Constant Curiosity

Norman Do

Not all numbers were created equal. Mathematically minded folk are all aware of the ubiquity of Archimedes' constant π, the importance of Euler's constant *e* and the beauty of the golden ratio ϕ. However, let us spare a thought for a few of the lesser known mathematical constants — ones which might not permeate the various fields of mathematics but have nevertheless been immortalised in the mathematical literature in one way or another. In this seminar, we will consider a few of these numerical curios and their rise to fame.

24 March, 2006

- These are the first few terms of the Fibonacci sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

- These are the first few terms of the Fibonacci sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

The Fibonacci sequence

The Fibonacci sequence is defined by the rules

- These are the first few terms of the Fibonacci sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

The Fibonacci sequence

The Fibonacci sequence is defined by the rules

- $F_1 = 1$;

- These are the first few terms of the Fibonacci sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

The Fibonacci sequence

The Fibonacci sequence is defined by the rules

- $F_1 = 1$;
- $F_2 = 1$;

- These are the first few terms of the Fibonacci sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

The Fibonacci sequence

The Fibonacci sequence is defined by the rules

- $F_1 = 1$;
- $F_2 = 1$;
- $F_{n+1} = F_n + F_{n-1}$ for $n > 1$.

- Binet's formula lets you calculate the nth term of the Fibonacci sequence.

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

- Binet's formula lets you calculate the nth term of the Fibonacci sequence.

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

- Fact: For really really humongous n,

$$F_n \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n .$$

- Binet's formula lets you calculate the nth term of the Fibonacci sequence.

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

- Fact: For really really humongous n,

$$F_n \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n .$$

- Fact: The Fibonacci sequence grows approximately exponentially and its growth factor is the golden ratio.

$$\lim_{n \to \infty} \sqrt[n]{F_n} = \frac{1+\sqrt{5}}{2}$$

- Now, let's spice up the Fibonacci sequence with a bit of randomness!

- Now, let's spice up the Fibonacci sequence with a bit of randomness!

Vibonacci sequences

A Vibonacci sequence is defined by the rules

- Now, let's spice up the Fibonacci sequence with a bit of randomness!

Vibonacci sequences

A Vibonacci sequence is defined by the rules

- $V_1 = 1$;

- Now, let's spice up the Fibonacci sequence with a bit of randomness!

Vibonacci sequences

A Vibonacci sequence is defined by the rules

- $V_1 = 1$;
- $V_2 = 1$;

- Now, let's spice up the Fibonacci sequence with a bit of randomness!

Vibonacci sequences

A Vibonacci sequence is defined by the rules

- $V_1 = 1$;
- $V_2 = 1$;
- $V_{n+1} = V_n \pm V_{n-1}$ for $n > 1$,

- Now, let's spice up the Fibonacci sequence with a bit of randomness!

Vibonacci sequences

A Vibonacci sequence is defined by the rules

- $V_1 = 1$;
- $V_2 = 1$;
- $V_{n+1} = V_n \pm V_{n-1}$ for $n > 1$,

  where the sign is chosen by the flip of a coin for each n.

- All heads: (HHHHHHHHHH. . .)
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

- All heads: (HHHHHHHHHH...)
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...
- All tails: (TTTTTTTTTT...)
  1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ...

- All heads: (HHHHHHHHHH. . .)
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .
- All tails: (TTTTTTTTTT. . .)
  1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, . . .
- Random: (TTHHHHTHHH. . .)
  1, 1, 0, -1, -1, -2, -3, -5, -2, -7, -9, -16, -7, 9, 16, 25, 41, 66, 25, 91, 66, -25, -91, -116, -25, 91, 116, 25, -91, -116, -25, 91, 116, 25, -91, -116, -207, -323, -530, . . .

- All heads: (HHHHHHHHHH. . .)
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .
- All tails: (TTTTTTTTTT. . .)
  1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, . . .
- Random: (TTHHHHTHHH. . .)
  1, 1, 0, -1, -1, -2, -3, -5, -2, -7, -9, -16, -7, 9, 16, 25, 41, 66, 25, 91, 66, -25, -91, -116, -25, 91, 116, 25, -91, -116, -25, 91, 116, 25, -91, -116, -207, -323, -530, . . .
- It seems like the signs are switching willy-nilly.

- All heads: (HHHHHHHHHH...)
  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...
- All tails: (TTTTTTTTTT...)
  1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ...
- Random: (TTHHHHTHHH...)
  1, 1, 0, -1, -1, -2, -3, -5, -2, -7, -9, -16, -7, 9, 16, 25, 41, 66, 25, 91, 66, -25, -91, -116, -25, 91, 116, 25, -91, -116, -25, 91, 116, 25, -91, -116, -207, -323, -530, ...
- It seems like the signs are switching willy-nilly.
- It seems like the magnitudes are growing larger and larger, on average.

Viswanath's Theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n\to\infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

Viswanath's Theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n\to\infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

- What does almost surely mean?

Viswanath's Theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n\to\infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

- What does almost surely mean?
- What is the number $1.3198824\ldots$?

Viswanath's Theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n\to\infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

- What does almost surely mean?
- What is the number $1.3198824\ldots$?
  - Simple answer: It's Viswanath's constant!

Viswanath's Theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n\to\infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

- What does almost surely mean?
- What is the number $1.3198824\ldots$?
  - Simple answer: It's Viswanath's constant!
  - Honest answer: We don't know!
    You'd guess that Viswanath's constant is related to the golden ratio — but no one has found such a relationship.

Viswanath's Theorem

If $V_1, V_2, V_3, \ldots$ is a Vibonacci sequence, then almost surely

$$\lim_{n \to \infty} \sqrt[n]{|V_n|} = 1.3198824\ldots.$$

- What does almost surely mean?
- What is the number $1.3198824\ldots$?
  - Simple answer: It's Viswanath's constant!
  - Honest answer: We don't know!
    You'd guess that Viswanath's constant is related to the golden ratio — but no one has found such a relationship.
- Viswanath's Theorem tells us that there is some semblance of order appearing in the randomness.

"Though this be madness, yet there is method in 't."

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The prime gap sequence

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \ldots$$

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The prime gap sequence

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \ldots$$

- The largest known prime is the number $2^{30,402,457} - 1$.

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The prime gap sequence

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \ldots$$

- The largest known prime is the number $2^{30,402,457} - 1$.
- Unsolved prime problems

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The prime gap sequence

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \ldots$$

- The largest known prime is the number $2^{30,402,457} - 1$.
- Unsolved prime problems
  - Twin Prime Conjecture: Are there infinitely many pairs of primes which differ by 2?

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The prime gap sequence

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \ldots$$

- The largest known prime is the number $2^{30,402,457} - 1$.
- Unsolved prime problems
  - Twin Prime Conjecture: Are there infinitely many pairs of primes which differ by 2?
  - Goldbach Conjecture: Can every even integer greater than 2 be written as the sum of 2 primes?

- The primes are somewhat elusive beasts among the menagerie of natural numbers.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

- The prime gap sequence

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \ldots$$

- The largest known prime is the number $2^{30,402,457} - 1$.
- Unsolved prime problems
  - Twin Prime Conjecture: Are there infinitely many pairs of primes which differ by 2?
  - Goldbach Conjecture: Can every even integer greater than 2 be written as the sum of 2 primes?
  - Riemann Hypothesis Does the Riemann zeta function $\zeta(s)$ have non-trivial zeroes which do not lie on the line $\text{Re}(s) = \frac{1}{2}$?

A crazy formula for primes

The set of positive values taken on by the following bizarre polynomial in 26 variables is precisely the set of primes, where the variables $a, b, c, \ldots, z$ vary over the non-negative integers.

$$(k+2)(1-(wz+h+j-q)^2-((gk+2g+k+1)(h+j)+h-z)^2-(2n+p+q+z-e)^2-(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2-(e^3(e+2)(a+1)^2+1-o^2)^2-((a^2-1)y^2+1-x^2)^2-(16r^2y^4(a^2-1)+1-u^2)^2-(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2-(n+l+v-y)^2-((a^2-1)l^2+1-m^2)^2-(ai+k+1-l-i)^2-(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2-(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2-(z+pl(a-p)+t(2ap-p^2-1)-pm)^2)$$

- What if we're not so fussy? Instead of generating all primes, is there a formula which generates only primes?

- What if we're not so fussy? Instead of generating all primes, is there a formula which generates only primes?
- Euler's attempt: $x^2 + x + 41$

- What if we're not so fussy? Instead of generating all primes, is there a formula which generates only primes?
- Euler's attempt: $x^2 + x + 41$

Mills' Theorem

There exists a positive constant M such that the expression

$$\left\lfloor M^{3^n} \right\rfloor$$

yields only primes for all positive integers n.

- So why can't we use Mills' Theorem to find the largest known prime?
  Because Mills didn't tell us what the number M is!

- So why can't we use Mills' Theorem to find the largest known prime?
  Because Mills didn't tell us what the number M is!
- If the Riemann Hypothesis is true (and most mathematicians believe that it is), then the smallest value of M which works in Mills' Theorem is

$$M = 1.3063778838630806904686144926026\ldots.$$

- So why can't we use Mills' Theorem to find the largest known prime?
  Because Mills didn't tell us what the number M is!
- If the Riemann Hypothesis is true (and most mathematicians believe that it is), then the smallest value of M which works in Mills' Theorem is

$$M = 1.306377883863080690468614492602\ldots.$$

- What is the number $1.306377883863080690468614492602\ldots$?

- So why can't we use Mills' Theorem to find the largest known prime?
  Because Mills didn't tell us what the number M is!
- If the Riemann Hypothesis is true (and most mathematicians believe that it is), then the smallest value of M which works in Mills' Theorem is

$$M = 1.3063778838630806904686144926 \ldots.$$

- What is the number $1.3063778838630806904686144926\ldots$?
  - Simple answer: It's Mills' constant!

- So why can't we use Mills' Theorem to find the largest known prime?
  Because Mills didn't tell us what the number M is!
- If the Riemann Hypothesis is true (and most mathematicians believe that it is), then the smallest value of M which works in Mills' Theorem is

$$M = 1.3063778838630806904686144926\ldots.$$

- What is the number $1.3063778838630806904686144926\ldots$?
  - Simple answer: It's Mills' constant!
  - Honest answer: We don't know!
    It is unknown whether Mills' constant is rational or not — it seems incredibly doubtful though.

A recipe for Mills' constant

A recipe for Mills' constant

1. Let $P_1 = 2$.

A recipe for Mills' constant

1. Let $P_1 = 2$.
2. For every positive integer n, let P_{n+1} be the next prime after P_n^3.

A recipe for Mills' constant

1. Let $P_1 = 2$.
2. For every positive integer n, let P_{n+1} be the next prime after P_n^3.
3. For every positive integer n, let $Q_n = \sqrt[3^n]{P_n}$.

A recipe for Mills' constant

1. Let $P_1 = 2$.
2. For every positive integer n, let P_{n+1} be the next prime after P_n^3.
3. For every positive integer n, let $Q_n = \sqrt[3^n]{P_n}$.
4. The numbers $Q_1, Q_2, Q_3, \ldots$ are increasing and converge to Mills' constant M.

What comes next?

1
11
21
1211
111221
312211
13112221
1113213211
31131211131221
⋮

- Each term of the sequence, except the first, describes the digits appearing in the previous term. For example, to generate the term after 312211, we scan along its digits and note that it is comprised of

  "one 3", "one 1", "two 2's", and "two 1's".

  So the next term is 13112221.

- Each term of the sequence, except the first, describes the digits appearing in the previous term. For example, to generate the term after 312211, we scan along its digits and note that it is comprised of

  "one 3", "one 1", "two 2's", and "two 1's".

  So the next term is 13112221.
- For obvious reasons, it has been coined the Look and Say Sequence.

- Let C_n denote the number of digits in the nth term of the Look and Say Sequence.

$$1, 2, 2, 4, 6, 6, 8, 10, 14, \ldots$$

- Let C_n denote the number of digits in the nth term of the Look and Say Sequence.

$$1, 2, 2, 4, 6, 6, 8, 10, 14, \ldots$$

- It seems like the sequence grows larger and larger, on average.

- Let C_n denote the number of digits in the nth term of the Look and Say Sequence.

$$1, 2, 2, 4, 6, 6, 8, 10, 14, \ldots$$

- It seems like the sequence grows larger and larger, on average.
- Does the number

$$\lim_{n \to \infty} \sqrt[n]{C_n}$$

exist? If so, what is it?

Conway's Theorem

If C_n denotes the number of digits in the nth term of the Look and Say Sequence, then

$$C = \lim_{n \to \infty} \sqrt[n]{C_n}$$

exists and is approximately

$$1.303577269034296391257099112152551890730 \ldots .$$

- What is the number $1.303577269034296391257099112152551890730 7\ldots$?

- What is the number $1.303577269034296391257099112152551890730 7\ldots$?
  - Simple answer: It's Conway's constant!

- What is the number 1.303577269034296391257099112152551890730 7...?
  - Simple answer: It's Conway's constant!
  - Honest answer: It's the unique positive real root of the following irreducible polynomial.

$$x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11} - 2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6.$$

- What is the number
  1.303577269034296391257099112152551890730 7...?
  - Simple answer: It's Conway's constant!
  - Honest answer: It's the unique positive real root of the following irreducible polynomial.

$$x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} - 4x^{11} - 2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6.$$

- This must certainly be one of the most bizarre of the algebraic numbers to appear in the mathematical literature!

- Main idea: Often, a string of digits can be broken down into substrings which evolve via the Look and Say rule without interfering with each other

- Main idea: Often, a string of digits can be broken down into substrings which evolve via the Look and Say rule without interfering with each other
- In particular, from the eighth term onwards, every term of the Look and Say Sequence is comprised of a combination of 92 substrings which never interfere with each other.

- Main idea: Often, a string of digits can be broken down into substrings which evolve via the Look and Say rule without interfering with each other
- In particular, from the eighth term onwards, every term of the Look and Say Sequence is comprised of a combination of 92 substrings which never interfere with each other.
- Conway names these substrings the atomic elements, giving each an atomic number and its corresponding name from the periodic table. He then calls the process of applying the Look and Say rule "audioactive decay".

Atomic Number	Element	String
1	Hydrogen	22
2	Helium	13112221133211322112211213322112
3	Lithium	312211322212221121123222122
4	Beryllium	111312211312113221133211322112211213322112
5	Boron	1321132122211322212221121123222112
⋮	⋮	⋮
92	Uranium	3

- Read my article at

  `http://www.austms.org.au/Publ/Gazette/`

- E-mail me at

  `norm@ms.unimelb.edu.au`