M4112: The Sun

Exercise Sheet 1

Q1. Show that in any star in equilibrium, the function

$$P(r)+\frac{GM^2(r)}{8\pi r^4}$$

decreases outward. Hence show that the central pressure P_c satisfies

$$P_c > \frac{GM^2}{8\pi R^4}$$

in the limit $P(R) \to 0$.

Q2. In the crudest approximation, we may say that

$$\frac{dP}{dr} \sim \frac{P_c - P_s}{R_c - R_s} \sim \frac{P}{R}.$$

By continuing in this cavalier fashion, show that radiative stars (assume κ is constant) obey a mass-luminosity relation of the form $L \sim M^3$.

Q3. Radiative energy transfer can be written in the form

$$\mathbf{F_{rad}} = -k_{rad}\vec{\nabla}T.$$

(a) Show that $k_{rad} = \frac{4ac}{3}\frac{T^3}{\rho\kappa_{rad}}$, where κ_{rad} is the usual Rosseland mean.

(b) Likewise, electron conduction can be accounted for by writing

$$\mathbf{F_{cond}} = -k_{cond}\vec{\nabla}T$$

where an appropriate κ_{cond} is now determined from quantum mechanics. If the total flux transport is to use the same equation

$$\mathbf{F_{tot}} = -k_{tot}\vec{\nabla}T$$

with $k_{tot} = \frac{4ac}{3}\frac{T^3}{\rho\kappa_{tot}}$, then determine an expression for κ_{tot}.

Q4. The diffusion approximation for radiative transfer gives

$$\mathbf{F} = -k_{rad}\vec{\nabla}T$$

where $k_{rad} = \frac{4ac}{3}\frac{T^3}{\kappa\rho}$. Note that the flux F is just the product of the internal energy per unit volume ($u = c_v T\rho$) and the effective radial velocity v of the photons (which is not c, because they do not travel radially). Hence the conservation of energy requires

$$\frac{\partial u}{\partial t} + \nabla\cdot(u\mathbf{v}) = 0.$$

Show that if c_v is constant, this reduces to the heat equation

$$\frac{\partial T}{\partial t} = k\nabla^2 T.$$

Q5. Consider a star with total pressure due to both gas and radiation.

(a) Show that

$$\frac{dP_{rad}}{dP} = \frac{\kappa L(r)}{4\pi c G m(r)}.$$

(b) Introduce the quantity

$$\eta(r) \equiv \frac{L(r)/m(r)}{L(R)/m(R)} = \frac{L(r)/m(r)}{L/M},$$

and define

$$\overline{\kappa\eta(r)} \equiv \frac{1}{P(r)} \int_0^{P(r)} \kappa\eta dP.$$

Hence show that

$$P_{rad}(r) = \frac{L}{4\pi c G M} P(r)\overline{\kappa\eta(r)}.$$

(c) The "Eddington Standard Model" is one where $\kappa\eta(r)$ is constant. Show that this means that

$$\beta \equiv \frac{P_{gas}}{P} = \text{constant},$$

and that therefore

$$P = \left[\left(\frac{\Re}{\mu}\frac{3}{a}\frac{1-\beta}{\beta^4}\right)^4\right]^{1/3} \rho^{4/3},$$

and that the resulting star is a polytrope of index $n = 3$.

Q6. Derive the following expressions for L, T, r and P as functions of mass m near the centre of a star. In these expressions, the subscript "c" refers to the central value, which is assumed to be constant for small values of m or r.

(a) $r(m) = \left(\frac{3m}{4\pi\rho_c}\right)^{1/3}$

(b) $L(m) = \varepsilon_c m$

(c) $P(m) = P_c - \frac{3G}{8\pi}\left(\frac{4\pi\rho_c}{3}\right)^{4/3} m^{2/3}$

(d) $T^4(m) = T_c^4 - \frac{1}{2ac}\left(\frac{3}{4\pi}\right)^{2/3} \kappa_c\varepsilon_c\rho_c^{4/3} m^{2/3}$ for radiative cores.

(e) $\ln T(m) = \ln T_c - \nabla_{ad,c} G\frac{\rho_c^{4/3}}{P_c}\left(\frac{\pi}{6}\right)^{1/3} m^{2/3}$ for convective cores.

Q7. Investigate the temperature near the photosphere. Assume that the diffusion approximation is valid even near the surface where $\tau \to 0$, even though this is not strictly true. Recall that the optical depth τ is defined by

$$\tau = \int_r^\infty \kappa\rho dr.$$

(a) Write the diffusion approximation for radiative transfer in the form

$$\frac{dP_{rad}}{dr} = \text{something}.$$

(b) Integrate this equation from the true surface ($\tau = 0$) to an arbitrary optical depth τ. In the integrand, replace r and $L(r)$ by the surface values R and L, respectively. This approximation is valid because neither of these variables change very much near the surface. Hence show that

$$P_{rad}(\tau) = \frac{L\tau}{4\pi c R^2} + P_{rad}(0).$$

(c) Given that

$$P_{rad}(0) = \frac{2\sigma}{3c} T_e^4$$

(which is a result from the theory of radiative transport) show that

$$T^4(r) = \frac{1}{2} T_e^4 \left(1 + \frac{3}{2}\tau\right).$$

(d) Note that this relates the actual temperature to the optical depth, and is known as a "$T - \tau$" relation. Hence show that

$$\tau = \frac{2}{3} \text{ at the photosphere.}$$

Q8. From the definition of the gas and radiation pressures, show that

$$\left(\frac{\partial\beta}{\partial T}\right)_P = -\left(\frac{\partial(1-\beta)}{\partial T}\right)_P = -\frac{4}{T}(1-\beta)$$

$$\left(\frac{\partial\beta}{\partial P}\right)_T = -\left(\frac{\partial(1-\beta)}{\partial P}\right)_T = \frac{1}{P}(1-\beta)$$

Q9. We have seen that the diffusion approximation can be used for radiative energy transport. To do this we used

$$\mathbf{F} = -D\vec{\nabla}U$$

where $D = c/3\kappa\rho$ and $U = aT^4$. In reality, these expressions depend on the photon frequency ν. Hence we have

$$\mathbf{F}_\nu = -D_\nu \vec{\nabla} U_\nu$$

where $D = c/3\kappa_\nu\rho$ and $U = \frac{4\pi}{c}B(\nu, T)$ which is the Planck function.

(a) Show that the total flux is now given by

$$\mathbf{F} = -\left[\frac{4\pi}{3\rho}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}d\nu\right]\vec{\nabla}T.$$

(b) By equating this with the frequency-independent expression derived in lectures, show that the appropriately averaged opacity is

$$\frac{1}{\kappa} = \frac{\pi}{acT^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}d\nu.$$

Note: such an average is called the "Rosseland mean" after Sven Rosseland, who first showed that this was the way to remove the frequency dependence and maintain the simple diffusion approximation. Note that since

$$\frac{\partial B}{\partial T}d\nu = \frac{acT^3}{\pi}$$

then we have

$$\frac{1}{\kappa} = \frac{\int_o^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}d\nu}{\frac{\partial B}{\partial T}d\nu}.$$