LATTICE POINTS IN MODULI SPACES OF CURVES

Norman Do

The University of Melbourne

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In how many ways can you obtain a genus g surface by gluing together the edges of a given set of polygons? Norbury interprets this question as counting lattice points in the moduli space of curves and shows that the answer exhibits polynomial behaviour. The top degree and constant terms of these lattice point polynomials are known to store interesting geometric information. On the other hand, the intermediate coefficients remain a complete mystery. In this talk, we'll present some results concerning these polynomials, indicate some interesting connections to other areas, and consider what the intermediate coefficients might mean.

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- Let the polygons be numbered $1, 2, \ldots, n$ and have $b_1, b_2, \ldots, b_n$ edges.
- The edges of the polygons form a graph on the surface called a ribbon graph. We think of a ribbon graph as a cell decomposition of the surface.
- We won't allow two adjacent edges to be glued together — in other words, we won't allow vertices of degree one in the ribbon graph.
- Denote the enumeration by $N_{g,n}(b_1, b_2, \ldots, b_n)$.

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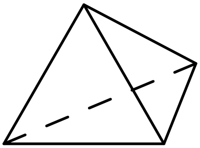
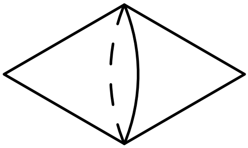
Example

You should be able to calculate that $N_{0,4}(3, 3, 3, 3) = 8$.



2 labellings



6 labellings

REPRESENTATION THEORY APPROACH

Let X be the set of oriented edges of the ribbon graph.

- $s_0 =$ the permutation on X which rotates anticlockwise about vertices
  $X/\langle s_0 \rangle = \{\text{vertices of the ribbon graph}\}$
- $s_1 =$ the permutation on X which flips edges
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- $s_2 = s_1^{-1} s_0^{-1}$
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The number $N_{g,n}(b_1, b_2, \ldots, b_n)$ counts triples $s_0 s_1 s_2 = \text{id}$ of permutations which satisfy the following.

- s_2 has cycle structure $(b_1, b_2, \ldots, b_n)$
- s_1 has cycle structure $(2, 2, \ldots, 2)$
- s_0 has V non-trivial cycles where $V - E + F = 2 - 2g$

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The Burnside formula expresses the answer as a sum over characters of the symmetric group — but this is not very useful for our purposes.

MATRIX INTEGRAL APPROACH

- Consider the following matrix integral where $V(M) = \sum \frac{t_k}{k} M^k$.

$$Z_N(t_1, t_2, \dots) = \int_{\mathcal{H}_N} \frac{\exp\left(-\frac{1}{2} N \text{ Tr } M^2\right) dM}{2^{N/2} \pi^{N^2/2}} \exp\left(N \text{ Tr } V(M)\right)$$

Here, $\mathcal{H}_N$ is the space of $N \times N$ Hermitian matrices with the Euclidean volume element dM.

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- The topological expansion of this matrix integral is

$$\log Z_N(t_1, t_2, \ldots) = \sum_{\Gamma \text{ a ribbon graph}} \frac{N^{2-2g}}{\#\text{Aut } \Gamma} t_{b_1} t_{b_2} \cdots t_{b_n}.$$

  Here, Γ is a ribbon graph on a genus g surface made from polygons with $b_1, b_2, \ldots, b_n$ edges. Also, Aut Γ is the automorphism group of Γ.

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  Here, Γ is a ribbon graph on a genus g surface made from polygons with $b_1, b_2, \ldots, b_n$ edges. Also, Aut Γ is the automorphism group of Γ.

- By specialising the variables, you can obtain the number $N_{g,n}(b_1, b_2, \ldots, b_n)$ as a coefficient of this generating function — but this is not very useful for our purposes. However, it indicates that we should count ribbon graphs with the weight $\frac{1}{\#\text{Aut } \Gamma}$.

COMBINATORIAL APPROACH (INSPIRED BY GEOMETRY)

Theorem (Norbury, 2008)

There exists a topological recursion in which $N_{g,n}$ relies on $N_{g-1,n+1}$, $N_{g,n-1}$, and $N_{g_1,n_1} \times N_{g_2,n_2}$ for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$. You can use this to compute $N_{g,n}(b_1, b_2, \ldots, b_n)$ from the following base cases.

- $N_{0,3}(b_1, b_2, b_3) = \begin{cases} 1 & \text{if } b_1 + b_2 + b_3 \text{ is even} \\ 0 & \text{if } b_1 + b_2 + b_3 \text{ is odd} \end{cases}$
- $N_{1,1}(b_1) = \begin{cases} \frac{1}{48}(b_1^2 - 48) & \text{if } b_1 \text{ is even} \\ 0 & \text{if } b_1 \text{ is odd} \end{cases}$

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Proof.

Think about what happens when you remove an edge from the graph. □

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Corollary

The count $N_{g,n}(b_1, b_2, \ldots, b_n)$ is an even symmetric quasi-polynomial of degree $6g - 6 + 2n$. So there exist polynomials $N^{(k)}_{g,n}(b_1, b_2, \ldots, b_n)$ such that

$$N_{g,n}(\underbrace{b_1, b_2, \ldots, b_k}_{\text{odd}}, \underbrace{b_{k+1}, b_{k+2}, \ldots, b_n}_{\text{even}}) = N^{(k)}_{g,n}(b_1, b_2, \ldots, b_n)$$

EXAMPLES OF LATTICE POINT POLYNOMIALS

Example

If k is odd, then $N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n) = 0$.

g	n	k	$N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$
0	3	0 or 2	1
1	1	0	$\frac{1}{48}(b_1^2 - 4)$
0	4	0 or 4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - 2)(b_1^2 + b_2^2 - 10)$
2	1	0	$\frac{1}{2^{16} \times 3^3 \times 5}(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(5b_1^2 - 32)$
3	1	0	$\frac{1}{2^{25} \times 3^6 \times 5^2 \times 7}(5b_1^4 - 188b_1^2 + 1152) \prod_{k=1}^{5}(b_1^2 - 4k^2)$

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Question

What do the coefficients mean?

MODULI SPACES OF CURVES

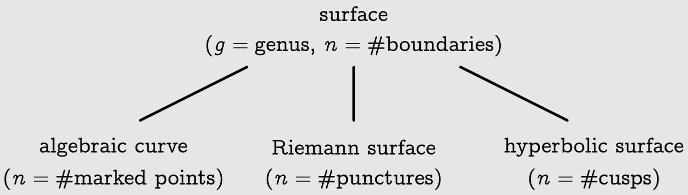
surface
(g = genus, n = #boundaries)

algebraic curve
(n = #marked points)

Riemann surface
(n = #punctures)

hyperbolic surface
(n = #cusps)

MODULI SPACES OF CURVES



- Moduli spaces of curves

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ smooth algebraic curves with distinct} \\ \text{points labelled from 1 up to } n \end{array} \right\}$$

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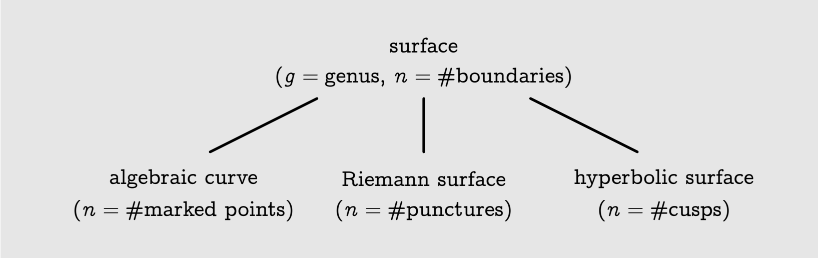

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- The dimension of $\mathcal{M}_{g,n}$ is $6g - 6 + 2n$. It's a Deligne–Mumford stack — so think of it as a complex orbifold.

DELIGNE–MUMFORD COMPACTIFICATION

- Deligne–Mumford compactification

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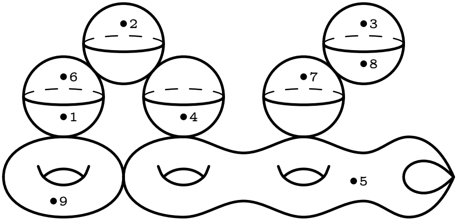
- A stable curve may be nodal but its components must satisfy $2g - 2 + n > 0$.



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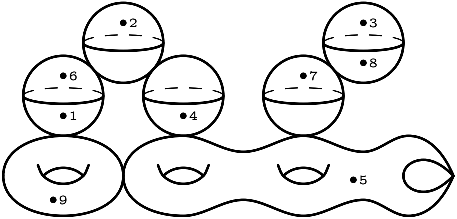
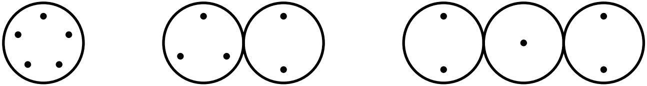


- The spaces $\overline{\mathcal{M}}_{g,n}$ are “stratified” by smaller moduli spaces of curves.



$$\overline{\mathcal{M}}_{0,5} = \mathcal{M}_{0,5} \cup \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} \cup \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$$

1 labelling | 10 labellings | 15 labellings

FROM ALGEBRAIC CURVES TO METRIC RIBBON GRAPHS

Theorem (Strebel)

Choose positive real numbers $r_1, r_2, \ldots, r_n$ and a Riemann surface S with punctures $p_1, p_2, \ldots, p_n$. There exists a unique quadratic differential on S whose non-closed horizontal trajectories form an embedded graph with complement punctured disks centred at $p_1, p_2, \ldots, p_n$ and with perimeters $r_1, r_2, \ldots, r_n$. The perimeters arise by integrating the square root of the quadratic differential along the edges of the graph.

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Corollary

Given positive real numbers $r_1, r_2, \ldots, r_n$, we can uniquely associate a point in $\mathcal{M}_{g,n}$ to a ribbon graph with

- every vertex of degree at least three;
- a length attached to every edge; and
- the perimeter of face k is r_k.

LATTICE POINTS IN MODULI SPACES OF CURVES

Idea

Interpret ribbon graphs with integer edge lengths as lattice points in moduli spaces of curves. So $N_{g,n}(b_1, b_2, \ldots, b_n)$ counts lattice points in $\mathcal{M}_{g,n}$. This gives a discrete approximation to the volume of the moduli space, which is known to store interesting topological information.

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Theorem (Norbury, 20008)

- The top degree part of the quasi-polynomial $N_{g,n}(b_1, b_2, \ldots, b_n)$ stores all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}$$

  Here, $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ and $a_1 + a_2 + \cdots + a_n = 3g - 3 + n$.

- The quasi-polynomial $N_{g,n}$ satisfies $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$.

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- The quasi-polynomial $N_{g,n}$ satisfies $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$.

Corollary

Combining this theorem with the earlier recursion gives a new proof of the Witten–Kontsevich theorem, which governs all psi-class intersection numbers.

LATTICE POINTS IN COMPACTIFIED MODULI SPACES

New idea

Count lattice points in compactified moduli spaces of curves

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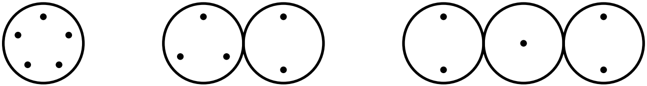
Example

Points in $\overline{\mathcal{M}}_{0,5}$ represent curves of the following types.



$$\overline{\mathcal{M}}_{0,5} = \underset{\text{1 labelling}}{\mathcal{M}_{0,5}} \cup \underset{\text{10 labellings}}{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}} \cup \underset{\text{15 labellings}}{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}$$

$$\begin{aligned}\overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) = {} & N_{0,5}(b_1, b_2, b_3, b_4, b_5) \\ & + \sum_{\text{10 terms}} N_{0,4}(b_i, b_j, b_k, 0) \cdot N_{0,3}(b_\ell, b_m, 0) \\ & + \sum_{\text{15 terms}} N_{0,3}(b_i, b_j, 0) \cdot N_{0,3}(b_k, 0, 0) \cdot N_{0,3}(b_\ell, b_m, 0)\end{aligned}$$

COMPACTIFIED LATTICE POINT POLYNOMIALS

Fact

- The count $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is an even symmetric quasi-polynomial of degree $6g - 6 + 2n$.
- The quasi-polynomials $N_{g,n}$ and $\overline{N}_{g,n}$ agree to leading order — so the top degree part of the quasi-polynomial $N_{g,n}(b_1, b_2, \ldots, b_n)$ stores all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
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1	2	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 + 48b_1^2 + 48b_2^2 + 84)$

QUESTIONS

Claim

The compactified enumeration $\overline{N}_{g,n}$ seems to be the right thing to study.

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The compactified enumeration $\overline{N}_{g,n}$ seems to be the right thing to study.

- What are we counting?
  We have a combinatorial interpretation for $N_{g,n}(b_1, b_2, \ldots, b_k, 0, 0, \ldots, 0)$, but only when k is positive.

QUESTIONS

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The compactified enumeration $\overline{N}_{g,n}$ seems to be the right thing to study.

- What are we counting?
  We have a combinatorial interpretation for $N_{g,n}(b_1, b_2, \ldots, b_k, 0, 0, \ldots, 0)$, but only when k is positive.

- Are the coefficients of $\overline{N}_{g,n}$ always positive?
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- The lattice point enumeration is part of a larger story which involves enumerative geometry, matrix integrals, factorisations in the symmetric group, integrable systems, and so on. What are the consequences of these connections?