CONSTANT CURIOSITY

Norm Do

Not all numbers were created equal. Mathematically minded folk are all aware of the ubiquity of Archimedes' constant π, the importance of Euler's constant *e* and the beauty of the golden ratio ϕ. However, let's spare a thought for a few of the lesser known mathematical constants — ones which might not permeate the various fields of mathematics but have nevertheless been immortalized in the mathematical literature in one way or another. In this seminar, we'll consider a few of these numerical curios and their rise to fame.

13 February 2010

The Fibonacci sequence

Consider the following sequence of positive integers.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

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The Fibonacci sequence is defined by the rules

- $F_1 = 1$,
- $F_2 = 1$
- $F_{n+1} = F_n + F_{n-1}$ for $n > 1$.

How large are Fibonacci numbers?

Binet's formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

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- The Fibonacci sequence grows exponentially and its growth factor is the golden ratio.

$$\lim_{n\to\infty} \sqrt[n]{F_n} = \frac{1+\sqrt{5}}{2}$$

Vibonacci sequences

Now let's spice up the Fibonacci sequence with a bit of randomness!

Vibonacci sequences

A Vibonacci sequence is defined by the rules

- $V_1 = 1$,
- $V_2 = 1$,
- $V_{n+1} = V_n \pm V_{n-1}$ for $n > 1$,

where the sign is chosen by the flip of a coin for each n.

Examples of Vibonacci sequences

- All heads (HHHHHHHHHH. . .)
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .
- All tails (TTTTTTTTTT. . .)
  1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, . . .

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  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...
- All tails (TTTTTTTTTT...)
  1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ...
- Random (TTHHHHTHHH...)
  1, 1, 0, -1, -1, -2, -3, -5, -2, -7, -9, -16, -7, 9, 16, 25, 41, 66, 25, 91, 66, -25, -91, -116, -25, 91, 116, 25, -91, -116, -25, 91, 116, 25, -91, -116, -207, -323, -530, ...

It seems like the signs are switching willy-nilly and that the magnitudes are growing larger and larger, on average.

How large are Vibonacci numbers?

Viswanath's theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n\to\infty} \sqrt[n]{|V_n|} = 1.13198824\ldots.$$

By the phrase almost surely, we mean with probability 1.

What is the number $1.3198824\ldots$?

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- Simple answer: It's Viswanath's constant!

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Viswanath's theorem tells us that there's some semblance of order appearing in all the randomness!

A generalization of Vibonacci sequences

The Embree-Trefethen theorem

Consider a sequence defined by the rules

- $X_1 = 1$,
- $X_2 = 1$,
- $X_{n+1} = X_n \pm bX_{n-1}$ for $n > 1$,

where the sign is chosen by the flip of a coin for each n. Then there exists a positive real number $V(b)$ such that almost surely

$$\lim_{n\to\infty} \sqrt[n]{|X_n|} = V(b).$$

- In this notation, Viswanath's constant is simply $V(1)$.

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- In this notation, Viswanath's constant is simply $V(1)$.
- Computational evidence suggests that the function V is very crazy — in fact, it's probably a fractal!

The randomness of primes

The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

The largest known prime is the number $2^{43,112,609} - 1$.

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Here are three unsolved prime problems...

- Twin prime conjecture: Are there infinitely many pairs of primes which differ by 2?
- Goldbach conjecture: Can every even integer greater than 2 be written as the sum of two primes?
- Riemann hypothesis Do all of the non-trivial zeroes of the Riemann zeta function lie on the line $\text{Re}(s) = \frac{1}{2}$?

A crazy formula for primes

The set of positive values taken on by the following bizarre polynomial in 26 variables is precisely the set of primes, where the variables $a, b, c, \ldots, z$ vary over the non-negative integers.

$$(k+2)(1-(wz+h+j-q)^2-((gk+2g+k+1)(h+j)+h-z)^2-(2n+p+q+z-e)^2-(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2-(e^3(e+2)(a+1)^2+1-o^2)^2-((a^2-1)y^2+1-x^2)^2-(16r^2y^4(a^2-1)+1-u^2)^2-(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2-(n+l+v-y)^2-((a^2-1)l^2+1-m^2)^2-(ai+k+1-l-i)^2-(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2-(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2-(z+pl(a-p)+t(2ap-p^2-1)-pm)^2)$$

A simple formula for primes

- What if we're not so fussy? Instead of generating all primes, is there a formula which generates only primes?
- Euler realized that the polynomial $n^2 + n + 41$ generates primes for $n = 0, 1, 2, \ldots, 39$.

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Mills' theorem

There exists a positive constant M such that the expression

$$\left\lfloor M^{3^n} \right\rfloor$$

yields only primes for all positive integers n.

Mills' constant

- So why can't we use Mills' theorem to find large primes? Because Mills didn't tell us what M is!

- If the Riemann Hypothesis is true (and most mathematicians believe that it is), then the smallest value of M which works in Mills' theorem is

$$M = 1.3063778838630806904686144\ldots.$$

- What is the number $1.3063778838630806904686144\ldots$?

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Where does Mills' constant come from?

A recipe for Mills' constant

- Let $P_1 = 2$.
- For every positive integer n, let P_{n+1} be the next prime after P_n^3.
- For every positive integer n, let $Q_n = \sqrt[3^n]{P_n}$.
- The numbers $Q_1, Q_2, Q_3, \ldots$ are increasing and converge to Mills' constant.

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If the Riemann hypothesis is true (and most mathematicians believe that it is), then there is a prime between any two consecutive perfect cubes. Then for every positive integer n, we have

$$\lfloor M^{3^n} \rfloor = P_n.$$

What comes next?

1
11
21
1211
111221
312211
13112221
1113213211
31131211131221
13211311123113112211
⋮

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Each sequence describes the digits appearing in the previous sequence.

So, for example, to generate the sequence after 312211, we scan along its digits and note that it consists of one 3, one 1, two 2s and two 1s.

So the next term is 13112221.

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For obvious reasons, these are called look-and-say sequences.

How long are look-and-say sequences?

- Let C_n denote the number of digits in the n^{th} look-and-say sequence. These numbers seem to grow larger and larger, on average.

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Conway's theorem

The limit

$$C = \lim_{n \to \infty} \sqrt[n]{C_n}$$

exists and is approximately $1.303577269034296391 25\ldots$.

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What is the number $1.303577269034296391 25\ldots$?

- Simple answer: It's Conway's constant!
- Honest answer: It's the unique positive real root of the following irreducible polynomial.

$$x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11} - 2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6$$

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This has to be one of the most bizarre of the algebraic numbers to appear in the mathematical literature!

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- Conway's idea: Often, a string of digits can be broken down into substrings which evolve via the look-and-say rule without interfering with each other.
- From the eighth term onwards, every look-and-say sequence is comprised of a combination of 92 substrings which never interfere with each other.
- Conway calls these substrings the atomic elements and calls the process of applying the look-and-say rule audioactive decay.

Atomic Number	Element	String
1	Hydrogen	22
2	Helium	13112221133211322112211213322112
3	Lithium	312211322212221121123222122
4	Beryllium	111312211312113221133211322112211213322112
5	Boron	1321132122211322212221121123222112
⋮	⋮	⋮
92	Uranium	3

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- A more natural hands-free approach is to represent a real number a by its continued fraction.

$$a = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Here $a_0 = \lfloor a \rfloor$ and $a_1, a_2, a_3, \ldots$ is a sequence of positive integers which is finite if a is rational and infinite if a is irrational.

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- Let's write continued fractions using the more compact notation $a = [a_0; a_1, a_2, a_3, \ldots]$.

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Khinchin's theorem

For almost all real numbers $a = [a_0; a_1, a_2, a_3, \ldots]$, the limit

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \ldots a_n}$$

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- What is the number $2.6854520010\ldots$?
  - Simple answer: It's Khinchin's constant!
  - Honest answer: It's $\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2+2k}\right)^{\log_2 k}$.

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- It's easy to find real numbers which don't obey Khinchin's theorem — rational numbers, solutions to integer quadratic equations and the number e.
- No one has managed to find a single number which does obey Khinchin's theorem without constructing it using its continued fraction!

The end

Many thanks to the organizers of SUMM for inviting me to speak and to you for staying awake while I've been speaking!

For more information, feel free to

- read my articles at `http://www.math.mcgill.ca/ndo`,
- email me at `ndo@math.mcgill.ca`, or
- see me after the show.