

# The metric tensor in Riemann normal coordinates

In this notebook we compute the recursive sequences

$$g_{ab,d\underline{e}} = (g_{cb}\Gamma^c_{a(d),\underline{e}}) + (g_{ac}\Gamma^c_{b(d),\underline{e}})$$

$$(n+3)\Gamma^a_{d(b,c\underline{e})} = (n+1)\left(R^a_{(bcd,\underline{e})} - (\Gamma^a_{f(c}\Gamma^f_{bd),\underline{e}})\right)$$

for  $n = 1, 2, 3, \dots$ . Note in these equations that the (extended) index  $\underline{e}$  contains  $n$  normal indices.

We can then construct a Taylor series for the metric using

$$\begin{aligned} g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \dots \\ &= g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!} g_{ab,\underline{c}} x^{\underline{c}} \end{aligned}$$

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# =====

::KeepHistory(false).
::PostDefaultRules( @@collect_terms!(%), @@sumflatten!(%) ).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v}::Indices.

\nabla_{\#}::PartialDerivative.
\partial_{\#}::PartialDerivative.

g_{a b}::Metric.
\delta^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R^{a}_{b c}{}^d::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial).
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R_{a b c d}::Depends(\partial).
R^{a}_{b c d}::Depends(\partial).
\Gamma^{a}_{b c}::Depends(\partial).

# =====
#   compute the derivatives
# =====

# --- derivatives of g_{ab} -----

gab00:=g_{a b}:

gab01:=g_{c b}\Gamma^{c}_{a d} + g_{a c}\Gamma^{c}_{b d}:

gab02:=\partial_{e}{ @(gab01) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_{d}{g_{a b}} -> @(gab01)):
@distributed! (%):

gab03:=\partial_{f}{ @(gab02) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_{d}{g_{a b}} -> @(gab01)):
@distributed! (%):

gab04:=\partial_{g}{ @(gab03) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_{d}{g_{a b}} -> @(gab01)):
@distributed! (%):

gab05:=\partial_{h}{ @(gab04) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_{d}{g_{a b}} -> @(gab01)):
@distributed! (%):

# --- derivatives of \Gamma -----

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eq01:= (1/3) A^{b}A^{c} ( R^{a}_{b c d} - \Gamma^{a}_{c e}\Gamma^{e}_{b d} ):

eq02:= (6/4) A^{a}\partial_{a}{ @(eq01) }:
@distribute! (%):
@prodrule! (%):
@unwrap! (%):
@distribute! (%):

eq03:= (12/10) A^{a}\partial_{a}{ @(eq02) }:
@distribute! (%):
@prodrule! (%):
@unwrap! (%):
@distribute! (%):

eq04:= (20/18) A^{a}\partial_{a}{ @(eq03) }:
@distribute! (%):
@prodrule! (%):
@unwrap! (%):
@distribute! (%):

eq05:= (30/28) A^{a}\partial_{a}{ @(eq04) }:
@distribute! (%):
@prodrule! (%):
@unwrap! (%):
@distribute! (%):

# =====
#   switch to Riemann normal coordinates
# =====

# --- isolate \Gamma by replacing all derivatives of \Gamma with a new symbol T -----
# --- derivatives of g_{ab}

@substitute! (gab02) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute! (gab03) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute! (gab04) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute! (gab05) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):

@substitute! (gab03) (\partial_{d e}{\Gamma^{a}_{b c}} -> T_{d e}^{a}_{b c}):

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@substitute!(gab04)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):
@substitute!(gab05)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):

@substitute!(gab04)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):
@substitute!(gab05)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):

@substitute!(gab05)(\partial_{d e f g}\{\Gamma^a_{b c}\} -> T_{d e f g}^a_{b c}):

# --- derivatives of \Gamma^a_{b c}

@substitute!(eq02)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):
@substitute!(eq03)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):
@substitute!(eq04)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):
@substitute!(eq05)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):

@substitute!(eq03)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):
@substitute!(eq04)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):
@substitute!(eq05)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):

@substitute!(eq04)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):
@substitute!(eq05)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):

@substitute!(eq05)(\partial_{d e f g}\{\Gamma^a_{b c}\} -> T_{d e f g}^a_{b c}):

# --- now set \Gamma to zero -----

@substitute!!(gab01)(\Gamma^a_{b c} -> 0):
@substitute!!(gab02)(\Gamma^a_{b c} -> 0):
@substitute!!(gab03)(\Gamma^a_{b c} -> 0):
@substitute!!(gab04)(\Gamma^a_{b c} -> 0):
@substitute!!(gab05)(\Gamma^a_{b c} -> 0):

@substitute!!(eq01)(\Gamma^a_{b c} -> 0):
@substitute!!(eq02)(\Gamma^a_{b c} -> 0):
@substitute!!(eq03)(\Gamma^a_{b c} -> 0):
@substitute!!(eq04)(\Gamma^a_{b c} -> 0):
@substitute!!(eq05)(\Gamma^a_{b c} -> 0):

# --- now re-introduce the derivatives of \Gamma -----
# --- derivatives of g_{ab}

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@substitute!(gab02)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(gab03)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(gab04)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(gab05)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):

@substitute!(gab03)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(gab04)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(gab05)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):

@substitute!(gab04)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):
@substitute!(gab05)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):

@substitute!(gab05)(T_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}):

# --- derivatives of \Gamma^{a}_{b c}

@substitute!(eq02)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(eq03)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(eq04)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(eq05)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):

@substitute!(eq03)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(eq04)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(eq05)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):

@substitute!(eq04)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):
@substitute!(eq05)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):

@substitute!(eq05)(T_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}):

# =====
#   prepare first six terms in the Taylor series expansion of g_{ab}(x)
# =====

term0:= @(gab00): @distribute!(%):
term1:= @(gab01) A^d: @distribute!(%):
term2:= @(gab02) A^d A^e: @distribute!(%):
term3:= @(gab03) A^d A^e A^f: @distribute!(%):
term4:= @(gab04) A^d A^e A^f A^g: @distribute!(%):
term5:= @(gab05) A^d A^e A^f A^g A^h: @distribute!(%):

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# --- imported from symm-riemann.lib -----
qderiv01:="import symm-riemann.lib pderiv01":
@run(qderiv01){"/Users/leo/local/sh/cdbfile"}:

qderiv02:="import symm-riemann.lib pderiv02":
@run(qderiv02){"/Users/leo/local/sh/cdbfile"}:

qderiv03:="import symm-riemann.lib pderiv03":
@run(qderiv03){"/Users/leo/local/sh/cdbfile"}:

qderiv04:="import symm-riemann.lib pderiv04":
@run(qderiv04){"/Users/leo/local/sh/cdbfile"}:

qderiv05:="import symm-riemann.lib pderiv05":
@run(qderiv05){"/Users/leo/local/sh/cdbfile"}:

# =====
#   replace partial derivs of \Gamma with products and derivs of Riemann tensor
# =====

@substitute!(eq03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(eq03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(eq04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(eq04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(eq04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(eq04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(eq05)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!(eq05)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!(eq05)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(eq05)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(eq05)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(eq05)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

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@substitute!(qderiv02)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(qderiv02)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(qderiv03)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(qderiv03)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(qderiv03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(qderiv03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!!(qderiv04)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!!(qderiv04)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!!(qderiv04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!!(qderiv04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!!(qderiv04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!!(qderiv04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(term2)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(term2)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(term3)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(term3)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(term3)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(term3)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!!(term4)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!!(term4)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!!(term4)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!!(term4)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!!(term4)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!!(term4)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{d b}\} -> @(eq04)):
@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{b d}\} -> @(eq04)):

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@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!!(term5)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!!(term5)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!!(term5)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!!(term5)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute!(%):

# =====
#   replace partial derivatives of Riemann tensor with its covariant derivatives
# =====

@substitute!(qderiv03)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):

@substitute!(qderiv04)(A^{b}A^{d}A^{e}A^{f}\partial_{e f}\{R^{u}_{b d v}\} -> @(qderiv02)):
@substitute!(qderiv04)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):

@eliminate_metric!(term1):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

@eliminate_metric!(term2):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

@substitute!(term3)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):
@eliminate_metric!(%):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

@substitute!(term4)(A^{b}A^{d}A^{e}A^{f}\partial_{e f}\{R^{u}_{b d v}\} -> @(qderiv02)):
@substitute!(term4)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):

```



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@eliminate_metric! (%):
@prodsort! (%):
@rename_dummies! (%):
@canonicalise! (%):

@substitute! (term5) (A^{b}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}\{R^{u}_{b d v}\} -> @(qderiv03)):
@substitute! (term5) (A^{b}A^{d}A^{e}A^{f}\partial_{e f}\{R^{u}_{b d v}\} -> @(qderiv02)):
@substitute! (term5) (A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute! (%):
@eliminate_metric! (%):
@prodsort! (%):
@rename_dummies! (%):
@canonicalise! (%):

# =====
#   force all indices on R to be downstairs
# =====

@substitute! (term0) (R^{a}_{b c d} -> R_{a b c d},
    R_{a}^{b}_{c d} -> R_{a b c d},
    R^{a}_{b c}^{d} -> R_{a b c d},
    R_{a b}^{c}_{d} -> R_{a b c d},
    R_{a b c}^{d} -> R_{a b c d},
    R^{a}_{b}^{c}_{d} -> R_{a b c d},
    R^{a}_{b}^{c}_{d} -> R_{a b c d},
    \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
    \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
    \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

@substitute! (term1) (R^{a}_{b c d} -> R_{a b c d},
    R_{a}^{b}_{c d} -> R_{a b c d},
    R^{a}_{b c}^{d} -> R_{a b c d},
    R_{a b}^{c}_{d} -> R_{a b c d},
    R_{a b c}^{d} -> R_{a b c d},
    R^{a}_{b}^{c}_{d} -> R_{a b c d},
    R^{a}_{b}^{c}_{d} -> R_{a b c d},
    \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
    \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
    \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

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@substitute!(term2)(R^{a}_{b c d} -> R_{a b c d},
  R_{a}^{b}_{c d} -> R_{a b c d},
  R^{a}_{b c}_{d} -> R_{a b c d},
  R_{a b}^{c}_{d} -> R_{a b c d},
  R_{a b c}^{d} -> R_{a b c d},
  R^{a}_{b}^{c}_{d} -> R_{a b c d},
  R^{a}_{b}^{c}_{d} -> R_{a b c d},
  \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
  \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
  \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

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@substitute!(term3)(R^{a}_{b c d} -> R_{a b c d},
  R_{a}^{b}_{c d} -> R_{a b c d},
  R^{a}_{b c}_{d} -> R_{a b c d},
  R_{a b}^{c}_{d} -> R_{a b c d},
  R_{a b c}^{d} -> R_{a b c d},
  R^{a}_{b}^{c}_{d} -> R_{a b c d},
  R^{a}_{b}^{c}_{d} -> R_{a b c d},
  \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
  \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
  \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

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@substitute!(term4)(R^{a}_{b c d} -> R_{a b c d},
  R_{a}^{b}_{c d} -> R_{a b c d},
  R^{a}_{b c}_{d} -> R_{a b c d},
  R_{a b}^{c}_{d} -> R_{a b c d},
  R_{a b c}^{d} -> R_{a b c d},
  R^{a}_{b}^{c}_{d} -> R_{a b c d},
  R^{a}_{b}^{c}_{d} -> R_{a b c d},
  \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
  \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
  \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

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@substitute!(term5)(R^{a}_{b c d} -> R_{a b c d},
  R_{a}^{b}_{c d} -> R_{a b c d},
  R^{a}_{b c}_{d} -> R_{a b c d},
  R_{a b}^{c}_{d} -> R_{a b c d},

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R_{a b c}^{d} -> R_{a b c d},
R^{a}_{b}{}^{c}{}_{d} -> R_{a b c d},
R^{a}_{b}{}^{c}{}_{d}{}_{e} -> R_{a b c d},
\nabla^{a}{}_{R_{b c d e}} -> \nabla_{a}{}_{R_{b c d e}},
\nabla^{a}{}_{f}{}_{R_{b c d e}} -> \nabla_{a f}{}_{R_{b c d e}},
\nabla^{a}{}_{f g}{}_{R_{b c d e}} -> \nabla_{a f g}{}_{R_{b c d e}}):

```

```
# =====
#   the metric expressed in terms of the Riemann tensor and its covariant derivatives
# =====
```

```
metric:=@(term0)
+ (1/1) @(term1)
+ (1/2) @(term2)
+ (1/6) @(term3)
+ (1/24) @(term4)
+ (1/120) @(term5):
```

```
@substitute!(%)(A^{a} -> x^{a}):
@canonicalise!(%):
```

```
@print["\Btag{01}g_{ab}(x)="\~@(metric)~"+\BigO{\eps^6}\Etag{01}"];
```

$$g_{ab}(x) = \left( g_{ab} - \frac{1}{3} x^c x^d R_{acbd} - \frac{1}{6} x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45} x^c x^d x^e x^f R_{acd g} R_{b e f g} - \frac{1}{20} x^c x^d x^e x^f \nabla_{cd} R_{a e b f} + \frac{1}{45} x^c x^d x^e x^f x^g R_{acd h} \nabla_e R_{b f g h} + \frac{1}{45} x^c x^d x^e x^f x^g R_{bcd h} \nabla_e R_{a f g h} - \frac{1}{90} x^c x^d x^e x^f x^g \nabla_{cde} R_{a f b g} \right) + \mathcal{O}(\epsilon^6)$$

```
# =====
#   rewrite the metric in the form  $g_{ab}(x) = g_{ab} + g_{abc} x^c + g_{abcd} x^c x^d + g_{abcde} x^c x^d x^e + \dots$ 
# =====
```

```
gab:=          @(term0):
gabc:=          @(term1): @sym!({_a,_b}):
gab cd:=        1/2 @(term2): @sym!({_a,_b}):
gab cde:=        1/6 @(term3): @sym!({_a,_b}):
gab cdef:=       1/24 @(term4): @sym!({_a,_b}):
gab cdefg:=      1/120 @(term5): @sym!({_a,_b}):
```

```
@substitute!(gab)(A^{a} -> 1); "gab.del"
@substitute!(gabc)(A^{a} -> 1); "gabc.del"
@substitute!(gab cd)(A^{a} -> 1); "gab cd.del"
@substitute!(gab cde)(A^{a} -> 1); "gab cde.del"
@substitute!(gab cdef)(A^{a} -> 1); "gab cdef.del"
@substitute!(gab cdefg)(A^{a} -> 1); "gab cdefg.del"
```

```
@print["g_{abc}="~@(gabc)];  
@print["g_{abcd}="~@(gabcd)];  
@print["g_{abcde}="~@(gabcde)];  
@print["g_{abcdef}="~@(gabcdef)];  
@print["g_{abcdefg}="~@(gabcdefg)];
```

```
# =====
#   the metric in the form  $g_{ab}(x) = g_{ab} + g_{abc} x^c + g_{abcd} x^c x^d + g_{abcde} x^c x^d x^e + \dots$ 
# =====
```

$$g_{ab}(x) = g_{ab} + g_{abc} x^c + g_{abcd} x^c x^d + g_{abcde} x^c x^d x^e + g_{abcdef} x^c x^d x^e x^f + g_{abcdefg} x^c x^d x^e x^f x^g + \mathcal{O}(\epsilon^6)$$

$$g_{abc} = 0$$

$$g_{abcd} = \left( -\frac{1}{6} R_{acbd} - \frac{1}{6} R_{bcad} \right)$$

$$g_{abcde} = \left( -\frac{1}{12} \nabla_c R_{adbce} - \frac{1}{12} \nabla_c R_{bdae} \right)$$

$$g_{abcdef} = \left( \frac{1}{45} R_{acd} R_{befg} + \frac{1}{45} R_{bcd} R_{aefg} - \frac{1}{40} \nabla_{cd} R_{aebf} - \frac{1}{40} \nabla_{cd} R_{beaf} \right)$$

$$g_{abcdefg} = \left( \frac{1}{45} R_{acd} \nabla_e R_{bfgh} + \frac{1}{45} R_{bcd} \nabla_e R_{afgh} - \frac{1}{180} \nabla_{cde} R_{afbg} - \frac{1}{180} \nabla_{cde} R_{bfag} \right)$$

```

# =====
#   compute the first partial  $g_{ab,c}(x) = g_{abc} + g_{abcd} x^d + g_{abdc} x^d + \dots$ 
#   =====

x^{a}::Depends(\partial).

g_{a b c d}::TableauSymmetry(shape={2}, indices={2,3}).
g_{a b c d e}::TableauSymmetry(shape={3}, indices={2,3,4}).
g_{a b c d e f}::TableauSymmetry(shape={4}, indices={2,3,4,5}).
g_{a b c d e f g}::TableauSymmetry(shape={5}, indices={2,3,4,5,6}).

metric:=g_{a b} + g_{a b c d} x^c x^d
           + g_{a b c d e} x^c x^d x^e
           + g_{a b c d e f} x^c x^d x^e x^f
           + g_{a b c d e f g} x^c x^d x^e x^f x^g:

dmetric:=\partial_{c}{@(metric)}:
@distribute! (%):
@prodrule! (%):
@unwrap! (%):
@substitute! (%) (\partial_{c}{g_{a b}} -> 0):
@substitute! (%) (\partial_{a}{x^{b}}->\delta^{b}_{a}):
@eliminate_kr! (%):
@rename_dummies! (%):
@canonicalise! (%):

@print["\Btag{02}g_{ab,c}(x)="~@(dmetric)~"+\Big0{\eps^6}\Etag{02}"];

# gabdefg as calculated earlier was not symmetrised over cdefg, do so now

@sym!(gabcd){_c,_d}:
@sym!(gabcde){_c,_d,_e}:
@sym!(gabcdef){_c,_d,_e,_f}:
@sym!(gabcdefg){_c,_d,_e,_f,_g}:

@substitute!(metric)(g_{a b c d} -> @(gabcd), g_{a b c d e} -> @(gabcde),
                    g_{a b c d e f} -> @(gabcdef), g_{a b c d e f g} -> @(gabcdefg)):
@distribute! (%):
@canonicalise! (%); "metric.del"

```

```

@substitute!(dmetric)(g_{a b c d} -> @(gabcd), g_{a b c d e} -> @(gabcde),
                    g_{a b c d e f} -> @(gabcdef), g_{a b c d e f g} -> @(gabcdefg));

@distribute!(%):
@canonicalise!(%); "dmetric.del"

@print["\Btag{03}g_{ab,c}(x)="\~@(dmetric)~"+\Big0{\eps^6}\Etag{03}"];

```

$$g_{ab,c}(x) = (2g_{abcd}x^d + 3g_{abcde}x^d x^e + 4g_{abcdef}x^d x^e x^f + 5g_{abcdefg}x^d x^e x^f x^g) + \mathcal{O}(\epsilon^6)$$

$$\begin{aligned}
g_{ab,c}(x) = & \left( -\frac{1}{3} R_{acbd}x^d - \frac{1}{3} R_{adbc}x^d - \frac{1}{6} \nabla_c R_{adbe}x^d x^e - \frac{1}{6} \nabla_d R_{acbe}x^d x^e - \frac{1}{6} \nabla_d R_{aebc}x^d x^e + \frac{2}{45} R_{acde}R_{bf dg}x^e x^f x^g - \frac{2}{45} R_{adce}R_{bf eg}x^d x^f x^g + \frac{2}{45} R_{adef}R_{bceg}x^d x^f x^g \right. \\
& - \frac{2}{45} R_{adef}R_{bgce}x^d x^f x^g - \frac{1}{10} \nabla_{cd} R_{aebf}x^d x^e x^f - \frac{1}{20} \nabla_{de} R_{acbf}x^d x^e x^f - \frac{1}{20} \nabla_{de} R_{afbc}x^d x^e x^f + \frac{1}{45} R_{acde} \nabla_f R_{bg dh}x^e x^f x^g x^h - \frac{1}{45} R_{adce} \nabla_f R_{bge h}x^d x^f x^g x^h \\
& + \frac{1}{45} R_{adef} \nabla_c R_{bge h}x^d x^f x^g x^h + \frac{1}{45} R_{adef} \nabla_g R_{bce h}x^d x^f x^g x^h - \frac{1}{45} R_{adef} \nabla_g R_{bhce}x^d x^f x^g x^h + \frac{1}{45} R_{bcde} \nabla_f R_{ag dh}x^e x^f x^g x^h - \frac{1}{45} R_{bdce} \nabla_f R_{age h}x^d x^f x^g x^h \\
& + \frac{1}{45} R_{bdef} \nabla_c R_{age h}x^d x^f x^g x^h + \frac{1}{45} R_{bdef} \nabla_g R_{ace h}x^d x^f x^g x^h - \frac{1}{45} R_{bdef} \nabla_g R_{ahce}x^d x^f x^g x^h - \frac{1}{30} \nabla_{cde} R_{af bg}x^d x^e x^f x^g - \frac{1}{90} \nabla_{def} R_{acbg}x^d x^e x^f x^g \\
& \left. - \frac{1}{90} \nabla_{def} R_{agbc}x^d x^e x^f x^g \right) + \mathcal{O}(\epsilon^6)
\end{aligned}$$



```

# =====
#   the partial derivatives of the Riemann tensor in terms of its covariant derivatives
# =====

# using @canonicalise after index gymnastics gives more compact results than using it here

# @canonicalise!(qderiv01):
# @canonicalise!(qderiv02):
# @canonicalise!(qderiv03):
# @canonicalise!(qderiv04):

tmp01:= @(qderiv01): @prodsort!(%): @rename_dummies!(%):
tmp02:= @(qderiv02): @prodsort!(%): @rename_dummies!(%):
tmp03:=2 @(qderiv03): @prodsort!(%): @rename_dummies!(%):
tmp04:=5 @(qderiv04): @prodsort!(%): @rename_dummies!(%):

# this use of @canonicalise gives more compact results than if used ealier, see above

@canonicalise!(tmp01): @factor_out!!(A^{a}): @substitute!(A^{a} -> 1):
@canonicalise!(tmp02): @factor_out!!(A^{a}): @substitute!(A^{a} -> 1):
@canonicalise!(tmp03): @factor_out!!(A^{a}): @substitute!(A^{a} -> 1):
@canonicalise!(tmp04): @factor_out!!(A^{a}): @substitute!(A^{a} -> 1):

tmp11:= -@(tmp01): @substitute!(\nabla_{m}{R^{b}}_{c d}) -> \nabla_{m}{R^{b}}_{c d e}):
tmp12:= -@(tmp02): @substitute!(\nabla_{m n}{R^{b}}_{c d}) -> \nabla_{m n}{R^{b}}_{c d e}):
tmp13:= -@(tmp03): @substitute!(\nabla_{m n o}{R^{b}}_{c d}) -> \nabla_{m n o}{R^{b}}_{c d e}):
tmp14:= -@(tmp04): @substitute!(\nabla_{m n o p}{R^{b}}_{c d}) -> \nabla_{m n o p}{R^{b}}_{c d e}):

@print["\Btag{21}R^{u}_{(bc\Dot v},a)"]="~@(tmp11)~"\Etag{21}";
@print["\Btag{22}R^{u}_{(cd\Dot v},ab)"]="~@(tmp12)~"\Etag{22}";
@print["\Btag{23}2 R^{u}_{(de\Dot v},abc)"]="~\killL"~@(tmp13)~"\killR\Etag{23}";
@print["\Btag{24}5 R^{u}_{(ef\Dot v},abcd)"]="~\killL"~@(tmp14)~"\killR\Etag{24}";

```

$$R^u_{(bc\dot{v},a)} = \nabla_a R^u_{bcv}$$

$$R^u_{(cd\dot{v},ab)} = \nabla_{ab} R^u_{cdv}$$

$$2R^u_{(de\dot{v},abc)} = 2\nabla_{abc} R^u_{dev} - R_{vabf} \nabla_c R^u_{def} + R^u_{abf} \nabla_c R_{vdef}$$

$$5R^u_{(ef\dot{v},abcd)} = 5\nabla_{abcd} R^u_{efv} - 7R_{vabg} \nabla_{cd} R^u_{efg} + 7R^u_{abg} \nabla_{cd} R_{vefg}$$

```
# =====  
#   export metric and dmetric  
# =====  
  
com:="open metric.lib":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib metric.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib dmetric.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib gab.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib gabc.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib gabcd.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib gabcde.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib gabcdef.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric.lib gabcdefg.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:
```