M4112 THE SUN
THE MIXING LENGTH THEORY

This is NOT EXAMINABLE!

You should go through this ONCE to see how it works...
It used to be an assignment....I have left it in that form!

If we have radiation and convection carrying the energy then the total flux is just the sum of these. Define ∇_{rad} to be the gradient needed to carry a given total flux by radiation:

$$F_{rad} + F_{con} = \frac{4acG}{3} \frac{mT^4}{\kappa P r^2} \nabla_{rad}.$$

Note that the radiation actually carries a flux

$$F_{rad} = \frac{4acG}{3} \frac{mT^4}{\kappa P r^2} \nabla$$

where ∇ is the actual gradient. Note that ∇ is unknown: indeed, it is our aim to find it !

Consider a blob of material with temperatures excess DT over its surroundings. Here DX is as defined in lectures:

$$DX = \left[\left(\frac{dX}{dr} \right)_e - \left(\frac{dX}{dr} \right)_s \right] \Delta r.$$

Let this blob be moving radially with speed v in pressure equilibrium ($DP = 0$).

(i) Show that local flux of convective energy is

$$F_{con} = \rho c_P v DT.$$

We need to replace vDT by some suitable and appropriate average over the whole convective zone. At the start of the motion $DT = 0$ and $v = 0$. Obviously DT and v increase as the blob rises. We shall assume that after moving an average distance of one "mixing length" ℓ_m the element mixes with its surroundings and loses its identity. Elements passing through a sphere of constant r will have many different v's and DT's because they will have started from different distances from the current position. But these distances must be bounded by zero and ℓ_m. We shall take $\ell_m/2$ to be the appropriate average.

(ii) Show that we get

$$\frac{1}{T} DT = (\nabla - \nabla_e) \frac{\ell_m}{2H_P}.$$

For $DP = 0$, and assuming that the convection takes place in a region of constant composition (*i.e.* $D\mu = 0$) then

$$\frac{1}{\rho}D\rho = \frac{\partial \ln \rho}{\partial \ln T}\frac{DT}{T}$$
$$= -\delta\frac{DT}{T},$$

where

$$\delta = -\frac{\partial \ln \rho}{\partial \ln T}.$$

The (radial) buoyancy force per unit mass is just $k_r = -gD\rho/\rho$. But, on average, only half this force will have acted on the element as it rises over its average motion of $\ell_m/2$.

(iii) Show that the work done is

$$W = \text{Force } \times \text{distance}$$
$$= \frac{1}{8}g\delta\frac{\ell_m^2}{H_p}(\nabla - \nabla_e).$$

Now, suppose that half of this energy goes into the kinetic motion of the blob and half is taken up by pushing aside the matter ahead of the blob. Then

$$\text{kinetic energy per mass} = \frac{1}{2}v^2 = \frac{1}{2}W.$$

(iv) Hence show that

$$v^2 = g\delta(\nabla - \nabla_e)\frac{\ell_m^2}{8H_P}$$

and

$$F_{con} = \rho c_P T\sqrt{g\delta}\frac{\ell_m^2}{4\sqrt{2}}H_P^{-3/2}(\nabla - \nabla_e)^{3/2}.$$

We now need the change of temperature of the element as it moves. Let the element have a surface area S, volume V and diameter d. The total temperature change is due to adiabatic expansion (or compression) plus radiative losses to its surrounds. The radiative flux due to a temperature gradient ∇T is just

$$\mathbf{f} = \frac{-4ac}{3}\frac{T^3}{\kappa\rho}\nabla T$$

in the diffusion approximation, which will be assumed valid here also. But here we are interested in the flux due to ∇T perpendicular to the surface of the element. Denote this by $\frac{dT}{dn}$ where the n means normal to the surface S. After rising a distance Δr the element has a temperature:

$$DT = \left[\left(\frac{dT}{dr}\right)_e - \left(\frac{dT}{dr}\right)_s\right]\Delta r$$

higher than its surroundings. It will be sufficiently accurate for our purposes to take $\frac{dT}{dn} \simeq DT/(d/2)$. Now, since the flux of radiation is lost from the whole surface S we finally obtain the radiative losses per unit time from the blob, λ, as

$$\begin{aligned}\lambda &= Sf \\ &= \frac{8acT^3}{3\kappa\rho} DT \frac{S}{d}.\end{aligned}$$

(v) Hence show that:

$$\left(\frac{dT}{dr}\right)_e = \left(\frac{dT}{dr}\right)_{ad} - \frac{\lambda}{\rho V c_P v}.$$

(vi) Further, show that for a spherical blob of diameter d:

$$\nabla_e - \nabla_{ad} = \frac{8ac}{3\kappa}\left(\frac{T}{\rho}\right)^2 \frac{H_P}{v c_P} DT \frac{6}{d^2}.$$

(vii) Thus derive:

$$\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{8acT^3}{\kappa\rho^2 v c_P}\left(\frac{\ell_m}{d^2}\right).$$

Numerical calculations of three-dimensional convection indicate that the convective blobs are not spheres, of course. We try to include these effects by replacing the factor $8\ell_m/d^2$ by $6/\ell_m$, which results from such calculations. In that case we replace the last equation by

$$\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{6acT^3}{\kappa\rho^2 v c_P \ell_m}.$$

We will use this equation from now on.

We now have 5 equations

$$F_{rad} + F_{con} = \frac{4acG}{3}\frac{mT^4}{\kappa P r^2}\nabla_{rad} \tag{1}$$

$$F_{rad} = \frac{4acG}{3}\frac{mT^4}{\kappa P r^2}\nabla \tag{2}$$

$$v^2 = g\delta(\nabla - \nabla_e)\frac{\ell_m^2}{8H_P} \tag{3}$$

$$F_{con} = \rho c_P T\sqrt{g\delta}\frac{\ell_m^2}{4\sqrt{2}}H_P^{-3/2}(\nabla - \nabla_e)^{3/2} \tag{4}$$

$$\frac{\nabla_e - \nabla_{ad}}{\nabla - \nabla_e} = \frac{6acT^3}{\kappa\rho^2 c_P \ell_m v} \tag{5}$$

for the 5 unknowns

$$F_{rad}, F_{con}, v, \nabla_e, \nabla$$

in terms of the local variables

$$P, T, \rho, c_P, \nabla_{ad}, \nabla_{rad}, g, \delta.$$

What we do *not* have is a value for ℓ_m ! It is traditional, and not entirely ridiculous, to take a multiple of the pressure scale-height:

$$\ell_m = \alpha H_P$$

where $\alpha \simeq$1–2.

Let us define two dimensionless variables U and W by

$$U = \frac{3acT^3}{c_P \rho^2 \kappa \ell_m^2} \sqrt{\frac{8H_P}{g\delta}}$$

$$W = \nabla_{rad} - \nabla_{ad}.$$

(viii) From these derive the following equalities:

$$\nabla_e - \nabla_{ad} = 2U\sqrt{\nabla - \nabla_e}.$$

$$(\nabla - \nabla_e)^{3/2} = \frac{8U}{9}(\nabla_{rad} - \nabla).$$

For a given W the solutions depend critically on U. Lets now write (2) and (4) as:

$$F_{rad} = \sigma_{rad}\nabla$$

and

$$F_{con} = \sigma_{con}(\nabla - \nabla_e)^{3/2}.$$

Here the σ's are just conductivities for each process.

(ix) Hence show that:

$$U = \frac{9}{8}\frac{\sigma_{rad}}{\sigma_{con}} = \text{ratio of conductivities.}$$

(x) Now show that:

$$\begin{aligned} U \to 0 \text{ means } F_{rad} &\ll F_{con} \\ \nabla_{rad} &\gg \nabla_{ad} \\ \nabla &= \nabla_{ad}. \end{aligned}$$

(xi) Likewise, show that:

$$\begin{aligned} U \to \infty \text{ means } F_{con} &\ll F_{rad} \\ \nabla &= \nabla_{rad}. \end{aligned}.$$

Note that for $0 < U < \infty$ we have $\nabla_{rad} > \nabla > \nabla_{ad}$. The actual gradient is determined by solving (numerically) equations (1) – (5). The gradient is said to be "super-adiabatic". This is true of the convection in stellar envelopes.

To get a feel for these quantities, here are some typicl numbers: For an ideal monatomic gas $\delta = 1, \mu = 1, c_P/\Re = 5/2$. Take $\ell_m = H_P$, and let $r = R_\odot/2, m = M_\odot/2, T = 10^7$K, $\kappa = 1\text{cm}^2\text{g}^{-1}$ and $\rho = 1\text{g cm}^{-3}$. Then

$$U \simeq 10^{-8} \text{ and } \nabla - \nabla_{ad} \simeq 10^{-5}!$$

In practice $\nabla - \nabla_{ad}$ is so small that it can often be set to zero. *i.e.* we can assume $\nabla = \nabla_{ad}$ wherever there is convection.

Returning to the Temperature Equation we can write it in a more suggestive, and versatile, form as

$$\begin{aligned}\frac{\partial T}{\partial m} &= \frac{dT}{dP}\frac{\partial P}{\partial m}\\ &= \frac{T}{P}\frac{d\ln T}{d\ln P}\frac{\partial P}{\partial m}\\ &= -\frac{T}{P}\frac{Gm}{4\pi r^4}\nabla\end{aligned}$$

where

$$\nabla = \begin{cases}\nabla_{rad}, & \text{if radiative;}\\ \nabla_{ad}, & \text{if convective.}\end{cases}$$

In some cases we may need the degree of super-adiabaticity, in which case we have to solve the mixing length theory (hereafter MLT) equations for ∇ in the convective zone.

Now lets actually solve these equations !

Define ξ by

$$\xi - U = (\nabla - \nabla_e)^{1/2}.$$

(xii) Show that the super-adiabaticity is given by:

$$x \equiv \nabla - \nabla_{ad} = \xi^2 - U^2.$$

(xiii) Combine these equations into a cubic for ξ in terms of U and $W = \nabla_{rad} - \nabla_{ad}$:

$$(\xi - U)^3 + \frac{8U}{9}\left(\xi^2 - U^2 - W\right) = 0. \tag{6}$$

(xiv) Define Γ by

$$\Gamma \equiv \frac{(\nabla - \nabla_e)^{1/2}}{2U} = \frac{\nabla - \nabla_e}{\nabla_e - \nabla_{ad}}.$$

By considering a spherical blob with diameter $d = \ell_m$ and surface area A, show that

$$\Gamma = \frac{1}{3}\frac{F_{con}A}{\lambda}.$$

Hence we see that Γ is related to the convective efficiency, being essentially the ratio of energy transported to energy lost.

The solution to the mixing length theory is the determination of ∇ (or ξ or x) for a given U or W. Before solving equation (6) lets look at its limiting behaviour. First, divide the $\log U$–$\log W$ plane into regions of high and low convective efficiency, by finding the line $\Gamma = 1$.

(xv) First, show that

$$W = U^2 \left(9\Gamma^3 + 4\Gamma^2 + 4\Gamma\right).$$

(xvi) Hence show that $\Gamma = 1$ corresponds to $W = 17U^2$, and that

$$\Gamma\left(1+\Gamma\right) = \frac{x}{4U^2},$$

and hence when $x \gg U^2$ show that

$$x \simeq \left(\frac{8}{9}UW\right)^{2/3}.$$

(xvii) Similarly, show that when $x \ll U^2$ then $x \simeq W$.

(xviii) Now solve equation (6) numerically, for $-4 \leq \log U \leq 4$ and $-4 \leq \log W \leq 4$. Plot the solution as contours of $x = 10^{-4}, 10^{-3}, \ldots 10^{4}$ in the $\log U$–$\log W$ plane. Also plot lines of constant $\Gamma = 10^{-4}, 1$ and 10^{2}. You should your choice of numerical technique, the convergence criterion, your choice of initial guesses and why you believe that your solution is correct !

Please hand me your work by Friday August 30.

JCL