

CONSTANT CURIOSITY

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Not all numbers were created equal. Mathematically minded folk are all aware of the ubiquity of Archimedes' constant π , the importance of Euler's constant e and the beauty of the golden ratio ϕ . However, let's spare a thought for a few of the lesser known mathematical constants — ones which might not permeate the various fields of mathematics but have nevertheless been immortalised in the mathematical literature in one way or another. In this seminar, we'll consider a few of these numerical curios and their rise to fame.

The Fibonacci sequence

Consider the following sequence of numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

This is the **Fibonacci sequence**, which is defined by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n > 1.$$

Binet's formula. We can calculate terms of the Fibonacci sequence without knowing any of the previous terms.

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

How large are Fibonacci numbers?

For humongous n , the second term in Binet's formula is tiny.

$$F_n \approx \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

So the Fibonacci sequence grows exponentially and its growth factor is the golden ratio.

$$\lim_{n \rightarrow \infty} \sqrt[n]{F_n} = \frac{1 + \sqrt{5}}{2}$$

Vibonacci sequences

A **Vibonacci sequence** is defined by

$$V_1 = 1, \quad V_2 = 1, \quad V_{n+1} = V_n \pm V_{n-1},$$

where the sign is chosen by the flip of a coin for each $n > 1$.

- **All heads** (HHHHHHHHHHH...)

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

- **All tails** (TTTTTTTTTTT...)

1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ...

- **Random** (TTHHHHTHHH...)

1, 1, 0, -1, -1, -2, -3, -5, -2, -7, -9, -16, -7, 9, 16,
25, 41, 66, 25, 91, 66, -25, -91, -116, -25, 91, 116, 25,
-91, -116, -25, 91, 116, 25, -91, -116, -207, -323, ...

How large are Vibonacci numbers?

Viswanath's theorem. If V_1, V_2, V_3, \dots is a Vibonacci sequence, then with probability 1 we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|V_n|} = 1.13198824\dots$$

What is the number 1.3198824...?

- Simple answer: It's **Viswanath's constant!**
- Honest answer: We don't know!

Viswanath's theorem tells us that there's some semblance of order appearing in all the randomness.

A generalisation of Fibonacci sequences

Embree–Trefethen theorem. Consider a sequence defined by

$$X_1 = 1, \quad X_2 = 1, \quad X_{n+1} = X_n \pm bX_{n-1},$$

where the sign is chosen by the flip of a coin for each $n > 1$. There is a number $F(b)$ such that with probability 1 we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|X_n|} = F(b).$$

- Viswanath's constant is simply $F(1)$.
- Computational evidence suggests that the function F is totally crazy. In fact, its graph is probably a fractal!

The randomness of primes

The primes are somewhat elusive beasts among the menagerie of natural numbers.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ...

The largest known prime is the number $2^{43,112,609} - 1$.

Here are three unsolved problems concerning primes.

- **Twin prime conjecture.** Are there infinitely many pairs of primes which differ by two?
- **Goldbach conjecture.** Is every even integer greater than two the sum of two primes?
- **Riemann hypothesis.** Do all of the non-trivial zeroes of the Riemann zeta function lie on the line $\text{Re}(z) = \frac{1}{2}$?

A crazy formula for primes

The set of positive values taken on by the following bizarre polynomial is precisely the set of primes, where the variables a, b, c, \dots, z vary over the non-negative integers.

$$\begin{aligned} & (k+2)(1 - (wz + h + j - q)^2 - ((gk + 2g + k + 1)(h + j) + h - \\ & z)^2 - (2n + p + q + z - e)^2 - (16(k+1)^3(k+2)(n+1)^2 + \\ & 1 - f^2)^2 - (e^3(e+2)(a+1)^2 + 1 - o^2)^2 - ((a^2 - 1)y^2 + 1 - \\ & x^2)^2 - (16r^2y^4(a^2 - 1) + 1 - u^2)^2 - (((a + u^2(u^2 - a))^2 - \\ & 1)(n + 4dy)^2 + 1 - (x + cu)^2)^2 - (n + l + v - y)^2 - ((a^2 - 1)l^2 + \\ & 1 - m^2)^2 - (ai + k + 1 - l - i)^2 - (p + l(a - n - 1) + b(2an + \\ & 2a - n^2 - 2n - 2) - m)^2 - (q + y(a - p - 1) + s(2ap + 2a - \\ & p^2 - 2p - 2) - x)^2 - (z + pl(a - p) + t(2ap - p^2 - 1) - pm)^2 \end{aligned}$$

A simple formula for primes

- Instead of generating all primes, is there a formula which generates only primes?
- Euler realized that the polynomial $n^2 + n + 41$ generates primes for $n = 0, 1, 2, \dots, 39$.

Mills' theorem. There is a constant M such that

$$\lfloor M^{3^n} \rfloor$$

is a prime for every positive integer n .

Mills' constant

So why can't we use Mills' theorem to find larger and larger primes? Because Mills never told us what M is!

If the Riemann Hypothesis is true — and most mathematicians believe that it is — then the smallest possible value of M is

$$M = 1.3063778838630806904686144\dots$$

What is the number 1.3063778838630806904686144...?

- Simple answer: It's **Mills' constant!**
- Honest answer: We don't know!

Where does Mills' constant come from?

- Let $P_1 = 2$.
- For every n , let P_{n+1} be the smallest prime larger than P_n^3 .
- For every n , let $Q_n = \sqrt[3^n]{P_n}$.
- The sequence Q_1, Q_2, Q_3, \dots is increasing and converges to Mills' constant.

If the Riemann hypothesis is true, then there is a prime between any two consecutive perfect cubes. Then for every n ,

$$\lfloor M^{3^n} \rfloor = P_n.$$

What comes next?

1
11
21
1211
111221
312211
13112221
1113213211
31131211131221
13211311123113112211
⋮

Each sequence describes the digits appearing in the previous sequence.

Reading from left to right, the sequence 312211 contains one 3, one 1, two 2s, two 1s.

So the next term is 13112221.

For obvious reasons, these are called **look-and-say sequences**.

How long are look-and-say sequences?

Let C_n be the number of digits in the n th sequence.

1, 2, 2, 4, 6, 6, 8, 10, 14, 20, 26, 34, 46, 62, 78, 102, ...

Does $\lim_{n \rightarrow \infty} \sqrt[n]{C_n}$ exist and, if so, what is it?

Conway's theorem. The limit $\lim_{n \rightarrow \infty} \sqrt[n]{C_n}$ exists and is

1.30357726903429639125....

Conway's constant

What is the number 1.30357726903429639125...?

- Simple answer: It's **Conway's constant!**
- Honest answer: It's the unique positive root of the following irreducible polynomial!

$$\begin{aligned} &x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60} - \\ &x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} + \\ &x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - \\ &5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - \\ &6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - \\ &7x^{21} + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - \\ &12x^{12} - 4x^{11} - 2x^{10} + 5x^9 + x^7 - 7x^6 + 7x^5 - 4x^4 + 12x^3 - 6x^2 + 3x - 6 \end{aligned}$$

Audioactive decay

- Conway noticed that from the eighth term onwards, every look-and-say sequence is a combination of 92 strings which **never interfere with each other**.
- The strings are called atomic elements and the process of applying the look-and-say rule is called **audioactive decay**.

NUMBER	ELEMENT	STRING
1	Hydrogen	22
2	Helium	13112221133211322112211213322112
3	Lithium	312211322212221121123222122
⋮	⋮	⋮
90	Thorium	1113
91	Protactinium	13
92	Uranium	3

Continued fractions

Representing real numbers in decimal notation is unnatural.

A natural hands-free approach is to represent a real number x by its **continued fraction**.

$$x = x_0 + \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \cdots}}}$$

Here, $x_0 = \lfloor x \rfloor$ and x_1, x_2, x_3, \dots is a sequence of positive integers that is finite for rational x and infinite for irrational x .

Examples of continued fractions

Let's write continued fractions using the compact notation

$$x = [x_0; x_1, x_2, x_3, \dots].$$

- $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \dots]$
- $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \dots]$
- $\phi = [1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \dots]$
- $\sqrt{19} = [4; 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \dots]$

What can be said about the behaviour of the sequence x_1, x_2, x_3, \dots for a randomly chosen real number x ?

Khinchin's constant

Khinchin's theorem. Consider a randomly chosen real number $x = [x_0; x_1, x_2, x_3, \dots]$. Then with probability 1 the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{x_1 x_2 \cdots x_n}$$

exists and is always equal to 2.6854520010....

What is the number 2.6854520010...?

- Simple answer: It's **Khinchin's constant!**

- Honest answer: It's $\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2 + 2k}\right)^{\log_2 k}$.

Why is Khinchin's theorem surprising?

- Any sequence of positive integers x_1, x_2, x_3, \dots corresponds to some number x . For many of these,

$$\lim_{n \rightarrow \infty} \sqrt[n]{x_1 x_2 \dots x_n}$$

doesn't exist. And even if the limit does exist, the probability that it equals Khinchin's constant is zero.

- It's easy to find numbers which don't obey Khinchin's theorem — rational numbers, solutions to integer quadratic equations, and the number e .
- No one knows a number which obeys Khinchin's theorem without constructing it from its continued fraction!

The end

Many thanks to MUMS for inviting me to speak and to you for listening to me speak.

For more information,

- read my articles at <http://www.ms.unimelb.edu.au/~nndo>,
- email me at normdo@gmail.com, or
- speak to me after the talk.