THE GEOMETRY AND COMBINATORICS OF MODULI SPACES OF CURVES

Melbourne University Algebra - Geometry - Topology Seminar (30/08/10)

Norman Do

* TOY EXAMPLE

$$\mathcal{M}_\Delta = \{\text{triangles}\} / \text{congruence}$$
$$= \left\{ (a,b,c) \in \mathbb{R}_+^3 \;\middle|\; \begin{array}{l} a+b>c \\ b+c>a \\ c+a>b \end{array} \right\} / S_3$$

enumerative geometry $\longleftrightarrow$ intersection theory (cohomology)

* MODULI SPACES OF CURVES

surfaces
(g = genus, n = #boundaries)

algebraic curves $\longleftrightarrow$ Riemann surfaces $\longleftrightarrow$ hyperbolic surfaces

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable curves with } n \\ \text{smooth points labelled } 1,2,\ldots,n \end{array} \right\} / \text{isomorphism}$$

e.g.

$\in \overline{\mathcal{M}}_{5,6}$

stable = allow nodes, but require $\chi = 2 - 2g - n < 0$ on a component
$\dim \overline{\mathcal{M}}_{g,n} = 2(3g - 3 + n)$

* HYPERBOLIC GEOMETRIC CONSTRUCTION

$$\mathcal{M}_{g,n}(\underline{L}) = \left\{ \begin{array}{l} \text{genus } g \text{ hyperbolic surfaces with } n \\ \text{geodesic boundaries of lengths } \underline{L} = (L_1, L_2, \ldots, L_n) \end{array} \right\} / \text{isometry}$$

e.g.

$\in \mathcal{M}_{2,3}(L_1, L_2, L_3)$

Parts decomposition: $3g-3+n$ curves
1 length + 1 twist / curve

Teichmüller space (deformations): $\mathcal{T}_{g,n}(\underline{L}) = \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$

Moduli space (structures): $\mathcal{M}_{g,n}(\underline{L}) = \mathcal{T}_{g,n}(\underline{L}) / MCG_{g,n}$

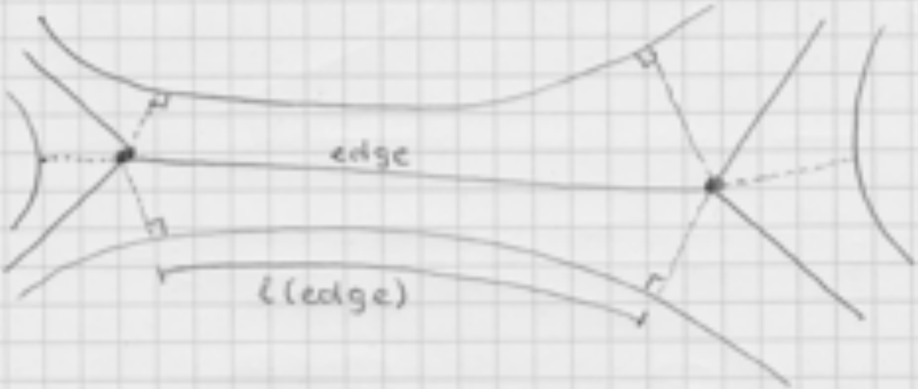
Weil-Petersson form: $\omega = \sum_{k=1}^{3g-3+n} \underset{\text{length}}{dl_k} \wedge \underset{\text{twist}}{d\tau_k}$

Fact: As $\underline{L}$ varies, symplectic structure of $\mathcal{M}_{g,n}(\underline{L})$ varies, but smooth structure does not.
In fact, $\mathcal{M}_{g,n}(\underline{L}) \cong \mathcal{M}_{g,n}$ for all $\underline{L}$.

* COMBINATORIAL MODULI SPACE

Given $S \in \mathcal{M}_{g,n}(\underline{L})$, consider the spine

$\Gamma(S) = \{ p \in S \mid$ there are ≥ 2 shortest paths from p to boundary$\}$



metric ribbon graph of type (g,n) = 1-skeleton of cell decomposition of genus g surface with n faces and a positive real assigned to each edge

$\mathcal{MRG}_{g,n}(\underline{L}) = \left\{ \begin{array}{l} \text{metric ribbon graphs of type } (g,n) \\ \text{with perimeters given by } \underline{L} \end{array} \right\} / \text{equivalence}$

Theorem (Bowditch-Epstein): $\mathcal{M}_{g,n}(\underline{L}) \cong \mathcal{MRG}_{g,n}(\underline{L})$

homeomorphic as orbifolds

* INTERSECTION THEORY ON $\overline{\mathcal{M}}_{g,n}$

π forgets $n+1$

$\mathcal{L}$ = line bundle of cotangents

$\psi_k = c_1(\sigma_k^* \mathcal{L}) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$

$\kappa_m = \pi_*(e^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$

where $e = c_1(\mathcal{L}(\sum_{k=1}^{n} \sigma_k(\overline{\mathcal{M}}_{g,n})))$

Goal: Compute $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \in \mathbb{Q}$, where $|\alpha| = 3g-3+n$

Witten - Kontsevich theorem:

natural generating function for psi-class intersection numbers = a KdV tau function

* MIRZAKHANI'S PROOF

Theorem: $[\omega] = \frac{1}{2}L_1^2\psi_1 + \frac{1}{2}L_2^2\psi_2 + \cdots + \frac{1}{2}L_n^2\psi_n + 2\pi^2\kappa_1 \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{R})$

Proof: Use symplectic reduction

Formula: $V_{g,n}(\underline{L}) = \int_{\mathcal{M}_{g,n}(\underline{L})} \frac{\omega^{3g-3+n}}{(3g-3+n)!}$

$$= \sum_{|\alpha|+m=3g-3+n} \frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\cdots\psi_n^{\alpha_n}\kappa_1^m}{2^{|\alpha|}\alpha_1!\,\alpha_2!\cdots\alpha_n!\,m!} L_1^{2\alpha_1}L_2^{2\alpha_2}\cdots L_n^{2\alpha_n}$$

Recursion: $V_{g,n}$ depends on $V_{g,n-1}$

$V_{g-1,n+1}$

$V_{g_1,n_1} \times V_{g_2,n_2}$ for $g_1+g_2=g$, $n_1+n_2=n+1$

(This is a "topological recursion" à la Eynard and Orantin.)

FORMULA + RECURSION = WITTEN - KONTSEVICH

* VOLUME RELATIONS

$V_{1,1}(L) = \frac{1}{48}(L^2 + 4\pi^2)$

$V_{0,4}(\underline{L}) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2 + 4\pi^2)$

$V_{2,1}(L) = \frac{1}{2^{16}\cdot 3^3\cdot 5}(L^2+4\pi^2)(L^2+12\pi^2)(5L^4 + 384\pi^2L^2 + 6960\pi^4)$

$V_{1,2}(\underline{L}) = \frac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2)$

Note that $V_{1,1}(2\pi i) = V_{0,4}(0,0,0,2\pi i) = V_{2,1}(2\pi i) = V_{1,2}(0, 2\pi i) = 0$.

Something interesting happens at $L = 2\pi i$.

Theorem (Do - Norbury):

$$V_{g,n+1}(\underline{L}, 2\pi i) = \sum_{k=1}^{n} \int_0^{L_k} L_k V_{g,n}(\underline{L}) \, dL_k$$

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\underline{L}, 2\pi i) = 2\pi i \, (2g-2+n) \, V_{g,n}(\underline{L})$$

$$\frac{\partial^2 V_{g,n+1}}{\partial L_{n+1}^2}(\underline{L}, 2\pi i) = \sum_{k=1}^{n} L_k \frac{\partial V_{g,n}}{\partial L_k}(\underline{L}) - (4g-4+n) V_{g,n}(\underline{L})$$

Corollary: $V_{g,0} = \frac{1}{4\pi i (g-1)} \frac{\partial V_{g,1}}{\partial L}(2\pi i)$

Proofs: 1) Use Mirzakhani to translate into intersection numbers ☑

2) Use Mirzakhani's recursion ☑

3) $2\pi i \longrightarrow$ hyperbolic cone surfaces with cone angle approaching 2π ☐

Subtle since Teichmüller theory breaks down with cone angles larger than π. $\boxed{?}$

Questions: Are there other volume relations?

Is there a natural forgetful map

$$\mathcal{M}_{g,n+1}(\underline{L}, L_{n+1}) \longrightarrow \mathcal{M}_{g,n}(\underline{L}) \quad ?$$

* COUNTING LATTICE POINTS

Question: Given n polygons, in how many ways can you glue edges to get a genus g surface?

$N_{g,n}(\underline{b})$ = # metric ribbon graphs of type (g,n) with integer edge lengths and perimeters given by $\underline{b}$, weighted by $\frac{1}{\# \mathrm{Aut}}$

Recursion (Norbury): $N_{g,n}$ depends on $N_{g,n-1}$
$N_{g-1,n+1}$
$N_{g_1,n_1} \times N_{g_2,n_2}$ for $g_1+g_2=g$, $n_1+n_2=n+1$

This "topological recursion" is a discrete version of Mirzakhani's

Corollary: $N_{g,n}$ is a quasi-polynomial, degree $2(3g-3+n)$, even and quasi-symmetric

$N_{g,n}$ counts lattice points in $\mathcal{MRG}_{g,n}(\underline{b})$, so $N_{g,n}(\underline{b}) \sim c \, V_{g,n}(\underline{b})$.

Examples: Take b's to be even

$N_{1,1}(b) = \frac{1}{48}(b^2 - 4)$

$N_{0,4}(\underline{b}) = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$

$N_{2,1}(b) = \frac{1}{2^{16} \cdot 3^2 \cdot 5}(b^2 - 4)(b^2 - 16)(b^2 - 36)(5b^2 - 32)$

$N_{1,2}(\underline{b}) = \frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$

Partial formula (Norbury):

$$[b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}]\, N_{g,n}(\underline{b}) = \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}}{2^{5g-6+2n}\, \alpha_1!\, \alpha_2! \cdots \alpha_n!} \qquad \text{for } |\alpha| = 3g-3+n$$

$N_{g,n}(\underline{0}) = \chi(\mathcal{M}_{g,n})$

PARTIAL FORMULA + RECURSION = WITTEN - KONTSEVICH

Question: What are the intermediate coefficients?

✱ THE COMPACTIFIED COUNT

Example:

$$\overline{\mathcal{M}}_{0,5} = \mathcal{M}_{0,5} \cup \overbrace{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}}^{10 \text{ copies}} \cup \overbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}^{15 \text{ copies}}$$

$$\overline{N}_{0,5}(\underline{b}) = N_{0,5}(\underline{b}) + \sum_{10 \text{ terms}} N_{0,4}(b_i, b_j, b_k, 0) \times N_{0,3}(b_l, b_m, 0)$$

$$+ \sum_{15 \text{ terms}} N_{0,3}(b_i, b_j, 0) \times N_{0,3}(b_k, 0, 0) \times N_{0,3}(b_l, b_m, 0)$$

Fact: $\overline{N}_{g,n}$ is a quasi-polynomial, degree $2(3g-3+n)$, even and quasi-symmetric

Partial formula:

$$[b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}]\, \overline{N}_{g,n}(\underline{b}) = \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}}{2^{5g-6+2n}\, \alpha_1!\, \alpha_2! \cdots \alpha_n!} \qquad \text{for } |\alpha| = 3g-3+n$$

$\overline{N}_{g,n}(\underline{0}) = \chi(\overline{\mathcal{M}}_{g,n})$

Examples: Take b's to be even

$$\overline{N}_{1,1}(b) = \frac{1}{48}(b^2 + 20)$$

$$\overline{N}_{0,4}(\underline{b}) = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$$

$$\overline{N}_{1,2}(\underline{b}) = \frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 + 48b_1^2 + 48b_2^2 + 192)$$

Claim: $\overline{N}_{g,n}$ is the "right" count

Questions:

Are the coefficients of $\overline{N}_{g,n}$ always positive?

Is there a "topological recursion" for $\overline{N}_{g,n}$?

What are the intermediate coefficients of $\overline{N}_{g,n}$?

intersection numbers on $\overline{\mathcal{M}}_{g,n}$, Hirzebruch - Riemann - Roch theorem, dimensions of spaces of sections, toric geometry, geometric quantisation

What can be said about relations between $N_{g,n}$ / $\overline{N}_{g,n}$ and matrix integrals, integrable hierarchies, factorisations in the symmetric group, Burnside's formula, free fermion expressions, ...?