+1}/\partial L_{n+1}(\mathbf{L}, 2\pi i)$ is

$$\sum_{j=0}^{m} \frac{(2j+2)(2\pi i)^{2j+1}}{2^{|\alpha|+j+1}\alpha!(j+1)!(m-j)!} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^{j+1} \omega^{m-j}$$
$$= 2\pi i \frac{2^{m-|\alpha|}\pi^{2m}}{\alpha!\, m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi^\alpha \psi_{n+1}^{j+1} \kappa_1^{m-j}$$

and the coefficient of $L_1^{2\alpha_1}, \ldots, L_n^{2\alpha_n}$ in $V_{g,n}(\mathbf{L})$ is

$$\frac{2^{m-|\alpha|}\pi^{2m}}{\alpha!\, m!} \int_{\overline{\mathcal{M}}_{g,n}} \psi^\alpha \kappa_1^m$$

so the equivalence follows. □

Completion of the proof of Theorem 2 It suffices to prove the relations (11) and (12). Notice that when $m = 0$ (11) and (12) are, respectively, the string and dilaton equations which were proven by Witten in [7]. The method of proof for the more general identities is similar.

Let $\pi : X \to B$ be a bundle of stable curves with n disjoint sections and π^*X the pull-back bundle with $n+1$ sections. Blow up π^*X along the intersections of images of sections to get a bundle of stable curves over X with $n+1$ disjoint sections $s_1, \ldots, s_{n+1}$. Our aim is to compare ψ and κ classes in $H^*(X)$ and $H^*(B)$.

Take the integrand of the left-hand side of (11) and consider its image under the umkehr map.

$$\begin{aligned}\pi_! \left\{ \sum_{j=0}^{m} (-1)^j \binom{m}{j} \psi_{n+1}^{j} \kappa_1^{m-j} \prod_{k=1}^{n} \psi_k^{\alpha_k} \right\} &= \pi_! \left\{ (\kappa_1 - \psi_{n+1})^m \prod_{k=1}^{n} \psi_k^{\alpha_k} \right\} \\ &= \pi_! \left\{ (\pi^* \kappa_1^m) \prod_{k=1}^{n} \left(\pi^* \psi_k^{\alpha_k} + \mathrm{PD}[s_k(B)] \cdot \pi^* \psi_k^{\alpha_k - 1} \right) \right\} \\ &= \kappa_1^m \sum_{k=1}^{n} \psi_1^{\alpha_1}, \ldots, \psi_k^{\alpha_k - 1}, \ldots, \psi_n^{\alpha_n}.\end{aligned}$$

To get from the first line to the second line we have used the pull-back formulae (8a) and (9a) and the fact that π^* is a ring homomorphism, so in particular $(\pi^*\eta)^m = \pi^*(\eta^m)$. To get from the second line to the third line we have used the fact that $\pi_! : H^*(X) \to H^*(B)$ is an $H^*(B)$ module homomorphism, i.e. $\pi_!(\xi \pi^* \eta) = \pi_!(\xi)\eta$, together with the explicit evaluations

$$\pi_!(1) = 0, \quad \pi_!(s_i(B)) = 1$$

most easily calculated from the Poincare duality description of $\pi_!$. Thus, in the product the image under $\pi_!$ of the highest degree term $\pi^*(\kappa_1^m \psi^\alpha)$ is zero, the image of the second highest degree term constitutes the expression in the third line, and the lower order terms vanish since they contain products $\mathrm{PD}[s_j(B)] \cdot \mathrm{PD}[s_k(B)] = 0$ because the images of s_j and s_k are disjoint.

Since

$$\int_X \eta = \int_B \pi_! \eta$$

choose $X = \overline{\mathcal{M}}_{g,n+1}$ and $B = \overline{\mathcal{M}}_{g,n}$, so (11) follows.

The proof of (12) is similar. Again apply the umkehr map to the integrand of the left-hand side of (12).

$$\begin{aligned}\pi_! \left\{ \sum_{j=0}^{m} (-1)^j \binom{m}{j} \psi_{n+1}^{j+1} \kappa_1^{m-j} \prod_{k=1}^{n} \psi_k^{\alpha_k} \right\} &= \pi_! \left\{ \psi_{n+1} \cdot (\kappa_1 - \psi_{n+1})^m \prod_{k=1}^{n} \psi_k^{\alpha_k} \right\} \\ &= \pi_! \left\{ \psi_{n+1} \cdot (\pi^* \kappa_1^m) \prod_{k=1}^{n} \left(\pi^* \psi_k^{\alpha_k} + \mathrm{PD}[s_k(B)] \cdot \pi^* \psi_k^{\alpha_k - 1} \right) \right\} \\ &= (2g-2+n) \kappa_1^m \prod_{k=1}^{n} \psi_k^{\alpha_k}.\end{aligned}$$

To go from the second line to the third line note that ψ_{n+1} coincides with the twisted Euler class e_{n+1} that satisfies (7) and hence $\pi_! \psi_{n+1} = 2g - 2 + n$ and $\psi_{n+1} \cdot \mathrm{PD}[s_k(B)] = 0$. Thus the top degree term constitutes the expression in the third line, and all lower degree terms vanish. ◻

Equations 2 and 3 suggest that a direct analysis of the moduli space of cone surfaces with cone angle $\theta \approx 2\pi$, or more accurately an infinitesimal analysis near 2π, gives rise to intriguing phenomena. Equation 3 seems plausible since the removed cone point is free to wander around each hyperbolic surface with area $2\pi(2g - 2 + n)$, so the change in volume is related to integrating over the smaller moduli space and along each fibre. Intuition for Eq. 2 seems less obvious.

Proof of Theorem 3 The first intersection number identity in the proof of Theorem 2 applies to $n = 0$ yielding

$$\sum_{j=0}^{3g-2} (-1)^j \binom{3g-2}{j} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^j \kappa_1^{3g-2-j} = 0. \tag{13}$$

Following the proof of Lemma 1, (13) is equivalent to the equation $V_{g,1}(2\pi i) = 0$ and hence the polynomial factorises into $V_{g,1}(L) = (L^2 + 4\pi^2)P_g(L)$ for some polynomial $P_g(L)$, proving (4).

The second intersection number identity in the proof of Theorem 2 applies to $n = 0$ to prove

$$\sum_{j=0}^{3g-3} (-1)^j \binom{3g-3}{j} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{j+1} \kappa_1^{3g-3-j} = (2g-2) \int_{\overline{\mathcal{M}}_{g,0}} \kappa_1^{3g-3} \tag{14}$$

and again as in Lemma 2 (14) is equivalent to the equation

$$V'_{g,1}(2\pi i) = 2\pi i(2g-2)V_{g,0}$$

thus expressing $V_{g,0}$ in terms of Mirzakhani's volumes. Since $V_{g,1}(L)$ vanishes at $L = 2\pi i$ we can also write the derivative as follows:

$$\begin{aligned} 2\pi i(2g-2)V_{g,0} = \left.\frac{dV_{g,1}}{dL}\right|_{L=2\pi i} &= \lim_{L\to 2\pi i} \frac{V_{g,1}(L)}{L - 2\pi i} \\ &= \lim_{L\to 2\pi i} \frac{4\pi i V_{g,1}(L)}{L^2 + 4\pi^2} = 4\pi i P_g(2\pi i) \end{aligned}$$

completing the proof of Theorem 3. □

4 Use of recursion relations

4.1 Low genus calculations

Proof of Theorem 4 The volume $V_{0,n+1}(L_1, \ldots, L_{n+1})$ is a degree $n-2$ symmetric polynomial in $L_1^2, \ldots, L_{n+1}^2$ and we need to show it is uniquely determined by evaluation at $L_{n+1} = 2\pi i$, since this is determined by $V_{0,n}(L_1, \ldots, L_n)$ via (2). This follows from the elementary fact that a symmetric polynomial $f(x_1, \ldots, x_n)$ of degree less than n is uniquely determined by evaluation of one variable at any $a \in \mathbb{C}$, $f(x_1, \ldots, x_{n-1}, a)$. To see this, suppose otherwise. Any symmetric $g(x_1, \ldots, x_n)$ of degree less than n that evaluates at a as f does, satisfies

$$\begin{aligned} f(x_1, \ldots, x_{n-1}, a) - g(x_1, \ldots, x_{n-1}, a) &= (x_n - a)P(x_1, \ldots, x_n) \\ &= Q(x_1, \ldots, x_n) \prod_{j=1}^{n} (x_j - a) \end{aligned}$$

but the degree is less than n so the difference is identically 0.

The proof for genus 1 is similar. The degree of $V_{1,n+1}$ as a polynomial in $L_1^2, \ldots, L_{n+1}^2$ is equal to $n+1$ so the proof of the genus 0 case shows that (2) determines $V_{1,n+1}$ from $V_{1,n}$

up to the constant c in $V_{1,n+1} + c\prod_{j=1}^{n}(L_j^2 + 4\pi^2)$. Now use (3) to determine c, and hence $V_{1,n+1}$.

Theorem 4 can be converted to an algorithm for calculating $V_{0,n}(\mathbf{L})$. The algorithm using (2) turns out to be much more efficient than the algorithm coming from Mirzakhani's relation (1) in genus 0, which needs $V_{0,n-1}$ and pairs V_{0,n_1}, V_{0,n_2} for all $n_1 + n_2 = n + 1$, to produce $V_{0,n}$. We have included a simple MAPLE routine in the appendix for calculating $V_{0,n}$ using (2). (The notion of a "more efficient" algorithm is not so precise here. We have merely compared the speeds of different calculations on MAPLE.)

In genus 0, the string equation—(11) with $m = 0$—leads to an explicit formula for the top coefficients, or equivalently the following formula for genus 0 intersection numbers without kappa classes:

$$\int\limits_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}, \ldots, \psi_n^{\alpha_n} = \binom{n-3}{\alpha_1, \ldots, \alpha_n}.$$

It seems reasonable to guess that when $g = 0$ (11) might be used to get an explicit combinatorial description of all genus 0 intersection numbers with powers of κ_1, or equivalently all coefficients of $V_{0,n}(\mathbf{L})$. Zograf [15] has recursion relations between the constant coefficients $V_{0,n}(\mathbf{0})$.

4.2 Higher derivatives

We expect to have expressions for higher derivatives $\partial^k V_{g,n+1}/\partial L_{n+1}^k$ evaluated at $L_{n+1} = 2\pi i$. Evidence comes from the fact that (2) and (3) use generalised versions of the string and dilaton equations. The Virasoro relations are a sequence of relations for the top degree terms of $V_{g,n}(\mathbf{L})$, with first two relations in the sequence the string and dilaton equations, so may also have versions in terms of evaluations of derivatives of the volume polynomial at $L_{n+1} = 2\pi i$. The Virasoro relations recursively determine the top degree coefficients of the volume polynomials by using the relations in a clever way. In recent work [16], Mulase and Safnuk showed how to extend the Virasoro relations to the full volume polynomials. It would be desirable to instead determine the polynomials recursively by relying on the more straightforward expansion of a function around a point. It would be interesting to know if one can express the results [16] in terms of derivatives of the volume polynomial at $L_{n+1} = 2\pi i$.

In principle, we can use Mirzakhani's recursion relation to get expressions for higher derivatives of the volume evaluated at $L_{n+1} = 2\pi i$. Differentiate the equation

$$\frac{\partial(L_{n+1}V_{g,n+1})}{\partial L_{n+1}} = \mathcal{A}_{g,n+1} + \mathcal{B}_{g,n+1}$$

to get

$$\frac{\partial^2(L_{n+1}V_{g,n+1})}{\partial L_{n+1}^2} = \frac{\partial\mathcal{A}_{g,n+1}}{\partial L_{n+1}} + \frac{\partial\mathcal{B}_{g,n+1}}{\partial L_{n+1}}$$

and evaluate at L_{n+1}. Substitute the equation for the first derivative, to get the following equation for the second derivative. Put $\mathcal{E} = \sum_{j=1}^{n} L_j\partial/\partial L_j$, the Euler vector field:

$$\frac{\partial^2 V_{g,n+1}}{\partial L_{n+1}^2}(\mathbf{L}, 2\pi i) = \mathcal{E}\cdot V_{g,n}(\mathbf{L}) - (4g - 4 + n)V_{g,n}(\mathbf{L}).$$

By taking higher derivatives of Mirzakhani's relation we can recursively get equations for higher derivatives. The strength of (2) and (3) is the simplification of Mirzakhani's relations

(1). It is not clear that the higher derivative relations obtained by the method above possess this same strength. This leads to the following question: when does $\partial^k V_{g,n+1}/\partial L_{n+1}^k(\mathbf{L}, 2\pi i)$ depend only on $V_{g,n}(\mathbf{L})$?

Appendix

MAPLE routine for calculating $V_{0,n}(\mathbf{L})$

```
>  # input: symmetric polynomial f in n variables L1,...,Ln
# output: symmetric polynomial S in n+1 variables L1,…,L(n+1)
# satisfying S(L(n+1)=0)=f
sym:=proc(f) local i,j,k,m,S,T,T1,prod,sum,epsilon:
S:=f:
epsilon:=array[1,...,100]:
for i from 1 to 100 do epsilon[i]:=0: od:
while epsilon[n+1]<1 do
T:=subs(seq(L||j=(1-epsilon[j])*L||j,j=1,...,n),f):
T1:=0:
for i from 1 to n do
prod:=1:
for j from i+1 to n+1 do
prod:=prod*(1-epsilon[j])
od:
T1:=T1+prod*subs(L||i=L||(n+1),T):
od:
sum:=0: for k from 1 to n do sum:=sum+epsilon[k] od:
S:=S+(-1)^sum*T1:
for k from 1 to 100 do
if epsilon[k]=1 then epsilon[k]:=0
else epsilon[k]:=1: k:=100 end if:
od:
od:
S:=simplify(S):
end:

> # calculate the genus zero volumes recursively from evalua-
tion
# of V_(0,n+1) at L(n+1)=2*Pi*I
for n from 3 to 12 do
P:=0:
for j from 1 to n do
P:=P+int(L||j*V[n],L||j)
od:
Q0:=P:
C0:=simplify(coeff(Q0,Pi,0)):
sim:=sym(C0):
V[n+1]:=sim:
for k from 1 to n-2 do
P||k:=sim-C||(k-1):
Q||k:=subs(L||(n+1)=2*Pi*I,Q||(k-1)-P||k*Pi^(2*k-2)):
C||k:=simplify(coeff(Q||k,Pi,2*k)):
sim:=sym(C||k):
V[n+1]:=V[n+1]+sim*Pi^(2*k):
od:
od:
```

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