

```

::KeepHistory(false).
::PostDefaultRules( @@collect_terms!(%), @@sumflatten!(%) ).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,t,u#,v#}::Indices.

\nabla{#}::PartialDerivative.
\partial{#}::PartialDerivative.

{g_{a b},x^{a},\Delta{x}^{a},R_{a b c d},
 \nabla_{a}{R_{p q r s}},\nabla_{a b}{R_{p q r s}},\nabla_{a b c}{R_{p q r s}},
 \partial_{a}{R_{p q r s}},\partial_{a b}{R_{p q r s}},\partial_{a b c}{R_{p q r s}}}:SortOrder.

# === metric =====

metric:="import metric.lib metric":
@run(metric){"/Users/leo/local/sh/cdbfile"}:

poly:=@(metric):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:

tmp:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05):
@prodsort!(%): @rename_dummies!(%):
metric:= 180 @(tmp):

@print["\Btag{01a}180g_{ab}(x)=\killL"~"@ (metric)~"\killR+\Big0{\eps^6}\Etag{01a}"];

```

$$\begin{aligned}
180g_{ab}(x) = & 180g_{ab} - 60x^cx^dR_{acbd} - 30x^cx^dx^e\nabla_cR_{adb e} + 8x^cx^dx^ex^fR_{acgd}R_{begf} - 9x^cx^dx^ex^f\nabla_{cd}R_{aebf} \\
& + 4x^cx^dx^ex^fx^gR_{achd}\nabla_eR_{bfhg} + 4x^cx^dx^ex^fx^gR_{bchd}\nabla_eR_{afhg} - 2x^cx^dx^ex^fx^g\nabla_{cde}R_{afbg} + \mathcal{O}(\epsilon^6)
\end{aligned}$$

```
# === inverse metric =====

metric:="import metric-inv.lib metric":
@run(metric){"/Users/leo/local/sh/cdbfile"}:

poly:=@(metric):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:

tmp:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05):
@prodsort!(%): @rename_dummies!(%):
metric:= 180 @(tmp):

@print["\Btag{02a}180g^{ab}(x)=\killL"~"@metric)~"\killR+\Big0{\eps^6}\Etag{02a}"];

```

$$180g^{ab}(x) = 180g^{ab} + 60x^cx^dR^a{}_c{}^b{}_d + 30x^cx^dx^e\nabla_cR^a{}_d{}^b{}_e + 12x^cx^dx^ex^fR^a{}_{cdg}R^b{}_{efg} + 9x^cx^dx^ex^f\nabla_{cd}R^a{}_e{}^b{}_f \\ + 6x^cx^dx^ex^fx^gR^a{}_{cdh}\nabla_eR^b{}_{fgh} + 6x^cx^dx^ex^fx^gR^b{}_{cdh}\nabla_eR^a{}_{fgh} + 2x^cx^dx^ex^fx^g\nabla_{cde}R^a{}_f{}^b{}_g + \mathcal{O}(\epsilon^6)$$

```
# === generalised gamma in rnc =====

genx2rnc:="import gen-gamma.lib genx2rnc":
@run(genx2rnc){"/Users/leo/local/sh/cdbfile"}:

poly:=@(genx2rnc):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:

```

```
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:
```

```
tmp:= 2 @(term02):
term02:=@(tmp):
```

```
tmp:= 6 @(term03): @distribute!(%):
@factor_out!!(tmp)(\Delta{x}^a):
term03:=@(tmp):
```

```
tmp:= 24 @(term04): @distribute!(%):
@factor_out!!(tmp)(\Delta{x}^a):
term04:=@(tmp):
```

```
tmp:= 360 @(term05): @distribute!(%):
@factor_out!!(tmp)(\Delta{x}^a):
term05:=@(tmp):
```

```
genx2rnc:=@(term00) + @(term01) + (1/2) @(term02) + (1/6) @(term03) + (1/24) @(term04) + (1/360) @(term05):
@print["\Btag{03a}y^a(x)=\""~@(genx2rnc)~"+\BigO{\eps^6}\Etag{03a}"];
```

$$y^a(x) = \left(\Delta x^a + \frac{1}{2} \Delta x^b \Delta x^c \Gamma^a_{bc} + \frac{1}{6} \Delta x^b \Delta x^c \Delta x^d (\Gamma^a_{be} \Gamma^e_{cd} + \partial_b \Gamma^a_{cd}) + \frac{1}{24} \Delta x^b \Delta x^c \Delta x^d \Delta x^e (2 \Gamma^a_{bf} \partial_c \Gamma^f_{de} + \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \partial_{bc} \Gamma^a_{de}) \right. \\ \left. + \frac{1}{360} \Delta x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f (-4 \Gamma^a_{bg} \Gamma^g_{ch} \Gamma^h_{di} \Gamma^i_{ef} + 2 \Gamma^a_{bg} \Gamma^g_{ch} \partial_d \Gamma^h_{ef} + 3 \Gamma^a_{bg} \Gamma^g_{hi} \Gamma^h_{cd} \Gamma^i_{ef} + 6 \Gamma^a_{bg} \Gamma^h_{cd} \partial_h \Gamma^g_{ef} - 6 \Gamma^a_{bg} \Gamma^h_{cd} \partial_e \Gamma^g_{fh} + 9 \Gamma^a_{bg} \partial_{cd} \Gamma^g_{ef} \right. \\ \left. + 4 \Gamma^a_{gh} \Gamma^g_{bc} \Gamma^h_{di} \Gamma^i_{ef} + 13 \Gamma^a_{gh} \Gamma^g_{bc} \partial_d \Gamma^h_{ef} + \Gamma^g_{bc} \Gamma^h_{dg} \partial_h \Gamma^a_{ef} - 4 \Gamma^g_{bc} \Gamma^h_{dg} \partial_e \Gamma^a_{fh} + 7 \partial_g \Gamma^a_{bc} \partial_d \Gamma^g_{ef} + 2 \partial_b \Gamma^a_{cg} \partial_d \Gamma^g_{ef} + 3 \Gamma^g_{bc} \Gamma^h_{de} \partial_g \Gamma^a_{fh} \right. \\ \left. + 3 \Gamma^g_{bc} \Gamma^h_{de} \partial_f \Gamma^a_{gh} + 6 \Gamma^g_{bc} \partial_{dg} \Gamma^a_{ef} - 3 \Gamma^g_{bc} \partial_{de} \Gamma^a_{fg} + 3 \partial_{bcd} \Gamma^a_{ef} \right) + \mathcal{O}(\epsilon^6)$$

```
@substitute!(term02)(\Delta{x}^a -> 1):
@substitute!(term03)(\Delta{x}^a -> 1):
@substitute!(term04)(\Delta{x}^a -> 1):
@substitute!(term05)(\Delta{x}^a -> 1):
```

```

@print["\Btag{03b}
y^a= \Delta x^a
+ y^a_{bc}\Delta x^b\Delta x^c
+ y^a_{bcd}\Delta x^b\Delta x^c\Delta x^d
+ y^a_{bcde}\Delta x^b\Delta x^c\Delta x^d\Delta x^e
+ y^a_{bcdef}\Delta x^b\Delta x^c\Delta x^d\Delta x^e\Delta x^f
+ \BigO{\eps^6}\Etag{03b}"];

@print["\Btag{03c}\phantom{00}2y^a_{bc}=\sim@(term02)\sim\Etag{03c}"];
@print["\Btag{03d}\phantom{00}6y^a_{bcd}=\killL"\sim@(term03)\sim\killR\Etag{03d}"];
@print["\Btag{03e}\phantom{0}24y^a_{bcde}=\killL"\sim@(term04)\sim\killR\Etag{03e}"];
@print["\Btag{03f}360y^a_{bcdef}=\killL"\sim@(term05)\sim\killR\Etag{03f}"];

```

$$y^a = \Delta x^a + y^a_{bc} \Delta x^b \Delta x^c + y^a_{bcd} \Delta x^b \Delta x^c \Delta x^d + y^a_{bcde} \Delta x^b \Delta x^c \Delta x^d \Delta x^e + y^a_{bcdef} \Delta x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f + \mathcal{O}(\epsilon^6)$$

$$2y^a_{bc} = \Gamma^a{}_{bc}$$

$$6y^a_{bcd} = \Gamma^a{}_{be} \Gamma^e{}_{cd} + \partial_b \Gamma^a{}_{cd}$$

$$24y^a_{bcde} = 2\Gamma^a{}_{bf} \partial_c \Gamma^f{}_{de} + \Gamma^a{}_{fg} \Gamma^f{}_{bc} \Gamma^g{}_{de} + \Gamma^f{}_{bc} \partial_f \Gamma^a{}_{de} + \partial_{bc} \Gamma^a{}_{de}$$

$$\begin{aligned}
360y^a_{bcdef} = & -4\Gamma^a{}_{bg} \Gamma^g{}_{ch} \Gamma^h{}_{di} \Gamma^i{}_{ef} + 2\Gamma^a{}_{bg} \Gamma^g{}_{ch} \partial_d \Gamma^h{}_{ef} + 3\Gamma^a{}_{bg} \Gamma^g{}_{hi} \Gamma^h{}_{cd} \Gamma^i{}_{ef} + 6\Gamma^a{}_{bg} \Gamma^h{}_{cd} \partial_h \Gamma^g{}_{ef} - 6\Gamma^a{}_{bg} \Gamma^h{}_{cd} \partial_e \Gamma^g{}_{fh} \\
& + 9\Gamma^a{}_{bg} \partial_{cd} \Gamma^g{}_{ef} + 4\Gamma^a{}_{gh} \Gamma^g{}_{bc} \Gamma^h{}_{di} \Gamma^i{}_{ef} + 13\Gamma^a{}_{gh} \Gamma^g{}_{bc} \partial_d \Gamma^h{}_{ef} + \Gamma^g{}_{bc} \Gamma^h{}_{dg} \partial_h \Gamma^a{}_{ef} - 4\Gamma^g{}_{bc} \Gamma^h{}_{dg} \partial_e \Gamma^a{}_{fh} + 7\partial_g \Gamma^a{}_{bc} \partial_d \Gamma^g{}_{ef} \\
& + 2\partial_b \Gamma^a{}_{cg} \partial_d \Gamma^g{}_{ef} + 3\Gamma^g{}_{bc} \Gamma^h{}_{de} \partial_g \Gamma^a{}_{fh} + 3\Gamma^g{}_{bc} \Gamma^h{}_{de} \partial_f \Gamma^a{}_{gh} + 6\Gamma^g{}_{bc} \partial_{dg} \Gamma^a{}_{ef} - 3\Gamma^g{}_{bc} \partial_{de} \Gamma^a{}_{fg} + 3\partial_{bcd} \Gamma^a{}_{ef}
\end{aligned}$$

```

genGamma02:="import gen-gamma.lib genGamma02":
@run(genGamma02){"/Users/leo/local/sh/cdbfile"}:

poly:=@(genGamma02):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:

tmp:= 12 @(term02):
@factor_out!!(tmp)(x^{a}):
term02:=@(tmp):

tmp:= 180 @(term03):
@factor_out!!(tmp)(x^{a}):
term03:=@(tmp):

tmp:= 180 @(term04):
@factor_out!!(tmp)(x^{a}):
term04:=@(tmp):

tmp:=@(term00) + @(term01) + (1/12) @(term02) + (1/180) @(term03) + (1/180) @(term04):
genGamma02:= 180 @(tmp):

@print["\Btag{04a}180\genGammaA=\killL"~@(genGamma02)~"\killR+\BigO{\eps^6}\Etag{04a}"];

# -----

genGamma03:="import gen-gamma.lib genGamma03":
@run(genGamma03){"/Users/leo/local/sh/cdbfile"}:

poly:=@(genGamma03):

x^{a}::Weight(label=myterms,value=1).

```

```

\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:

tmp:= 60 @(term02):
@factor_out!!(tmp)(x^{a}):
term02:=@(tmp):

tmp:= 18 @(term03):
@factor_out!!(tmp)(x^{a}):
term03:=@(tmp):

tmp:=@(term00) + @(term01) + (1/60) @(term02) + (1/18) @(term03):
genGamma03:= 180 @(tmp):

@print["\Btag{04b}180\genGammaB=\killL"~"@(\genGamma03)~"\killR+\BigO{\eps^6}\Etag{04b}"];

# -----

genGamma04:="import gen-gamma.lib genGamma04":
@run(genGamma04){"/Users/leo/local/sh/cdbfile"}:

poly:=@(genGamma04):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:

tmp:= 18 @(term02):
@factor_out!!(tmp)(x^{a}):
term02:=@(tmp):

tmp:=@(term00) + @(term01) + (1/18) @(term02):

```

```

genGamma04:= 18 @(tmp):

@print["\Btag{04c}\phantom{0}18\genGammaC=\killL"~@(genGamma04)~"\killR+\Big0{\eps^6}\Etag{04c}"];

# -----

genGamma05:="import gen-gamma.lib genGamma05":
@run(genGamma05){"/Users/leo/local/sh/cdbfile"}:

poly:=@(genGamma05):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:

tmp:= 3 @(term01):
@factor_out!!(tmp)(x^{a}):
term01:=@(tmp):

tmp:=@(term00) + (1/3) @(term01):
genGamma05:= 3 @(tmp):

@print["\Btag{04d}\phantom{00}3\genGammaD="~@(genGamma05)~"+\Big0{\eps^6}\Etag{04d}"];

@print["\Btag{04e}\phantom{000}\genGammaE=\Big0{\eps^6}\Etag{04e}"];

```

$$\begin{aligned}
180\Gamma_{(bc)}^a(x) &= 120 x^d R^a{}_{bdc} + 15 x^d x^e (2 \nabla_b R^a{}_{dec} + 4 \nabla_d R^a{}_{bec} + \nabla^a R_{dbec}) \\
&\quad + x^d x^e x^f (32 R^a{}_{deg} R_{fbcg} - 16 R^a{}_{bdg} R_{ecfg} - 8 R^a{}_{dbg} R_{ecfg} + 18 \nabla_{db} R^a{}_{efc} + 18 \nabla_{de} R^a{}_{bfc} - 8 R^a{}_{gdb} R_{ecfg} + 9 \nabla^a{}_d R_{ebfc}) \\
&\quad + x^d x^e x^f x^g (16 R_{dbch} \nabla_e R^a{}_{fgh} + 6 R^a{}_{deh} \nabla_b R_{fcgh} + 16 R^a{}_{deh} \nabla_f R_{gbch} - 8 R^a{}_{bdh} \nabla_e R_{fcgh} - 4 R^a{}_{dbh} \nabla_e R_{fcgh} - 4 R_{dbeh} \nabla_c R^a{}_{fgh} - 8 R_{dbeh} \nabla_f R^a{}_{cgh} \\
&\quad - 4 R_{dbeh} \nabla_f R^a{}_{gch} + 6 \nabla_{deb} R^a{}_{fgc} + 4 \nabla_{def} R^a{}_{bgc} - 5 R^a{}_{deh} \nabla_h R_{fbgc} - 4 R^a{}_{hdb} \nabla_e R_{fcgh} - 4 R_{dbeh} \nabla^a R_{fcgh} - 4 R_{dbeh} \nabla_f R^a{}_{hgc} + 3 \nabla^a{}_{de} R_{fbgc}) + \mathcal{O}(\epsilon^6) \\
\\
180\Gamma_{(bcd)}^a(x) &= 90 x^e \nabla_b R^a{}_{ced} + 3 x^e x^f (8 R^a{}_{ebg} R_{fcdg} + 32 R^a{}_{beg} R_{fcdg} - 8 R^a{}_{bcg} R_{edfg} + 18 \nabla_{eb} R^a{}_{cfd} + 6 \nabla_{bc} R^a{}_{efd} + 24 R^a{}_{geb} R_{fcdg} + 3 \nabla^a{}_b R_{ecfd}) \\
&\quad + 10 x^e x^f x^g (2 R_{ebch} \nabla_d R^a{}_{fgh} + 2 R_{ebch} \nabla_h R^a{}_{fgd} + 4 R_{ebch} \nabla_f R^a{}_{dgh} + 4 R_{ebch} \nabla_f R^a{}_{hgd} + 2 R_{ebch} \nabla^a R_{fdgh} + 2 R^a{}_{beh} \nabla_c R_{fdgh} + 4 R^a{}_{beh} \nabla_f R_{gcdh} \\
&\quad - R^a{}_{beh} \nabla_h R_{fcgd} + 2 R^a{}_{heb} \nabla_c R_{fdgh} + 4 R^a{}_{heb} \nabla_f R_{gcdh} - R^a{}_{heb} \nabla_h R_{fcgd}) + \mathcal{O}(\epsilon^6)
\end{aligned}$$

$$18\Gamma_{(bcde)}^a(x) = 8x^f R^a{}_{bcg} R_{fdeg} + x^f x^g (2R^a{}_{bch} \nabla_d R_{fegh} + 4R^a{}_{bch} \nabla_f R_{gdeh} - R^a{}_{bch} \nabla_h R_{fdge} + 2R_{fbch} \nabla_d R^a{}_{geh} + 10R_{fbch} \nabla_d R^a{}_{hge} + 4R_{fbch} \nabla_g R^a{}_{deh} \\ + 8R_{fbch} \nabla_h R^a{}_{dge} + 2R_{fbch} \nabla^a R_{gdeh} + 12R_{fbch} \nabla_d R^a{}_{egh} + 6R^a{}_{bfh} \nabla_c R_{gdeh} + 6R^a{}_{hfb} \nabla_c R_{gdeh}) + \mathcal{O}(\epsilon^6)$$

$$3\Gamma_{(bcdef)}^a(x) = x^g (2R^a{}_{bch} \nabla_d R_{gefh} + 3R_{gbch} \nabla_d R^a{}_{efh}) + \mathcal{O}(\epsilon^6)$$

$$\Gamma_{(bcdefg)}^a(x) = \mathcal{O}(\epsilon^6)$$

```
# === geodesic ivp =====

ivp:="import geodesic-ivp.lib sol":
@run(ivp){"/Users/leo/local/sh/cdbfile"}:

poly:=@ (ivp):
@distribute! (%):

x^{a}::Weight(label=myterms,value=1).
\thetdotx^{a}::Weight(label=myterms,value=1).

term00:=@ (poly): @keep_weight!(term00){myterms}{0}:
term01:=@ (poly): @keep_weight!(term01){myterms}{1}:
term02:=@ (poly): @keep_weight!(term02){myterms}{2}:
term03:=@ (poly): @keep_weight!(term03){myterms}{3}:
term04:=@ (poly): @keep_weight!(term04){myterms}{4}:
term05:=@ (poly): @keep_weight!(term05){myterms}{5}:
term06:=@ (poly): @keep_weight!(term06){myterms}{6}:

ivp:=@ (term00) + @ (term01) + @ (term02) + @ (term03) + @ (term04) + @ (term05) + @ (term06):
@print["\Btag{06}x^{a}(s)="\~@ (ivp) ~"+\BigO{\eps^6}\Etag{06}"];

poly:=@ (ivp):
@distribute! (%):
@substitute! (%)(s -> 1):

\thetdotx^{a}::Weight(label=sterms,value=1).
```



```

term00:=@(poly): @keep_weight!(term00){sterms}{0}:
term01:=@(poly): @keep_weight!(term01){sterms}{1}:
term02:=@(poly): @keep_weight!(term02){sterms}{2}:
term03:=@(poly): @keep_weight!(term03){sterms}{3}:
term04:=@(poly): @keep_weight!(term04){sterms}{4}:
term05:=@(poly): @keep_weight!(term05){sterms}{5}:

tmp:= -360 @(term02):
@factor_out!!(tmp)(\thedotx^{a}):
term02:=@(tmp):

tmp:= -1080 @(term03):
@factor_out!!(tmp)(\thedotx^{a}):
term03:=@(tmp):

tmp:= -432 @(term04):
@factor_out!!(tmp)(\thedotx^{a}):
term04:=@(tmp):

tmp:= -360 @(term05):
@factor_out!!(tmp)(\thedotx^{a}):
term05:=@(tmp):

ivp:=@(term00) + s @(term01) - (1/360) s**2 @(term02)
      - (1/1080) s**3 @(term03)
      - (1/432) s**4 @(term04)
      - (1/360) s**5 @(term05):
@print["\Btag{07}x^{a}(s)=\""~@(ivp)~"+"~\Big0{\eps^6}\Etag{07}"];

```

$$\begin{aligned}
x^a(s) = & \left(x^a + s\dot{x}^a - \frac{1}{3} s^2 \dot{x}^b \dot{x}^c x^d R^a{}_{bcd} - \frac{1}{12} s^2 \dot{x}^b \dot{x}^c x^d x^e \nabla_b R^a{}_{dec} - \frac{1}{6} s^2 \dot{x}^b \dot{x}^c x^d x^e \nabla_d R^a{}_{bec} - \frac{1}{24} s^2 \dot{x}^b \dot{x}^c x^d x^e \nabla^a R_{dbec} - \frac{1}{12} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e \nabla_b R^a{}_{ced} \right. \\
& - \frac{4}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f R^a{}_{deg} R_{fbcg} + \frac{2}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f R^a{}_{bdg} R_{ecfg} + \frac{1}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f R^a{}_{dbg} R_{ecfg} - \frac{1}{20} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f \nabla_{db} R^a{}_{efc} - \frac{1}{20} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f \nabla_{de} R^a{}_{bfc} \\
& + \frac{1}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f R^a{}_{gdb} R_{ecfg} - \frac{1}{40} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f \nabla^a{}_d R_{ebfc} - \frac{1}{45} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f R^a{}_{ebg} R_{fcdg} - \frac{4}{45} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f R^a{}_{beg} R_{fcdg} + \frac{1}{45} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f R^a{}_{bcg} R_{edfg} \\
& - \frac{1}{20} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f \nabla_{eb} R^a{}_{cfd} - \frac{1}{60} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f \nabla_{bc} R^a{}_{efd} - \frac{1}{15} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f R^a{}_{geb} R_{fcdg} - \frac{1}{120} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f \nabla^a{}_b R_{ecfd} - \frac{1}{54} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f R^a{}_{bcg} R_{fdeg} \\
& - \frac{2}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R_{dbch} \nabla_e R^a{}_{fgh} - \frac{1}{60} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R^a{}_{deh} \nabla_b R_{fcgh} - \frac{2}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R^a{}_{deh} \nabla_f R_{gbch} + \frac{1}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R^a{}_{bdh} \nabla_e R_{fcgh} \\
& + \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R^a{}_{dbh} \nabla_e R_{fcgh} + \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R_{dbeh} \nabla_c R^a{}_{fgh} + \frac{1}{45} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R_{dbeh} \nabla_f R^a{}_{cgh} + \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R_{dbeh} \nabla_f R^a{}_{gch} \\
& - \frac{1}{60} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g \nabla_{deb} R^a{}_{fgc} - \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g \nabla_{def} R^a{}_{bgc} + \frac{1}{72} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R^a{}_{deh} \nabla_h R_{fbgc} + \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R^a{}_{hdb} \nabla_e R_{fcgh} \\
& + \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R_{dbeh} \nabla^a R_{fcgh} + \frac{1}{90} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g R_{dbeh} \nabla_f R^a{}_{hgc} - \frac{1}{120} s^2 \dot{x}^b \dot{x}^c x^d x^e x^f x^g \nabla^a{}_{de} R_{fbgc} - \frac{1}{54} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R_{ebch} \nabla_d R^a{}_{fgh} \\
& - \frac{1}{54} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R_{ebch} \nabla_h R^a{}_{fgd} - \frac{1}{27} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R_{ebch} \nabla_f R^a{}_{dgh} - \frac{1}{27} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R_{ebch} \nabla_f R^a{}_{hgd} - \frac{1}{54} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R_{ebch} \nabla^a R_{fdgh} \\
& - \frac{1}{54} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R^a{}_{beh} \nabla_c R_{fdgh} - \frac{1}{27} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R^a{}_{beh} \nabla_f R_{gcdh} + \frac{1}{108} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R^a{}_{beh} \nabla_h R_{fcgd} - \frac{1}{54} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R^a{}_{heb} \nabla_c R_{fdgh} \\
& - \frac{1}{27} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R^a{}_{heb} \nabla_f R_{gcdh} + \frac{1}{108} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e x^f x^g R^a{}_{heb} \nabla_h R_{fcgd} - \frac{1}{216} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R^a{}_{bch} \nabla_d R_{fegh} - \frac{1}{108} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R^a{}_{bch} \nabla_f R_{gdeh} \\
& + \frac{1}{432} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R^a{}_{bch} \nabla_h R_{fdge} - \frac{1}{216} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R_{fbch} \nabla_d R^a{}_{geh} - \frac{5}{216} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R_{fbch} \nabla_d R^a{}_{hge} - \frac{1}{108} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R_{fbch} \nabla_g R^a{}_{deh} \\
& - \frac{1}{54} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R_{fbch} \nabla_h R^a{}_{dge} - \frac{1}{216} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R_{fbch} \nabla^a R_{gdeh} - \frac{1}{36} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R_{fbch} \nabla_d R^a{}_{egh} - \frac{1}{72} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R^a{}_{bfh} \nabla_c R_{gdeh} \\
& \left. - \frac{1}{72} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f x^g R^a{}_{hfb} \nabla_c R_{gdeh} - \frac{1}{180} s^5 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \dot{x}^f x^g R^a{}_{bch} \nabla_d R_{gef h} - \frac{1}{120} s^5 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \dot{x}^f x^g R_{gbch} \nabla_d R^a{}_{ef h} \right) + \mathcal{O}(\epsilon^6)
\end{aligned}$$

$$\begin{aligned}
x^a(s) = & \left(x^a + s\dot{x}^a - \frac{1}{360} s^2 \dot{x}^b \dot{x}^c \left(120 x^d R^a_{ bdc} + 30 x^d x^e \nabla_b R^a_{ dec} + 60 x^d x^e \nabla_d R^a_{ bec} + 15 x^d x^e \nabla^a R_{dbec} + 32 x^d x^e x^f R^a_{ deg} R_{fbcg} - 16 x^d x^e x^f R^a_{ bdg} R_{ecfg} \right. \right. \\
& - 8 x^d x^e x^f R^a_{ dbg} R_{ecfg} + 18 x^d x^e x^f \nabla_{db} R^a_{ efc} + 18 x^d x^e x^f \nabla_{de} R^a_{ bfc} - 8 x^d x^e x^f R^a_{ gdb} R_{ecfg} + 9 x^d x^e x^f \nabla^a_{ d} R_{ebfc} + 16 x^d x^e x^f x^g R_{dbch} \nabla_e R^a_{ fgh} \\
& + 6 x^d x^e x^f x^g R^a_{ deh} \nabla_b R_{fcgh} + 16 x^d x^e x^f x^g R^a_{ deh} \nabla_f R_{gbch} - 8 x^d x^e x^f x^g R^a_{ bdh} \nabla_e R_{fcgh} - 4 x^d x^e x^f x^g R^a_{ dbh} \nabla_e R_{fcgh} - 4 x^d x^e x^f x^g R_{dbeh} \nabla_c R^a_{ fgh} \\
& - 8 x^d x^e x^f x^g R_{dbeh} \nabla_f R^a_{ cgh} - 4 x^d x^e x^f x^g R_{dbeh} \nabla_f R^a_{ gch} + 6 x^d x^e x^f x^g \nabla_{deb} R^a_{ fgc} + 4 x^d x^e x^f x^g \nabla_{def} R^a_{ bgc} - 5 x^d x^e x^f x^g R^a_{ deh} \nabla_h R_{fbgc} \\
& - 4 x^d x^e x^f x^g R^a_{ hdb} \nabla_e R_{fcgh} - 4 x^d x^e x^f x^g R_{dbeh} \nabla^a R_{fcgh} - 4 x^d x^e x^f x^g R_{dbeh} \nabla_f R^a_{ hgc} + 3 x^d x^e x^f x^g \nabla_{de} R_{fbgc} \Big) \\
& - \frac{1}{1080} s^3 \dot{x}^b \dot{x}^c \dot{x}^d \left(90 x^e \nabla_b R^a_{ ced} + 24 x^e x^f R^a_{ ebg} R_{fcgd} + 96 x^e x^f R^a_{ beg} R_{fcgd} - 24 x^e x^f R^a_{ bcg} R_{edfg} + 54 x^e x^f \nabla_{eb} R^a_{ cfd} + 18 x^e x^f \nabla_{bc} R^a_{ efd} \right. \\
& + 72 x^e x^f R^a_{ geb} R_{fcgd} + 9 x^e x^f \nabla^a_b R_{ecfd} + 20 x^e x^f x^g R_{ebch} \nabla_d R^a_{ fgh} + 20 x^e x^f x^g R_{ebch} \nabla_h R^a_{ fgd} + 40 x^e x^f x^g R_{ebch} \nabla_f R^a_{ dgh} + 40 x^e x^f x^g R_{ebch} \nabla_f R^a_{ hgd} \\
& + 20 x^e x^f x^g R_{ebch} \nabla^a R_{fdgh} + 20 x^e x^f x^g R^a_{ beh} \nabla_c R_{fdgh} + 40 x^e x^f x^g R^a_{ beh} \nabla_f R_{gcdh} - 10 x^e x^f x^g R^a_{ beh} \nabla_h R_{fcgd} + 20 x^e x^f x^g R^a_{ heb} \nabla_c R_{fdgh} \\
& + 40 x^e x^f x^g R^a_{ heb} \nabla_f R_{gcdh} - 10 x^e x^f x^g R^a_{ heb} \nabla_h R_{fcgd} \Big) \\
& - \frac{1}{432} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \left(8 x^f R^a_{ bcg} R_{fdeg} + 2 x^f x^g R^a_{ bch} \nabla_d R_{fegh} + 4 x^f x^g R^a_{ bch} \nabla_f R_{gdeh} - x^f x^g R^a_{ bch} \nabla_h R_{fdge} + 2 x^f x^g R_{fbch} \nabla_d R^a_{ geh} + 10 x^f x^g R_{fbch} \nabla_d R^a_{ hge} \right. \\
& + 4 x^f x^g R_{fbch} \nabla_g R^a_{ deh} + 8 x^f x^g R_{fbch} \nabla_h R^a_{ dge} + 2 x^f x^g R_{fbch} \nabla^a R_{gdeh} + 12 x^f x^g R_{fbch} \nabla_d R^a_{ egh} + 6 x^f x^g R^a_{ bfh} \nabla_c R_{gdeh} + 6 x^f x^g R^a_{ hfb} \nabla_c R_{gdeh} \Big) \\
& \left. - \frac{1}{360} s^5 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \dot{x}^f \left(2 x^g R^a_{ bch} \nabla_d R_{gef h} + 3 x^g R_{gbch} \nabla_d R^a_{ efh} \right) \right) + \mathcal{O}(\epsilon^6)
\end{aligned}$$

== geodesic bvp =====

@print["\Btag{08a}x^a(s) = x^a + s x_1^a + s^2 x_2^a + s^3 x_3^a + \BigO{\eps^4}\Btag{08a}"];

```
x1:="import geodesic-bvp.lib term01":
@run(x1){"/Users/leo/local/sh/cdbfile"}:
```

```
poly:=@(x1):
@distributed(%):
```

```
x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).
```

```
term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
```

```

term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:
term06:=@(poly): @keep_weight!(term06){myterms}{6}:

x1:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05) + @(term06):
@prodsort!(%): @rename_dummies!(%):
@print["\Btag{08b}x_1^a(s)="~@(x1)~"\Etag{08b}"];

# -----

x2="import geodesic-bvp.lib term02":
@run(x2){"/Users/leo/local/sh/cdbfile"}:

poly:=@(x2):
@distribute!(%):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:
term06:=@(poly): @keep_weight!(term06){myterms}{6}:

x2:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05) + @(term06):
@prodsort!(%): @rename_dummies!(%):
@print["\Btag{08c}x_2^a(s)="~@(x2)~"\Etag{08c}"];

# -----

x3="import geodesic-bvp.lib term03":
@run(x3){"/Users/leo/local/sh/cdbfile"}:

poly:=@(x3):
@distribute!(%):

```

```
x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).
```

```
term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:
term06:=@(poly): @keep_weight!(term06){myterms}{6}:
```

```
x3:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05) + @(term06):
@prodsort!(%): @rename_dummies!(%):
@print["\Btag{08d}x_3^a(s)="\~@(x3)~"\Etag{08d}"];
```

$$x^a(s) = x^a + sx_1^a + s^2x_2^a + s^3x_3^a + \mathcal{O}(\epsilon^4)$$

$$\begin{aligned}
x_1^a(s) = & \left(\Delta x^a + \frac{1}{3} x^b \Delta x^c \Delta x^d R^a_{cbd} + \frac{1}{12} x^b x^c \Delta x^d \Delta x^e \nabla_d R^a_{bce} + \frac{1}{6} x^b x^c \Delta x^d \Delta x^e \nabla_b R^a_{dce} + \frac{1}{24} x^b x^c \Delta x^d \Delta x^e \nabla^a R_{bdce} + \frac{1}{12} x^b \Delta x^c \Delta x^d \Delta x^e \nabla_c R^a_{dbe} \right. \\
& + \frac{4}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{bgc} R_{gedf} - \frac{2}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{egb} R_{gcdf} - \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{bge} R_{gcdf} + \frac{1}{20} x^b x^c x^d \Delta x^e \Delta x^f \nabla_{be} R^a_{cdf} \\
& + \frac{1}{20} x^b x^c x^d \Delta x^e \Delta x^f \nabla_{bc} R^a_{edf} + \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{gbe} R_{gcdf} + \frac{1}{40} x^b x^c x^d \Delta x^e \Delta x^f \nabla^a_b R_{cedf} + \frac{1}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{bgd} R_{gecf} \\
& + \frac{4}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{dgb} R_{gecf} - \frac{1}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{dge} R_{gbcf} + \frac{1}{20} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{bd} R^a_{ecf} + \frac{1}{60} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{de} R^a_{bcf} \\
& + \frac{7}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{gbd} R_{gecf} + \frac{1}{120} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla^a_d R_{becf} + \frac{1}{54} x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f R^a_{cgd} R_{geb f} + \frac{2}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hfbg} \nabla_c R^a_{dhe} \\
& + \frac{1}{60} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_f R_{hdeg} + \frac{2}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_d R_{h f e g} - \frac{1}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{f h b} \nabla_c R_{hdeg} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{b h f} \nabla_c R_{hdeg} \\
& - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_g R^a_{dhe} - \frac{1}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{ghe} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{ehg} + \frac{1}{60} x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla_{bcf} R^a_{deg} \\
& + \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla_{bcd} R^a_{feg} + \frac{1}{72} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_h R_{dfeg} + \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{hbf} \nabla_c R_{hdeg} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla^a R_{hdeg} \\
& + \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{heg} + \frac{1}{120} x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla^a_{bc} R_{dfeg} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_g R^a_{chd} + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_h R^a_{cdg} \\
& + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_c R^a_{ghd} + \frac{2}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_c R^a_{hdg} - \frac{1}{108} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla^a R_{hcdg} \\
& + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_f R_{hcdg} + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_c R_{hfdg} + \frac{1}{108} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_h R_{cfdg} \\
& + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_f R_{hcdg} + \frac{2}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_c R_{hfdg} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_h R_{cfdg} \\
& + \frac{1}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhe} \nabla_f R_{hbcg} + \frac{1}{108} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhe} \nabla_b R_{hfcg} + \frac{1}{432} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhe} \nabla_h R_{bfcg} \\
& + \frac{1}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_f R^a_{chg} - \frac{5}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_f R^a_{hcg} + \frac{1}{108} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_c R^a_{fhg} \\
& + \frac{7}{108} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_h R^a_{fcg} + \frac{1}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla^a R_{hfcg} + \frac{1}{36} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_f R^a_{ghc} \\
& + \frac{1}{72} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhb} \nabla_e R_{hfcg} + \frac{1}{24} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbd} \nabla_e R_{hfcg} + \frac{1}{180} x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{chd} \nabla_e R_{hfbg} \\
& \left. + \frac{1}{120} x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hcbd} \nabla_e R^a_{fhg} \right)
\end{aligned}$$

$$\begin{aligned}
x_2^a(s) = & \left(-\frac{1}{3} x^b \Delta x^c \Delta x^d R^a_{cbd} - \frac{1}{12} x^b x^c \Delta x^d \Delta x^e \nabla_d R^a_{bce} - \frac{1}{6} x^b x^c \Delta x^d \Delta x^e \nabla_b R^a_{dce} - \frac{1}{24} x^b x^c \Delta x^d \Delta x^e \nabla^a R_{bdce} - \frac{4}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{bgc} R_{gedf} \right. \\
& + \frac{2}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{egb} R_{gcdf} + \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{bge} R_{gcdf} - \frac{1}{20} x^b x^c x^d \Delta x^e \Delta x^f \nabla_{be} R^a_{cdf} - \frac{1}{20} x^b x^c x^d \Delta x^e \Delta x^f \nabla_{bc} R^a_{edf} \\
& - \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{gbe} R_{gcdf} - \frac{1}{40} x^b x^c x^d \Delta x^e \Delta x^f \nabla_b R_{cedf} - \frac{2}{9} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{gbd} R^g_{ecf} - \frac{2}{45} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hfbg} \nabla_c R^a_{dhe} \\
& - \frac{1}{60} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_f R_{hdeg} - \frac{2}{45} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_d R_{hdeg} + \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{fhh} \nabla_c R_{hdeg} + \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{bhf} \nabla_c R_{hdeg} \\
& + \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_g R^a_{dhe} + \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{ghe} + \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{ehg} - \frac{1}{60} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g \nabla_{bcf} R^a_{deg} \\
& - \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g \nabla_{bcd} R^a_{feg} - \frac{1}{72} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_h R_{dfeg} - \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbf} \nabla_c R_{hdeg} + \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hbcf} \nabla^a R_{hdeg} \\
& - \frac{1}{90} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{heg} - \frac{1}{120} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g \nabla_{bc} R_{dfeg} - \frac{1}{18} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_f R^h_{cdg} - \frac{1}{9} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_c R^h_{fdg} \\
& - \frac{1}{36} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla^h R_{cf dg} - \frac{1}{18} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^h_{ebf} \nabla_h R^a_{cdg} - \frac{1}{9} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^h_{ebf} \nabla_c R^a_{hdg} \\
& \left. + \frac{1}{36} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^h_{ebf} \nabla^a R_{hcdg} - \frac{1}{18} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbd} \nabla_e R^h_{fcg} \right)
\end{aligned}$$

$$\begin{aligned}
x_3^a(s) = & \left(-\frac{1}{12} x^b \Delta x^c \Delta x^d \Delta x^e \nabla_c R^a_{dbe} - \frac{1}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{bgd} R_{gecf} - \frac{4}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{dgb} R_{gecf} + \frac{1}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{dge} R_{gbcf} \right. \\
& - \frac{1}{20} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{bd} R^a_{ecf} - \frac{1}{60} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{de} R^a_{bcf} + \frac{1}{15} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{gbd} R_{gecf} - \frac{1}{120} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_d R_{becf} \\
& - \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_g R^a_{chd} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_h R^a_{cdg} - \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_c R^a_{ghd} \\
& + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_c R^a_{hdg} - \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla^a R_{hcdg} - \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_f R_{hcdg} \\
& - \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_c R_{hfdg} - \frac{1}{108} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_h R_{cf dg} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_f R_{hcdg} \\
& \left. + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_c R_{hfdg} + \frac{1}{108} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_h R_{cf dg} - \frac{1}{12} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^h_{dbe} \nabla_h R^a_{fcg} \right)
\end{aligned}$$

=== translated rnc =====

```

y05:="import geodesic-rnc.lib y05":
@run(y05){"/Users/leo/local/sh/cdbfile"}:

poly:=@(y05):
@distributed!(%):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{a}::Weight(label=myterms,value=1).

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:
term06:=@(poly): @keep_weight!(term06){myterms}{6}:

y05:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05) + @(term06):
@prodsort!(%): @rename_dummies!(%):
@print["\Btag{12a}y^a="~@(y05)~"+\BigO{\eps^6}\Etag{12a}"];

```


$$\begin{aligned}
y^a = & \left(\Delta x^a + \frac{1}{3} x^b \Delta x^c \Delta x^d R^a_{cbd} + \frac{1}{12} x^b x^c \Delta x^d \Delta x^e \nabla_d R^a_{bce} + \frac{1}{6} x^b x^c \Delta x^d \Delta x^e \nabla_b R^a_{dce} + \frac{1}{24} x^b x^c \Delta x^d \Delta x^e \nabla^a R_{bdce} + \frac{1}{12} x^b \Delta x^c \Delta x^d \Delta x^e \nabla_c R^a_{dbe} \right. \\
& + \frac{4}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{bgc} R_{gedf} - \frac{2}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{egb} R_{gcdf} - \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{bge} R_{gcdf} + \frac{1}{20} x^b x^c x^d \Delta x^e \Delta x^f \nabla_{be} R^a_{cdf} + \frac{1}{20} x^b x^c x^d \Delta x^e \Delta x^f \nabla_{bc} R^a_{edf} \\
& + \frac{1}{45} x^b x^c x^d \Delta x^e \Delta x^f R^a_{gbe} R_{gcdf} + \frac{1}{40} x^b x^c x^d \Delta x^e \Delta x^f \nabla_b R_{cedf} + \frac{1}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{bgd} R_{gecf} + \frac{4}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{dgb} R_{gecf} \\
& - \frac{1}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{dge} R_{gbcf} + \frac{1}{20} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{bd} R^a_{ecf} + \frac{1}{60} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{de} R^a_{bcf} + \frac{7}{45} x^b x^c \Delta x^d \Delta x^e \Delta x^f R^a_{gbd} R_{gecf} \\
& + \frac{1}{120} x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_d R_{becf} + \frac{1}{54} x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f R^a_{cgd} R_{gebf} + \frac{2}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hfbg} \nabla_c R^a_{dhe} + \frac{1}{60} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_f R_{hdeg} \\
& + \frac{2}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_d R_{hdeg} - \frac{1}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{fhh} \nabla_c R_{hdeg} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhf} \nabla_c R_{hdeg} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_g R^a_{dhe} \\
& - \frac{1}{45} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{ghe} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{ehg} + \frac{1}{60} x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla_{bcf} R^a_{deg} + \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla_{bcd} R^a_{feg} \\
& + \frac{1}{72} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{bhc} \nabla_h R_{dfeg} + \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R^a_{hbf} \nabla_c R_{hdeg} - \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla^a R_{hdeg} + \frac{1}{90} x^b x^c x^d x^e \Delta x^f \Delta x^g R_{hbcf} \nabla_d R^a_{heg} \\
& + \frac{1}{120} x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla_{bc} R_{dfeg} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_g R^a_{chd} + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_h R^a_{cdg} + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_c R^a_{ghd} \\
& + \frac{2}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla_c R^a_{hdg} - \frac{1}{108} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{hebf} \nabla^a R_{hcdg} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_f R_{hcdg} \\
& + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_c R_{hfdg} + \frac{1}{108} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{ehb} \nabla_h R_{cfdg} + \frac{1}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_f R_{hcdg} \\
& + \frac{2}{27} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_c R_{hfdg} + \frac{1}{54} x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbe} \nabla_h R_{cfdg} + \frac{1}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhe} \nabla_f R_{hbcg} \\
& + \frac{1}{108} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhe} \nabla_b R_{hfcg} + \frac{1}{432} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhe} \nabla_h R_{bfcg} + \frac{1}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_f R^a_{chg} \\
& - \frac{5}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_f R^a_{hcg} + \frac{1}{108} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_c R^a_{fhg} + \frac{7}{108} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_h R^a_{fcg} \\
& + \frac{1}{216} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla^a R_{hfcg} + \frac{1}{36} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdbe} \nabla_f R^a_{ghc} + \frac{1}{72} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{dhh} \nabla_e R_{hfcg} \\
& + \frac{1}{24} x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{hbd} \nabla_e R_{hfcg} + \frac{1}{180} x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R^a_{chd} \nabla_e R_{hfbg} + \frac{1}{120} x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hcbd} \nabla_e R^a_{fhg} \Big) + \mathcal{O}(\epsilon^6)
\end{aligned}$$

poly:=@{y05}:

\Delta{x}^{\{a\}}::Weight(label=y05terms,value=1).

```

term00:=@(poly): @keep_weight!(term00){y05terms}{0}:
term01:=@(poly): @keep_weight!(term01){y05terms}{1}:
term02:=@(poly): @keep_weight!(term02){y05terms}{2}:
term03:=@(poly): @keep_weight!(term03){y05terms}{3}:
term04:=@(poly): @keep_weight!(term04){y05terms}{4}:
term05:=@(poly): @keep_weight!(term05){y05terms}{5}:

# I couldn't get SortOrder to get all of the \Delta{x} to the front of every term, so
# I resorted to this kludge, using A and B an relying on the default lexicographic ordering.

@substitute!(term02)(\Delta{x}^a -> A^a, x^a -> B^a):
@substitute!(term03)(\Delta{x}^a -> A^a, x^a -> B^a):
@substitute!(term04)(\Delta{x}^a -> A^a, x^a -> B^a):
@substitute!(term05)(\Delta{x}^a -> A^a, x^a -> B^a):

y0502:=@(term02): @prodsort!!(%): @rename_dummies!(%):
y0503:=@(term03): @prodsort!!(%): @rename_dummies!(%):
y0504:=@(term04): @prodsort!!(%): @rename_dummies!(%):
y0505:=@(term05): @prodsort!!(%): @rename_dummies!(%):

@substitute!(y0502)(A^{a} -> 1, B^a -> x^a):
@substitute!(y0503)(A^{a} -> 1, B^a -> x^a):
@substitute!(y0504)(A^{a} -> 1, B^a -> x^a):
@substitute!(y0505)(A^{a} -> 1, B^a -> x^a):

# -----

poly:=@(y0502):

x^{a}::Weight(label=xterms,value=1).

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:
term02:=@(poly): @keep_weight!(term02){xterms}{2}:
term03:=@(poly): @keep_weight!(term03){xterms}{3}:
term04:=@(poly): @keep_weight!(term04){xterms}{4}:

tmp:= 24 @(term02):
@factor_out!!(tmp)(x^{a}):

```

```

term02:=@(tmp):

tmp:= 360 @(term03):
@factor_out!!(tmp)(x^{a}):
term03:=@(tmp):

tmp:= 360 @(term04):
@factor_out!!(tmp)(x^{a}):
term04:=@(tmp):

tmp:=@(term00) + @(term01) + (1/24) @(term02) + (1/360) @(term03) + (1/360) @(term04):
y0502:= 360 @(tmp):

# -----

poly:=@(y0503):

x^{a}::Weight(label=xterms,value=1).

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:
term02:=@(poly): @keep_weight!(term02){xterms}{2}:
term03:=@(poly): @keep_weight!(term03){xterms}{3}:

tmp:= 360 @(term02):
@factor_out!!(tmp)(x^{a}):
term02:=@(tmp):

tmp:= 108 @(term03):
@factor_out!!(tmp)(x^{a}):
term03:=@(tmp):

tmp:=@(term00) + @(term01) + (1/360) @(term02) + (1/108) @(term03):
y0503:= 1080 @(tmp):

# -----

poly:=@(y0504):

x^{a}::Weight(label=xterms,value=1).

```

```

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:
term02:=@(poly): @keep_weight!(term02){xterms}{2}:

tmp:= 432 @(term02):
@factor_out!!(tmp)(x^{a}):
term02:=@(tmp):

tmp:=@(term00) + @(term01) + (1/432) @(term02):
y0504:= 432 @(tmp):

# -----

poly:=@(y0505):

x^{a}::Weight(label=xterms,value=1).

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:

tmp:= 360 @(term01):
@factor_out!!(tmp)(x^{a}):
term01:=@(tmp):

tmp:=@(term00) + (1/360) @(term01):
y0505:= 360 @(tmp):

# -----

@print["\Btag{12b}
  y^a= \Delta x^a
        + y^a_{bc}\Delta x^b\Delta x^c
        + y^a_{bcd}\Delta x^b\Delta x^c\Delta x^d
        + y^a_{bcde}\Delta x^b\Delta x^c\Delta x^d\Delta x^e
        + y^a_{bcdef}\Delta x^b\Delta x^c\Delta x^d\Delta x^e\Delta x^f
        + \BigO{\eps^6}\Etag{12b}"];

@print["\Btag{12c}\phantom{0}360y^a_{bc}=\killL"~@(y0502)~"\killR\Etag{12c}"];
@print["\Btag{12d}1080y^a_{bcd}=\killL"~@(y0503)~"\killR\Etag{12d}"];

```

```
@print["\Btag{12e}\phantom{0}432y^a_{bcde}=\killL"~@(y0504)~"\killR\Etag{12e}"];
@print["\Btag{12f}\phantom{0}360y^a_{bcdef}=\killL"~@(y0505)~"\killR\Etag{12f}"];
```

$$y^a = \Delta x^a + y_{bc}^a \Delta x^b \Delta x^c + y_{bcd}^a \Delta x^b \Delta x^c \Delta x^d + y_{bcde}^a \Delta x^b \Delta x^c \Delta x^d \Delta x^e + y_{bcdef}^a \Delta x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f + \mathcal{O}(\epsilon^6)$$

$$\begin{aligned} 360y_{bc}^a = & 120 x^d R^a_{bdc} + 15 x^d x^e (2 \nabla_b R^a_{dec} + 4 \nabla_d R^a_{bec} + \nabla^a R_{dbec}) \\ & + x^d x^e x^f (32 R^a_{dge} R_{gbfc} - 16 R^a_{bgd} R_{gefc} - 8 R^a_{dgb} R_{gefc} + 18 \nabla_{db} R^a_{efc} + 18 \nabla_{de} R^a_{bfc} + 8 R^a_{gdb} R_{gefc} + 9 \nabla^a_d R_{ebfc}) \\ & + x^d x^e x^f x^g (16 R_{hbd} \nabla_e R^a_{fhg} + 6 R^a_{dhe} \nabla_b R_{hfgc} + 16 R^a_{dhe} \nabla_f R_{hbgc} - 8 R^a_{bhd} \nabla_e R_{hfgc} - 4 R^a_{dhb} \nabla_e R_{hfgc} - 4 R_{hdeb} \nabla_c R^a_{fhg} - 8 R_{hdeb} \nabla_f R^a_{chg} \\ & - 4 R_{hdeb} \nabla_f R^a_{ghc} + 6 \nabla_{deb} R^a_{fgc} + 4 \nabla_{def} R^a_{bgc} + 5 R^a_{dhe} \nabla_h R_{fbgc} + 4 R^a_{hdb} \nabla_e R_{hfgc} - 4 R_{hdeb} \nabla^a R_{hfgc} + 4 R_{hdeb} \nabla_f R^a_{hgc} + 3 \nabla^a_{de} R_{fbgc}) \end{aligned}$$

$$\begin{aligned} 1080y_{bcd}^a = & 90 x^e \nabla_b R^a_{ced} + 3 x^e x^f (8 R^a_{egb} R_{gcf d} + 32 R^a_{bge} R_{gcf d} - 8 R^a_{bgc} R_{gef d} + 18 \nabla_{eb} R^a_{cfd} + 6 \nabla_{bc} R^a_{efd} + 56 R^a_{geb} R_{gcf d} + 3 \nabla^a_b R_{ecfd}) \\ & + 10 x^e x^f x^g (2 R_{hbec} \nabla_d R^a_{fhg} + 4 R_{hbec} \nabla_h R^a_{fgd} + 4 R_{hbec} \nabla_f R^a_{dhg} + 8 R_{hbec} \nabla_f R^a_{hgd} - R_{hbec} \nabla^a R_{hfgd} + 2 R^a_{bhe} \nabla_c R_{hfgd} + 4 R^a_{bhe} \nabla_f R_{hcgd} \\ & + R^a_{bhe} \nabla_h R_{fcgd} + 4 R^a_{heb} \nabla_c R_{hfgd} + 8 R^a_{heb} \nabla_f R_{hcgd} + 2 R^a_{heb} \nabla_h R_{fcgd}) \end{aligned}$$

$$\begin{aligned} 432y_{bcde}^a = & 8 x^f R^a_{bgc} R_{gdf e} + x^f x^g (2 R^a_{bhc} \nabla_d R_{hfg e} + 4 R^a_{bhc} \nabla_f R_{hdge} + R^a_{bhc} \nabla_h R_{fdge} + 2 R_{hbf c} \nabla_d R^a_{ghe} - 10 R_{hbf c} \nabla_d R^a_{hge} + 4 R_{hbf c} \nabla_g R^a_{dhe} \\ & + 28 R_{hbf c} \nabla_h R^a_{dge} + 2 R_{hbf c} \nabla^a R_{hdge} + 12 R_{hbf c} \nabla_d R^a_{ehg} + 6 R^a_{bhf} \nabla_c R_{hdge} + 18 R^a_{hfb} \nabla_c R_{hdge}) \end{aligned}$$

$$360y_{bcdef}^a = x^g (2 R^a_{bhc} \nabla_d R_{hegf} + 3 R_{hbgc} \nabla_d R^a_{ehf})$$

```
# === geodesic lsq =====
{g_{a b},x^{a},\Delta{x}^{\{a\}},R_{a b c d},
 \nabla_{a}\{R_{p q r s}\},\nabla_{a b}\{R_{p q r s}\},\nabla_{a b c}\{R_{p q r s}\},
 \partial_{a}\{R_{p q r s}\},\partial_{a b}\{R_{p q r s}\},\partial_{a b c}\{R_{p q r s}\}}::SortOrder.

lsq:="import geodesic-lsq.lib lsq":
@run(lsq){"/Users/leo/local/sh/cdbfile"}:

poly:=@(lsq):

x^{a}::Weight(label=myterms,value=1).
\Delta{x}^{\{a\}}::Weight(label=myterms,value=1).
```

```

term00:=@(poly): @keep_weight!(term00){myterms}{0}:
term01:=@(poly): @keep_weight!(term01){myterms}{1}:
term02:=@(poly): @keep_weight!(term02){myterms}{2}:
term03:=@(poly): @keep_weight!(term03){myterms}{3}:
term04:=@(poly): @keep_weight!(term04){myterms}{4}:
term05:=@(poly): @keep_weight!(term05){myterms}{5}:
term06:=@(poly): @keep_weight!(term06){myterms}{6}:
term07:=@(poly): @keep_weight!(term07){myterms}{7}:

lsq:=@(term00) + @(term01) + @(term02) + @(term03) + @(term04) + @(term05) + @(term06) + @(term07):
@prodsort!(%): @rename_dummies!(%):
@print["\Btag{13a}L^2_{PQ}="~@(lsq)~"+\BigO{\eps^6}\Etag{13a}"];

```

$$\begin{aligned}
L_{PQ}^2 = & \left(g_{ab} \Delta x^a \Delta x^b - \frac{1}{3} x^a x^b \Delta x^c \Delta x^d R_{acbd} - \frac{1}{12} x^a x^b \Delta x^c \Delta x^d \Delta x^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c \Delta x^d \Delta x^e \nabla_a R_{bdce} + \frac{2}{45} x^a x^b x^c \Delta x^d \Delta x^e \Delta x^f R_{gabd} R_{gecf} \right. \\
& - \frac{1}{20} x^a x^b x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{ad} R_{becf} - \frac{11}{45} x^a x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f R_{gcad} R_{gebf} - \frac{1}{60} x^a x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f \nabla_{cd} R_{aebf} + \frac{2}{45} x^a x^b x^c x^d \Delta x^e \Delta x^f R_{gabe} R_{gcdf} \\
& - \frac{1}{20} x^a x^b x^c x^d \Delta x^e \Delta x^f \nabla_{ab} R_{cedf} + \frac{1}{45} x^a x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{heaf} \nabla_b R_{hcdg} + \frac{1}{45} x^a x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{habe} \nabla_f R_{hcdg} \\
& + \frac{1}{45} x^a x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g R_{habe} \nabla_e R_{hfdg} - \frac{1}{60} x^a x^b x^c x^d \Delta x^e \Delta x^f \Delta x^g \nabla_{abe} R_{cfdg} - \frac{2}{27} x^a x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdae} \nabla_f R_{hbcg} \\
& - \frac{1}{12} x^a x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdae} \nabla_h R_{bfcg} - \frac{5}{27} x^a x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hdae} \nabla_b R_{hfcg} + \frac{1}{54} x^a x^b \Delta x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{hcad} \nabla_e R_{hfbg} \\
& \left. + \frac{1}{18} x^a x^b x^c \Delta x^d \Delta x^e \Delta x^f \Delta x^g R_{habd} \nabla_e R_{hfcg} + \frac{2}{45} x^a x^b x^c x^d x^e \Delta x^f \Delta x^g R_{habf} \nabla_c R_{hdeg} - \frac{1}{90} x^a x^b x^c x^d x^e \Delta x^f \Delta x^g \nabla_{abc} R_{dfeg} \right) + \mathcal{O}(\epsilon^6)
\end{aligned}$$

```
poly:=@(lsq):
```

```
\Delta{x}^{\{a\}}::Weight(label=Dxterms,value=1).
```

```

term00:=@(poly): @keep_weight!(term00){Dxterms}{0}:
term01:=@(poly): @keep_weight!(term01){Dxterms}{1}:
term02:=@(poly): @keep_weight!(term02){Dxterms}{2}:
term03:=@(poly): @keep_weight!(term03){Dxterms}{3}:
term04:=@(poly): @keep_weight!(term04){Dxterms}{4}:

```

```

term05:=@(poly): @keep_weight!(term05){Dxterms}{5}:

{g_{a b},\Delta{x}^a,x^a,R_{a b c d},
 \nabla_a{R_{p q r s}},\nabla_a b{R_{p q r s}},\nabla_a b c{R_{p q r s}},
 \partial_a{R_{p q r s}},\partial_a b{R_{p q r s}},\partial_a b c{R_{p q r s}}}:SortOrder.

lsq02:=@(term02): @prodsort!(%): @rename_dummies!(%): @substitute!!(%)(\Delta{x}^a -> 1):
lsq03:=@(term03): @prodsort!(%): @rename_dummies!(%): @substitute!!(%)(\Delta{x}^a -> 1):
lsq04:=@(term04): @prodsort!(%): @rename_dummies!(%): @substitute!!(%)(\Delta{x}^a -> 1):
lsq05:=@(term05): @prodsort!(%): @rename_dummies!(%): @substitute!!(%)(\Delta{x}^a -> 1):

# -----

poly:=@(lsq02):

x^a::Weight(label=xterms,value=1).

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:
term02:=@(poly): @keep_weight!(term02){xterms}{2}:
term03:=@(poly): @keep_weight!(term03){xterms}{3}:
term04:=@(poly): @keep_weight!(term04){xterms}{4}:
term05:=@(poly): @keep_weight!(term05){xterms}{5}:

tmp:= 180 @(term04):
@factor_out!!(tmp)(x^a):
term04:=@(tmp):

tmp:= 90 @(term05):
@factor_out!!(tmp)(x^a):
term05:=@(tmp):

tmp:=@(term00) + @(term01) + @(term02) + @(term03) + (1/180) @(term04) + (1/90) @(term05):
lsq02:= 180 @(tmp):

# -----

poly:=@(lsq03):

x^a::Weight(label=xterms,value=1).

```

```

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:
term02:=@(poly): @keep_weight!(term02){xterms}{2}:
term03:=@(poly): @keep_weight!(term03){xterms}{3}:
term04:=@(poly): @keep_weight!(term04){xterms}{4}:

tmp:= 180 @(term03):
@factor_out!!(tmp)(x^{a}):
term03:=@(tmp):

tmp:= 180 @(term04):
@factor_out!!(tmp)(x^{a}):
term04:=@(tmp):

tmp:=@(term00) + @(term01) + @(term02) + (1/180) @(term03) + (1/180) @(term04):
lsq03:= 180 @(tmp):

# -----

poly:=@(lsq04):

x^{a}::Weight(label=xterms,value=1).

term00:=@(poly): @keep_weight!(term00){xterms}{0}:
term01:=@(poly): @keep_weight!(term01){xterms}{1}:
term02:=@(poly): @keep_weight!(term02){xterms}{2}:
term03:=@(poly): @keep_weight!(term03){xterms}{3}:

tmp:= -180 @(term02):
@factor_out!!(tmp)(x^{a}):
term02:=@(tmp):

tmp:= -108 @(term03):
@factor_out!!(tmp)(x^{a}):
term03:=@(tmp):

tmp:=@(term00) + @(term01) - (1/180) @(term02) - (1/108) @(term03):
lsq04:=540 @(tmp):

```



```

# -----

tmp:=(lsq05):
lsq05:= 54 @(tmp):

# -----

@print["\Btag{13b}L^2_{PQ}
      = f_{ab}\Delta x^a\Delta x^b
      + f_{abc}\Delta x^a\Delta x^b\Delta x^c
      + f_{abcd}\Delta x^a\Delta x^b\Delta x^c\Delta x^d
      + f_{abcde}\Delta x^a\Delta x^b\Delta x^c\Delta x^d\Delta x^e + \BigO{\eps^6}\Etag{13b}"];

@print["\Btag{13c}180f_{ab}=\killL"~@(lsq02)~"\killR\Etag{13c}"];
@print["\Btag{13d}180f_{abc}=\killL"~@(lsq03)~"\killR\Etag{13d}"];
@print["\Btag{13e}540f_{abcd}=\killL"~@(lsq04)~"\killR\Etag{13e}"];
@print["\Btag{13f}\phantom{0}54f_{abcde}=\killL"~@(lsq05)~"\Etag{13f}"];

```

$$L_{PQ}^2 = f_{ab}\Delta x^a\Delta x^b + f_{abc}\Delta x^a\Delta x^b\Delta x^c + f_{abcd}\Delta x^a\Delta x^b\Delta x^c\Delta x^d + f_{abcde}\Delta x^a\Delta x^b\Delta x^c\Delta x^d\Delta x^e + \mathcal{O}(\epsilon^6)$$

$$180f_{ab} = 180g_{ab} - 60x^cx^dR_{cadb} - 30x^cx^dx^e\nabla_cR_{daeb} + x^cx^dx^ex^f(8R_{gcda}R_{gefb} - 9\nabla_{cd}R_{eafb}) + 2x^cx^dx^ex^fx^g(4R_{hcda}\nabla_eR_{hfgb} - \nabla_{cde}R_{fagb})$$

$$180f_{abc} = -15x^dx^e\nabla_aR_{dbec} + x^dx^ex^f(8R_{gdea}R_{gbfc} - 9\nabla_{da}R_{ebfc}) + x^dx^ex^fx^g(4R_{hadb}\nabla_eR_{hfgc} + 4R_{hdea}\nabla_bR_{hfgc} + 4R_{hdea}\nabla_fR_{hbgc} - 3\nabla_{dea}R_{fbgc})$$

$$540f_{abcd} = -3x^ex^f(44R_{gaeb}R_{gcfd} + 3\nabla_{ab}R_{ecfd}) - 5x^ex^fx^g(8R_{haeb}\nabla_cR_{hfgd} + 9R_{haeb}\nabla_hR_{fcgd} + 20R_{haeb}\nabla_fR_{hcgd} - 6R_{hefa}\nabla_bR_{hcgd})$$

$$54f_{abcde} = x^fx^gR_{hafb}\nabla_cR_{hdge}$$