Counting lattice points in compactified moduli spaces of curves 17/05/11

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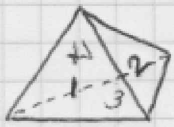
Joint with Paul Norbury (arXiv: 1012.5923 v2)

★ Ribbon graphs

Ribbon graph of type (g,n):
- 1-skeleton of genus g surface
- faces labelled $1, 2, 3, \ldots, n$
- degrees ≥ 3

1-skeleton $\longrightarrow$ graph
\+ \+
orientation $\longrightarrow$ cyclic ordering of edges at a vertex

Example:



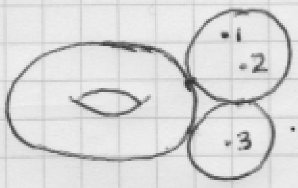
Metric ribbon graph: assign a positive length to each edge

★ Moduli spaces of curves

$$\begin{array}{ccc} \mathcal{M}_{g,n} & \subseteq & \overline{\mathcal{M}}_{g,n} \\ \| & & \| \\ \left\{\begin{array}{l}\text{genus } g \text{ smooth curves} \\ \text{with } n \text{ labelled points}\end{array}\right\} & & \left\{\begin{array}{l}\text{genus } g \text{ stable curves} \\ \text{with } n \text{ labelled points}\end{array}\right\} \end{array}$$

smooth curve

stable curve
(nodal and $\chi(\text{component}) < 0$)

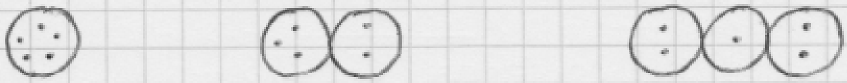


★ Stratification

Picture above suggests $\mathcal{M}_{1,2} \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} \subseteq \overline{\mathcal{M}}_{2,3}$

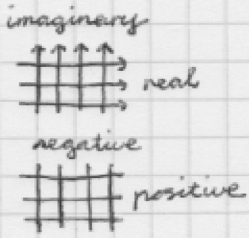
So $\overline{\mathcal{M}}_{g,n}$ is stratified by topology and labelling

Example: $\overline{\mathcal{M}}_{0,5} = \mathcal{M}_{0,5} \cup 10 \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} \cup 15 \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$



1 labelling   10 labellings   15 labellings

★ Strebel differentials



differential $\qquad f(z)\,dz = f(z(w))\,\frac{dz}{dw}\,dw$

quadratic differential $\qquad f(z)\,(dz)^2 = f(z(w))\left(\frac{dz}{dw}\right)^2 (dw)^2$

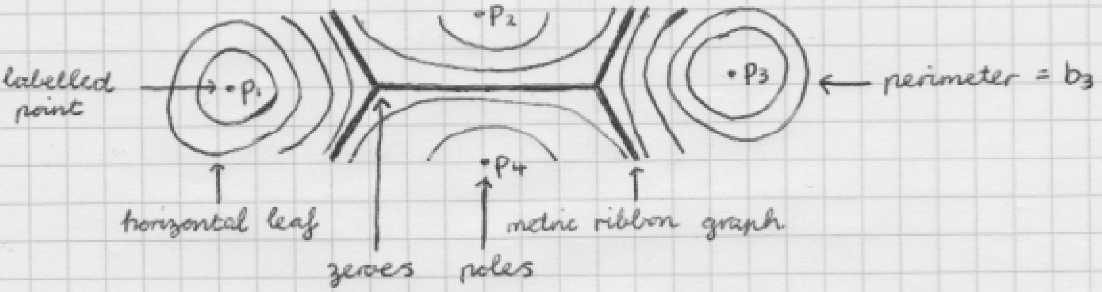
Strebel (quadratic) differential $\qquad$ require most horizontal leaves to close up

length (leaf) $= \int_{\text{leaf}} \sqrt{f(z)}\,dz$

Theorem: (Strebel)

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \left\{ \begin{matrix} \text{metric ribbon graphs} \\ \text{of type } (g,n) \end{matrix} \right\}$$

$(C, p_1, p_2, \ldots, p_n)$ $\longmapsto$ non-closed leaves of unique Strebel differential
$(b_1, b_2, \ldots, b_n)$ $\qquad$ on $C \setminus \{p_1, p_2, \ldots, p_n\}$ with perimeters $b_1, b_2, \ldots, b_n$



Corollary: For fixed $b_1, b_2, \ldots, b_n > 0$, $\mathcal{M}_{g,n} \cong \left\{ \begin{matrix} \text{metric ribbon graphs of type} \\ (g,n) \text{ with perimeters } b_1, b_2, \ldots, b_n \end{matrix} \right\}$.

Idea: $\quad$ lattice points in $\mathcal{M}_{g,n}$ $\longleftrightarrow$ metric ribbon graphs with integral edge lengths

volume of $\mathcal{M}_{g,n}$ $\longleftrightarrow$ # integral ribbon graphs

$\downarrow$ $\qquad\qquad$ $\downarrow$

Witten-Kontsevich
Mirzakhani
intersection theory on $\overline{\mathcal{M}}_{g,n}$

matrix models
topological recursion
representation theory of S_d

★ Counting ribbon graphs

$N_{g,n}(b_1, b_2, \ldots, b_n)$ = # integral ribbon graphs of type (g,n) with perimeters $b_1, b_2, \ldots, b_n$, weighted by $\frac{1}{\#\mathrm{Aut}}$

"=" # ways to make a genus g surface from a b_1-gon, a b_2-gon, ..., a b_n-gon

Theorem: (Norbury)

- $N_{g,n}(b_1, b_2, \ldots, b_n)$ is quasipolynomial in $b_1^2, b_2^2, \ldots, b_n^2$, depending on "parity class", of degree $\dim_{\mathbb{C}} \mathcal{M}_{g,n} = 3g-3+n$
- If $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 3g-3+n$, then $\left[\frac{b_1^{2\alpha_1}}{\alpha_1!} \frac{b_2^{2\alpha_2}}{\alpha_2!} \cdots \frac{b_n^{2\alpha_n}}{\alpha_n!}\right] N_{g,n}(b_1, b_2, \ldots, b_n)$ is

$$\left(\tfrac{1}{2}\right)^{5g-6+2n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}$$

  where $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ are "tautological" classes.
- $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$
- Tutte recursion (remove an edge): $N_{g,n+1}$ depends on

$$N_{g,n}, \qquad N_{g-1,n+2} \qquad \text{and} \qquad N_{g_1,n_1+1} \times N_{g_2,n_2+1} \quad \text{for} \quad \begin{matrix} g_1+g_2=g \\ n_1+n_2=n \end{matrix}$$

- String / dilaton equations: relate certain evaluations of $N_{g,n+1}$ to $N_{g,n}$
- Eynard-Orantin topological recursion: $N_{g,n}$ is stored in correlators $\omega_{g,n}$ for the spectral curve

$$\begin{matrix} x(z) = z + \frac{1}{z} \\ y(z) = z \end{matrix} \quad \longleftrightarrow \quad y^2 - xy + 1 = 0$$

Question: What are the intermediate coefficients of $N_{g,n}$?

★ Compact counting

Idea: Count lattice points in $\overline{\mathcal{M}}_{g,n}$ so $\overline{N}_{g,n} = N_{g,n}$ + "stable contribution"

Example: $\overline{\mathcal{M}}_{0,5} = \mathcal{M}_{0,5} \cup 10 \times \mathcal{M}_{0,4} \times \mathcal{M}_{0,3} \cup 15 \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$

$$\Rightarrow \overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) = N_{0,5}(b_1, b_2, b_3, b_4, b_5)$$

$$+ \sum_{\text{10 labellings}} N_{0,4}(b_i, b_j, b_k, 0)\, N_{0,3}(b_\ell, b_m, 0) + \sum_{\text{15 labellings}} N_{0,3}(b_i, b_j, 0)\, N_{0,3}(b_k, 0, 0)\, N_{0,3}(b_\ell, b_m, 0)$$

Question: What is $\overline{N}_{g,n}$ counting?

- Stable ribbon graphs - come from Strebel differentials on stable curves
- Stable maps $f: C \to \mathbb{P}^1$ in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \underbrace{b_1 + b_2 + \cdots + b_n}_{\text{degree}})$
  - f regular over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$
  - $f^{-1}(\infty) = \{p_1, p_2, \ldots, p_n\}$ with ramification order b_k at p_k
  - ramification profile over 1 is $(2, 2, \ldots, 2)$
  - points with ramification order 1 over 0 are nodes

Theorem: (Do - Norbury)

- $\overline{N}_{g,n}$ has same quasipolynomial behaviour as $N_{g,n}$
- $\overline{N}_{g,n}$ has same top degree coefficients as $N_{g,n}$
- $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$
- Tutte recursion
- String / dilaton equations

Questions:

- Is there a spectral curve and an Eynard-Orantin topological recursion?
- What are the intermediate coefficients?

Ideas:

- Interpret $N_{g,n}$ as a Hurwitz problem. Does $\overline{N}_{g,n}$ arise from the Gromov - Witten / Hurwitz correspondence?
- Since $\overline{N}_{g,n}$ counts stable maps, it should be possible to interpret it as a (relative) Gromov - Witten invariant.
- Multi-cut matrix models can be used to count variants of stable ribbon graphs. Can we find a spectral curve using this idea?