STELLAR STRUCTURE AND EVOLUTION NOTES

PART III: The Equations of Stellar Structure

Contents

I. Mass Conservation–the Radius equation

We'll often use mass as the independent variable. This is logical because one does not know *a priori* the radius of a model at a given time, so an Eulerian formulation is not practical. In any case, we'll write the equations with both r and $m = m(r)$ as the independent variable. One can always transform between one formulation and the other by using the mass conservation equation, below.

Trivially, since mass must be conserved in going a small distance dr between two concentric shells of a spherical star we must have:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

or

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}. \tag{R}$$

II. Hydrostatic Equilibrium–the Pressure equation

Consider a concentric shell of width dr in the star. The outward force due to pressure is $dPdA = \frac{\partial P}{\partial r} dAdr$, where dA is the surface area and dP is the difference in pressure across dr. The force inward due to gravity is just $(-Gm/r^2)dM = (-Gm/r^2)\rho dAdr$. Equating these gives us the equation of hydrostatic equilibrium:

$$\frac{\partial P}{\partial r} = -\left(\frac{Gm\rho}{r^2}\right).$$

Using equation (R):

$$\frac{\partial P}{\partial m} = -\left(\frac{Gm}{4\pi r^4}\right). \tag{P}$$

III. Energy Conservation–the Luminosity equation

If the nuclear energy generation is ϵ erg/gm/sec then, trivially:

$$\frac{\partial \ell}{\partial r} = 4\pi r^2 \rho \epsilon,$$

or

$$\frac{\partial \ell}{\partial m} = \epsilon.$$

But this has assumed that there is no change in energy due to sources other than nuclear. Often a star will be slowly contracting or expanding, doing work against (or having work done by) the gravitational field, together with the contribution from PdV. There could also be changes in the internal energy of the gas, such as release from radioactive decay. In

summary we may need a more general equation. We have used, in the above:

$$dQ = dU + PdV = 0$$

where

$$\frac{dQ}{dt} = \left(\epsilon_n - \frac{\partial \ell}{\partial m}\right),$$

where ϵ_n is the energy released by nuclear reactions. In general we have

$$dQ = dU + PdV = dU - \frac{P}{\rho^2}d\rho$$

Thus

$$\frac{dQ}{dt} = \left(\epsilon_n - \frac{\partial \ell}{\partial m}\right) = \frac{\partial u}{\partial t} - \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}$$

or

$$\frac{\partial \ell}{\partial m} = \epsilon_n + \epsilon_g,$$

where

$$\epsilon_g = -\frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}.$$

In some stages of evolution energy losses by neutrinos may also be important. Thus we add (rather, subtract) ϵ_ν:

$$\frac{\partial \ell}{\partial m} = \epsilon_n + \epsilon_g - \epsilon_\nu, \qquad (L)$$

or

$$\frac{\partial \ell}{\partial r} = 4\pi r^2 \rho(\epsilon_n + \epsilon_g - \epsilon_\nu).$$

It is common to write $\epsilon = \epsilon_n + \epsilon_g - \epsilon_\nu$.

IV. Energy Transport–the Temperature equations

IV.1 Energy Transport by Radiation: the Diffusion Approximation

Obviously the mean free path ℓ_γ of a photon must be inversely proportional to the density. *i.e.* $\ell_\gamma \propto 1/\rho$. We let the co-efficient of proportionality be $1/\kappa$, where κ is the "radiative opacity", or the "absorption co-efficient". We thus have:

$$\ell_\gamma = \frac{1}{\kappa\rho}.$$

Typically $\kappa \simeq 1\text{cm}^2\text{g}^{-1}$, and $\rho \simeq 1\text{g cm}^{-3}$. Thus we see that $\ell_\gamma \simeq 1\text{cm}$. *i.e.* stellar material is very opaque to radiation.

We can estimate a typical stellar temperature gradient by

$$\overline{\frac{dT}{dr}} \sim \frac{T_c - T_e}{R} \sim \frac{10^7\text{K} - 10^3\text{K}}{10^{11}\text{cm}} \sim 10^{-4}\text{K/cm}.$$

So a typical photon will, during its lifetime, see a temperature difference of $\Delta T \sim (dT/dr)\ell_\gamma \sim 10^{-4}\text{K}$. Thus we have $\ell_\gamma \ll R$, the stellar radius. In this case we can treat the radiation as a diffusion process, and the resulting theory is called the "diffusion approximation" to radiative transfer.

The diffusive flux $\mathbf{j}$ of particles per unit area per unit time between places of different particle density n is simply:

$$\mathbf{j} = -D\boldsymbol{\nabla} n \tag{1}$$

where D is the diffusion co-efficient

$$D = \frac{1}{3}v\ell \tag{2}$$

for a mean free-path ℓ. For radiative transfer we replace n by the energy density $U = aT^4$. Because of spherical symmetry $\boldsymbol{\nabla} U$ is purely radial:

$$\boldsymbol{\nabla} U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} = 4aT^3\frac{\partial T}{\partial r}\hat{\mathbf{r}}.$$

The flux is also radial, and of magnitude F, so that equation (1) becomes

$$\begin{aligned} F &= -\frac{1}{3}v\ell_\gamma\frac{\partial U}{\partial r} \\ &= -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial r}. \end{aligned}$$

Note that this is just a heat conduction equation:

$$\mathbf{F} = -k_{rad}\boldsymbol{\nabla} T$$

where

$$k_{rad} = \frac{4ac}{3}\frac{T^3}{\kappa\rho}.$$

To show this we note that F is measured in erg/cm^2/sec, which is the product of an energy density and a velocity. Thus $\mathbf{F} = u\mathbf{v}$. But from thermodynamics we know that the internal energy per unit volume is $c_v T\rho$. Conservation of energy then demands that

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot(u\mathbf{v}) = 0.$$

Now if c_v is constant this is equivalent to

$$T\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right) + \rho\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} T\right) = 0.$$

Now the first term in parenthesis is zero, due to mass conservation. Thus we have

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} T = 0,$$

which we can combine with the diffusion equation

$$\mathbf{v} = -k\boldsymbol{\nabla} T$$

to finally give us the heat equation:

$$\frac{\partial T}{\partial t} = k\nabla^2 T.$$

Now, since the flux $F = \ell/4\pi r^2$ we get

$$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{\kappa}{r^2}\frac{\rho\ell}{T^3}$$

or, in terms of m:

$$\frac{\partial T}{\partial m} = -\frac{3}{64\pi^2 ac}\frac{\kappa}{r^4}\frac{\ell}{T^3}. \qquad (T1)$$

Of course, near the surface of the star the diffusion approximation is no longer valid: ρ decreases, so ℓ_γ increases and the distance to be travelled is no longer $\gg \ell_\gamma$.

Note that we can re-write equation T1 in a more suggestive manner, to be used later. We have

$$F = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial r}.$$

If we multiply by

$$\frac{\partial r}{\partial P}\times\frac{\partial P}{\partial r}$$

we get

$$F = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial P}\left(\frac{-GM\rho}{r^2}\right)$$

which we write as

$$F = -\frac{4acG}{3}\frac{T^4 m}{\kappa P r^2}\nabla$$

where

$$\nabla = \frac{\partial \ln T}{\partial \ln P}.$$

At this stage we have not yet considered energy transport by conduction. We shall discuss this later, but clearly it satisfies a heat conduction equation (!) so we can still use equation T1, provided we use a suitable κ. In fact we take

$$\frac{1}{\kappa_{tot}} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}.$$

IV.2 Limits on Radiation: Convective Instability

Consider an element of stellar material e which rises in pressure equilibrium with its surroundings s from a point r at pressure P, temperature T, and density ρ to a new point at $r+dr$ with pressure $P+dP$, temperature $T+dT$ and density $\rho+d\rho$. Let us define the associated change in a quantity X due to this motion as

$$DX = \left[\left(\frac{dX}{dr}\right)_e - \left(\frac{dX}{dr}\right)_s\right]\Delta r.$$

Note that $DP = 0$ because we assume that the element is at all times in equilibrium with its surroundings. (Is this reasonable ? Why ?) The (radial) buoyancy force $\mathbf{B}$ per unit volume which is associated with a density difference $D\rho$ is just

$$\mathbf{B} = -\mathbf{g}D\rho.$$

(We can ignore the vectors, since all are directions are radial.) Clearly, if $D\rho < 0$ then $B > 0$ and the element continues to rise. But if $D\rho > 0$ then $B < 0$ and the element sinks back toward its original position. Thus, for stability we require:

$$\left(\frac{d\rho}{dr}\right)_e - \left(\frac{d\rho}{dr}\right)_s > 0.$$

This is fine, but virtually useless. In a calculation of a model star we don't know $d\rho/dr$. We'll now spend some time turning this condition into one which we *can* evaluate, one which involves only locally known variables.

We'll assume that the element moves adiabatically. Let $\rho = \rho(P, T, \mu)$, where μ is the mean molecular weight per free particle (in atomic mass units). Then

$$d\rho = \frac{\partial \rho}{\partial P} dP + \frac{\partial \rho}{\partial T} dT + \frac{\partial \rho}{\partial \mu} d\mu \tag{1}$$

or

$$\frac{d\rho}{\rho} = \frac{\partial \ln \rho}{\partial \ln P}\frac{dP}{P} + \frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{T} + \frac{\partial \ln \rho}{\partial \ln \mu}\frac{d\mu}{\mu}.$$

Thus our stability condition becomes:

$$\left(\frac{1}{P}\frac{\partial \ln \rho}{\partial \ln P}\frac{dP}{dr}\right)_e + \left(\frac{1}{T}\frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{dr}\right)_e + \left(\frac{1}{\mu}\frac{\partial \ln \rho}{\partial \ln \mu}\frac{d\mu}{dr}\right)_e - \left(\frac{1}{P}\frac{\partial \ln \rho}{\partial \ln P}\frac{dP}{dr}\right)_s - \left(\frac{1}{T}\frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{dr}\right)_s - \left(\frac{1}{\mu}\frac{\partial \ln \rho}{\partial \ln \mu}\frac{d\mu}{dr}\right)_s > 0.$$

Now the third term is zero because the element does not change its composition during its rise. Also, we have assumed that $DP = 0$ so the terms containing $\frac{dP}{dr}$ cancel. We thus have:

$$\left(\frac{1}{T}\frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{dr}\right)_e - \left(\frac{1}{T}\frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{dr}\right)_s - \left(\frac{1}{\mu}\frac{\partial \ln \rho}{\partial \ln \mu}\frac{d\mu}{dr}\right)_s > 0.$$

Let us now define the "pressure scale-height" H_P by

$$H_P = -P\frac{dr}{dP} = -\frac{dr}{d\ln P}.$$

Multiplying the previous equation by H_P gives:

$$\left(-\frac{P}{T}\frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{dP}\right)_e + \left(\frac{P}{T}\frac{\partial \ln \rho}{\partial \ln T}\frac{dT}{dP}\right)_s + \left(\frac{P}{\mu}\frac{\partial \ln \rho}{\partial \ln \mu}\frac{d\mu}{dP}\right)_s < 0,$$

which we may write as

$$\left(-\frac{\partial \ln \rho}{\partial \ln T}\frac{d\ln T}{d\ln P}\right)_e + \left(\frac{\partial \ln \rho}{\partial \ln T}\frac{d\ln T}{d\ln P}\right)_s + \left(\frac{\partial \ln \rho}{\partial \ln \mu}\frac{d\ln \mu}{d\ln P}\right)_s < 0,$$

or

$$\left(\frac{d\ln T}{d\ln P}\right)_s < \left(\frac{d\ln T}{d\ln P}\right)_e - \left(\frac{\frac{\partial \ln \rho}{\partial \ln \mu}}{\frac{\partial \ln \rho}{\partial \ln T}}\right)\left(\frac{d\ln \mu}{d\ln P}\right)_s.$$

With the following definitions:

$$\theta = \frac{\partial \ln \rho}{\partial \ln \mu}\Big/\frac{\partial \ln \rho}{\partial \ln T}$$

$$\nabla = \left(\frac{d\ln T}{d\ln P}\right)_s$$

$$\nabla_e = \left(\frac{d\ln T}{d\ln P}\right)_e$$

$$\nabla_\mu = \left(\frac{d\ln \mu}{d\ln P}\right)_s$$

then we obtain

$$\nabla < \nabla_e - \theta\nabla_\mu.$$

In the stable case the energy is transported by radiation. Thus $\nabla = \nabla_{rad}$. In keeping with our assumption that the element moves adiabatically we'll put $\nabla_e = \nabla_{ad}$. Thus we obtain the "Ledoux criterion":

$$\nabla_{rad} < \nabla_{ad} - \theta\nabla_\mu.$$

If $\nabla_\mu = 0$ we obtain the "Schwarzschild criterion":

$$\nabla_{rad} < \nabla_{ad}.$$

This tells us that the radiation will carry the energy until the temperature gradient exceeds $\nabla_{rad} = \nabla_{ad} - \theta\nabla_\mu$. Then convection will develop. So we have:

1. if $\nabla_{rad} < \nabla_{ad} - \theta\nabla_\mu$ then $\nabla = \nabla_{rad}$.
2. if $\nabla_{rad} > \nabla_{ad} - \theta\nabla_\mu$ then $\nabla = ?$

To determine ∇ in the convective case we need a theory of convection.

IV.3 Energy Transport by Convection: the "Mixing-Length" Theory

See Assignment 1 !

We can write the temperature equation in a more versatile form:

$$\begin{aligned}\frac{\partial T}{\partial m} &= \frac{dT}{dP}\frac{\partial P}{\partial m} \\ &= \frac{T}{P}\frac{d\ln T}{d\ln P}\frac{\partial P}{\partial m} \\ &= -\frac{T}{P}\frac{Gm}{4\pi r^4}\nabla\end{aligned}$$

where

$$\nabla = \begin{cases} \nabla_{rad}, & \text{if radiative;} \\ \nabla_{ad}, & \text{if convective.}\end{cases}$$

In some cases we may need the degree of super-adiabaticity, in which case we have to solve the mixing length theory (hereafter MLT) equations for ∇ in the convective zone.

V. Summary of the Equations

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}$$

$$\frac{\partial L}{\partial m} = \epsilon + \epsilon_g - \epsilon_\nu$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla$$

where

$$\epsilon_g = -\frac{\partial U}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}$$

and

$$\nabla = \begin{cases} \nabla_{rad} = \frac{3}{16\pi acG}\frac{\kappa LP}{mT^4} & \text{if } \nabla_{rad} < \nabla_{ad} - \theta\nabla_\mu; \\ \text{solution of MLT} = \begin{cases} \nabla_{ad}; & \text{in deep interior;} \\ \text{soln of eqns 1–5;} & \text{in envelope.} \end{cases} & \text{if } \nabla_{rad} > \nabla_{ad} - \theta\nabla_\mu. \end{cases}$$

Remember that

$$\nabla_{ad} = \left(\frac{d\ln T}{d\ln P}\right)_{\text{adiabatic}}$$

$$\nabla_\mu = \frac{d\ln T}{d\ln \mu}$$

and

$$\theta = \frac{\partial \ln \rho}{\partial \ln \mu}\Big/\frac{\partial \ln \rho}{\partial \ln T}.$$

Note that there is a slight inconsistency. We assumed that the blob moves adiabatically in deriving the instability condition, yet in the MLT itself we allowed for radiative losses. Also, we assumed a homogeneous composition in the MLT. These corrections are small when compared to the crudity of the theory !

VI. Changes in Chemical Composition

Let X_i be the mass fraction of element i. Then

$$X_i = \frac{m_i n_i}{\rho}$$

where n_i is the number density of element i. Then if nuclear reactions can produce element i from element j at a rate r_{ji}, and destroy element i by transforming it into element k at a rate r_{ik}, we have

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\left[\sum_j r_{ji} - \sum_k r_{ik}\right]$$

for $i = 1, 2, \ldots I$ where there are I species. Note that in a convective zone the timescale for convection is so quick that we can "always" take the zone to have homogeneous composition.

So, after making a model and finding X_i at each mass shell, we find the convective boundaries and homogenize (*i.e.* mix) the abundances in this region.

VII. Solving the Equations

VII.1 Simplifications

Now that we know the basic equations of stellar structure we'll look at how to solve them. But first we'll make a simplifying assumption, which will simplify the algebra but still illustrate the method of solution. We shall assume that the star is in "complete equilibrium", *i.e.* mechanical and thermal equilibrium. In this case eq uation (L) becomes (in the absence of neutrinos):

$$\frac{\partial \ell}{\partial m} = \epsilon$$

since complete thermal equilibrium means $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial \rho}{\partial t} = 0$. Thus the four equations determine the structure for a given composition $X_i(m)$. They thus become 4 ODE's as there is no longer any time dependence, and mass is the only independent variable. One may ask how models in complete equilibrium follow the evolution of a star. The answer is through the time dependence of the $X_i(m)$. Once the structure is determined, by solving the 4 ODE's, we then use the resulting run of physical variables to calculate the nuclear reaction rates. These, coupled with a Δt, determine the new $X_i(m)$ at $t + \Delta t$. With these new $X_i(m)$ we have changed the value of $\mu(m)$, thus altering the pressure. Also, such variables as the opacity and nuclear energy generation depend on the composition. So they will also have changed. Thus the 4 ODE's must be solved again, for the new structure. In this way we couple the structure of the star with the time-dependence of the composition, and recover the evolution of the model ! (Note that we have assumed that the "constitutive relations" such as $\kappa(\rho, T, X_i)$, $P(\rho, T, X_i)$, and $\nabla_{ad}(\rho, T, X_i)$ are known. In fact, we will spend quite a few lectures on determining these !)

VII.2 Central Boundary Conditions

Clearly we have both $r = 0$ and $\ell = 0$ at $m = 0$. The temperature, radius and pressure equations are actually singular at the origin. Whether this is a problem or not depends on how we difference the equations when we construct the numerical method. So lets talk about the behaviour of these functions near the origin.

Now $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$ can be written as

$$d(r^3) = \frac{3}{4\pi\rho} dm.$$

If we assume that the density is approximately constant for small values of m near the origin, and takes the value ρ_c, then we can integrate this equation:

$$r \simeq \left(\frac{3}{4\pi\rho_c}\right)^{1/3} m^{1/3}. \tag{1}$$

The hydrostatic support equation

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

can be written, using (1), as:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi}\left(\frac{3}{4\pi\rho_c}\right)^{-4/3} m^{-4/3}$$

which can be integrated to give

$$P - P_c = -\frac{3G}{8\pi}\left(\frac{3}{4\pi\rho_c}\right)^{4/3} m^{2/3}. \tag{2}$$

Of course, the pressure gradient must vanish at the centre, where the pressure is a maximum $= P_c$. Using (1) and (2):

$$\frac{dP}{dr} \sim \frac{m}{r^2} \sim r \rightarrow 0.$$

Consider now the ℓ equation:

$$\frac{d\ell}{dm} = \epsilon.$$

With the assumption that ϵ changes very little for small m near zero we can integrate this immediately to get

$$\ell \simeq \epsilon_c m.$$

Finally, for radiative transfer:

$$\frac{dT}{dm} = -\frac{3}{64\pi^2 ac}\frac{\kappa\ell}{r^4T^3}$$

which can be written as

$$4T^3 dT = -\frac{3\kappa_c\epsilon_c}{4\pi ac}\frac{\rho_c}{3}\left(\frac{4\pi\rho_c}{3}\right)^{1/3} m^{-1/3}dm.$$

Integrating yields:

$$T^4 - T_c^4 = -\frac{1}{2ac}\left(\frac{3}{4\pi}\right)^{2/3}\rho_c^{4/3}\kappa_c\epsilon_c m^{2/3}.$$

If the central region is convective, then

$$\frac{dT}{dm} = -\frac{GmT}{4\pi r^4 P}\nabla_{ad}$$

which can be integrated to give

$$\ln T - \ln T_c = -\left(\frac{\pi}{6}\right)^{1/3}\frac{G\nabla_{\mathrm{ad,c}}\,\rho_c^{4/3}}{P_c}m^{2/3}.$$

VII.3 Surface Boundary Conditions

The most naive surface boundary conditions are the so-called "radiative zero" conditions:

$$T = P = 0 \quad \text{at} \quad m = M.$$

In reality, of course, there is a very gradual decrease of density (and thus pressure) away from the star. So the condition $P = T = 0$ is not entirely satisfactory. On the other hand, we are not interested in the regions of extremely low density, where the stellar material merges with the surrounding interstelar medium. Can we not find some obvious physical boundary for the star ? A quite reasonable definition of the surface is the photosphere, that layer from which photons leave the star, on average, rather than suffering further collisions with stellar matter. This is, after all, the region of the star from which *all* our information is derived ! Detailed models of stellar atmospheres tell us that this point is at optical depth $\tau = 2/3$ where

$$\tau(r) = \int_r^\infty \kappa(r)\rho(r)dr.$$

Lets assume that the opacity is constant $= \bar{\kappa}$ just beyond the photosphere at $r = R$, where the largest contribution to the optical depth is found. Then we have

$$\frac{2}{3} = \bar{\kappa} \int_R^\infty \rho dr.$$

Now, from the pressure equation

$$P(R) = \int_R^\infty g\rho dr = g_0 \int_R^\infty \rho dr$$

where we have put $g_0 = GM/R^2 \simeq$ constant just beyond the photosphere. Combining these two equations yields

$$P(R) = \frac{GM}{R^2}\frac{2}{3}\frac{1}{\bar{\kappa}}.$$

Now, at the photosphere $T = T_{eff}$, be definition. Thus

$$L = 4\pi R^2 \sigma T_{eff}^4.$$

These two equations define $L(R)$ and $P(R)$, or rather, $L(M)$ and $P(M)$, which we take as the two surface boundary conditions. In reality, of course, the diffusion approximation fails below the photosphere, which introduces errors in the above integrations. This can be overcome by using a fitting point m_F deep enough for the diffusion approximation to be valid, but sufficiently near the surface for $\ell \simeq L$, the total luminosity. We won't consider this technique any further, and in what follows we shall, for simplicity, use the radiative zero conditions.

VII.4 Numerical Technique

Note that we have four equations for ℓ, T, P and r in terms of m. And, although we have four boundary conditions, they are split: two at the surface and two at the centre. Thus a naive application of a standard integration technique, such as the Runge-Kutta method, is not appropriate. One could, however, guess a central temperature and pressure, T_c and P_c and integrate outwards (or, equivalently, guess a surface L and R and integrate inwards). Of course, the results of this integration will *not* satisfy the surface boundary conditions unless our guesses for the central values were correct. We can then iterate on the initial guesses until the solution satisfies the surface boundary conditions. This is called the "shooting method" because one guesses the missing boundary conditions, itegrates from one boundary to the other (the "shooting" phase), and then estimates the corrections. The method is moderately slow, with much integration done to get a converged model. It can often diverge, or converge *very* slowly due to the sensitivity of the surface values to small changes in the interior.

A more common and efficient method is that due to Henyey and co-workers, and is called the "Henyey Method", but is really nothing more than the generalised Newton's method for multiple non-linear equations. In this we write the four equations as:

$$\frac{dy^i}{dm} = f^i(y^1, y^2, y^3, y^4)$$

where $i = 1, 2, 3, 4$ and

$$\begin{aligned} y^1 &= r \\ y^2 &= P \\ y^3 &= L \\ y^4 &= T. \end{aligned}$$

Consider dividing our star into $N-1$ mass shells (not necessarily equal !) m_j, $j = 1, \ldots, N$. Let all variables be defined at the shell boundaries. We now have N shell boundaries, with four unknown variables at each, namely the y^i_j for $i = 1, 2, 3, 4$ and $j = 1, \ldots, N$. But we know 4 y values, at the central and surface boundaries:

$$\begin{aligned} r_1 &= y^1_1 = 0 \\ \ell_1 &= y^3_1 = 0 \\ P_N &= y^2_N = 0 \\ T_N &= y^4_N = 0. \end{aligned}$$

So there are really $4(N-1)$ unknowns. Also, for the N boundaries there are $N-1$ zones across which the four de's apply, giving us a total of $4(N-1)$ equations. So the system is well posed.

Let us now construct

$$A^i_j = \frac{y^i_j - y^i_{j+1}}{m_j - m_{j+1}} - f^i(y^1_{j+\frac{1}{2}}, y^2_{j+\ half}, y^3_{j+\frac{1}{2}}, y^4_{j+\frac{1}{2}}).$$

The first term is a forward differenced approximation to $\frac{dy^i}{dm}$ at mesh point j. The second term is an approximation to f^i across the shell. For $y^i_{j+\frac{1}{2}}$ we can take either the arithmetic mean

$$y^i_{j+\frac{1}{2}} = \frac{y^i_j + y^i_{j+1}}{2},$$

or the geometric mean

$$y^i_{j+\frac{1}{2}} = \sqrt{y^i_j y^i_{j+1}}.$$

For simplicity, we shall take the arithmetic mean in the following discussion. Now, obviously we require $A^i_j = 0 \ \forall \ i$ and j. Suppose we have a guess for the y^i_j, perhaps from a previous calculation, or a model at a slightly earlier time. Then the A^i_j will not be zero. Lets take expansions of A^i_j by Taylor series to only first order in the changes δy^i_j in the y^i_j:

$$\begin{aligned} \delta A^i_j &= \sum_{k=1}^{4} \frac{\partial A^i_j}{\partial y^k_j} \delta y^k_j + \sum_{k=1}^{4} \frac{\partial A^i_j}{\partial y^k_{j+1}} \delta y^k_{j+1} \\ &= \frac{\partial A^i_j}{\partial y^1_j} \delta y^1_j + \frac{\partial A^i_j}{\partial y^2_j} \delta y^2_j + \frac{\partial A^i_j}{\partial y^3_j} \delta y^3_j + \frac{\partial A^i_j}{\partial y^4_j} \delta y^4_j \\ &\qquad + \frac{\partial A^i_j}{\partial y^1_{j+1}} \delta y^1_{j+1} + \frac{\partial A^i_j}{\partial y^2_{j+1}} \delta y^2_{j+1} + \frac{\partial A^i_j}{\partial y^3_{j+1}} \delta y^3_{j+1} + \frac{\partial A^i_j}{\partial y^4_{j+1}} \delta y^4_{j+1}. \end{aligned}$$

We wish to find the δA_j^i such that

$$A_j^i + \delta A_j^i = 0.$$

We may write this in matrix form

$$\mathbf{H}\delta = -\mathbf{A}$$

where

$$\boldsymbol{\delta} = (\, \delta y_1^2 \quad \delta y_1^4 \quad \delta y_2^1 \quad \delta y_2^2 \quad \delta y_2^3 \quad \delta y_2^4 \quad \ldots \quad \delta y_{N-1}^1 \quad \delta y_{N-1}^2 \quad \delta y_{N-1}^3 \quad \delta y_{N-1}^4 \quad \delta y_N^1 \quad \delta y_N^3 \,)^T ,$$

a $4(N-1)$ element column vector,

$$\mathbf{A} = (\, A_1^1 \quad A_1^2 \quad A_1^3 \quad A_1^4 \quad A_2^1 \quad A_2^2 \quad A_2^3 \quad A_2^4 \quad \ldots \quad A_{N-1}^1 \quad A_{N-1}^2 \quad A_{N-1}^3 \quad A_{N-1}^4 \,)^T ,$$

another $4(N-1)$ element column vector, and

$$H = \begin{pmatrix} \frac{\partial A_1^1}{\partial y_1^2} & \frac{\partial A_1^1}{\partial y_1^4} & \frac{\partial A_1^1}{\partial y_2^1} & \frac{\partial A_1^1}{\partial y_2^2} & \frac{\partial A_1^1}{\partial y_2^3} & \frac{\partial A_1^1}{\partial y_2^4} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial A_1^2}{\partial y_1^2} & \frac{\partial A_1^2}{\partial y_1^4} & \frac{\partial A_1^2}{\partial y_2^1} & \frac{\partial A_1^2}{\partial y_2^2} & \frac{\partial A_1^2}{\partial y_2^3} & \frac{\partial A_1^2}{\partial y_2^4} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \frac{\partial A_{N-1}^3}{\partial y_{N-1}^1} & \frac{\partial A_{N-1}^3}{\partial y_{N-1}^2} & \frac{\partial A_{N-1}^3}{\partial y_{N-1}^3} & \frac{\partial A_{N-1}^3}{\partial y_{N-1}^4} & \frac{\partial A_{N-1}^3}{\partial y_N^1} & \frac{\partial A_{N-1}^3}{\partial y_N^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \frac{\partial A_{N-1}^4}{\partial y_{N-1}^1} & \frac{\partial A_{N-1}^4}{\partial y_{N-1}^2} & \frac{\partial A_{N-1}^4}{\partial y_{N-1}^3} & \frac{\partial A_{N-1}^4}{\partial y_{N-1}^4} & \frac{\partial A_{N-1}^4}{\partial y_N^1} & \frac{\partial A_{N-1}^4}{\partial y_N^3} \end{pmatrix} ,$$

a $4(N-1)\times4(N-1)$ square matrix, with many zeros !

We solve this matrix equation using some standard method to obtain $\boldsymbol{\delta}$. We then add the δy_j^i to the y_j^i and check for convergence. *i.e.* check to see if the A_j^i are now sufficiently close to zero. If not, we repeat the process by constructing new A_j^i from the new y_j^i giving a new $\mathbf{H}$. Again we solve

$$\mathbf{H}\boldsymbol{\delta} = -\mathbf{A}$$

for some new $\boldsymbol{\delta}$. We decide upon convergence by requiring two condit ions be satisfied.

i) $\delta y_j^i / y_j^i < \epsilon_1 \ \forall \ i$ and j,

ii) $\Delta_j^i < \epsilon_2 \quad \forall \ i$ and j.

where

$$\Delta_j^i = \frac{A_j^i}{(y_j^i - y_{j+1}^i)/(m_j - m_{j+1})}$$

is a measure of the fractional error in the equations at each mesh point. The first condition says that the corrections must be small for all quantities, whilst the second condition states that the current values of the quantities must also satisfy the equations.

VII.5 An Illustrative Example

Now, to salvage some understanding from your current confusion, we'll consider an example. Divide a model star of mass M into 4 shells, or 3 zones:

At $m = m_1 = 0$	we have	$r = r_1 = 0,$	$P = P_1 = P_c,$	$\ell = \ell_1 = 0,$	$T = T_1 = T_c;$
At $m = m_2$	we have	$r = r_2,$	$P = P_2,$	$\ell = \ell_2,$	$T = T_2;$
At $m = m_3$	we have	$r = r_3,$	$P = P_3,$	$\ell = \ell_3,$	$T = T_3;$
At $m = m_4$	we have	$r = r_4 = R,$	$P = P_4 = 0,$	$\ell = \ell_4 = \mathcal{L},$	$T = T_4 = 0.$

Note that we do not know T_c, P_c, R or $\mathcal{L}$. So we have $4(N-2)+2+2 = 4(N-1) = 12$ variables. And also $A_j^i = 0$ for four equations and three interfaces gives us 12 equations. Now, what are the A_j^i ? Lets evaluate some.

A_1^1 is the r equation (superscript $= 1$) across shells 1 (subscript $= 1$) and 2. *i.e.*

$$A_1^1 = \frac{y_1^1 - y_2^1}{m_1 - m_2} - f^1(y^1_{\frac{3}{2}}, y^2_{\frac{3}{2}}, y^3_{\frac{3}{2}}, y^4_{\frac{3}{2}}) = 0.$$

i.e.

$$0 = \frac{r_1 - r_2}{m_1 - m_2} - \frac{\Re T_{\frac{3}{2}}}{4\pi P_{\frac{3}{2}} \mu r^2_{\frac{3}{2}}},$$

where we have put $P = \frac{\rho \Re T}{\mu}$. This is needed because ρ is not one of our fundamental variables, it is in fact merely an alternative fromulation of the equation of state (various thermodynamic variables are possible: some codes use ρ as a variable, others use entropy etc). Also, for simplicity, we will assume that μ is constant in what follows. (In fact it is not ! Although it will begin as constant, due to the Hayashi convective phase as the star approaches the Main Sequence, it will change due to nuclear burning. This simply means that we would replace μ by $\mu_{j+\frac{1}{2}}$ in the equations below.) Now, $r_1 = m_1 = \ell_1 = 0$. Thus

$$\frac{r_2}{m_2} - \frac{\Re(T_1 + T_2)}{\pi r_2^2 (P_1 + P_2)\mu} = 0.$$

For $A_1^2 = 0$ we have

$$-\frac{(P_1 - P_2)}{m_2} + \frac{2Gm_2}{\pi r_2^4} = 0.$$

For $A_1^3 = 0$ we have

$$\frac{\ell_2}{m_2} - \frac{(\epsilon_1 + \epsilon_2)}{2} = 0.$$

For $A_1^4 = 0$ we have

$$-\frac{(T_1 - T_2)}{m_2} + \frac{3(\kappa_1 + \kappa_2)\ell_2}{2\pi^2 acr_2^4 (T_1 + T_2)^3} = 0.$$

Well, the first shell is easy because $r_1 = m_1 = 0$, but the principle is the same for all. For example, for $A_2^1 = 0$ we get:

$$\left(\frac{r_2 - r_3}{m_2 - m_3}\right) + \frac{\Re(T_2 + T_3)}{\pi(r_2 + r_3)^2 (P_2 + P_3)\mu} = 0,$$

whilst for or $A_2^4 = 0$ we have

$$\left(\frac{T_2 - T_3}{m_2 - m_3}\right) + \frac{3(\kappa_2 + \kappa_3)(\ell_2 + \ell_3)}{2\pi^2 ac(r_2 + r_3)^4 (T_2 + T_3)^3} = 0.$$

We have $T_4 = P_4 = 0$ at $m_4 = M$, so the A_3^i are simpler, much like the A_1^i were.

So we now have 12 equations in 12 unknowns. We must construct the Henyey matrix $\mathbf{H}$, which is of the form:

$$\mathbf{H} = \begin{pmatrix}
\frac{\partial A_1^1}{\partial P_1} & \frac{\partial A_1^1}{\partial T_1} & \frac{\partial A_1^1}{\partial r_2} & \frac{\partial A_1^1}{\partial P_2} & \frac{\partial A_1^1}{\partial \ell_2} & \frac{\partial A_1^1}{\partial T_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial A_1^2}{\partial P_1} & \frac{\partial A_1^2}{\partial T_1} & \frac{\partial A_1^2}{\partial r_2} & \frac{\partial A_1^2}{\partial P_2} & \frac{\partial A_1^2}{\partial \ell_2} & \frac{\partial A_1^2}{\partial T_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial A_1^3}{\partial P_1} & \frac{\partial A_1^3}{\partial T_1} & \frac{\partial A_1^3}{\partial r_2} & \frac{\partial A_1^3}{\partial P_2} & \frac{\partial A_1^3}{\partial \ell_2} & \frac{\partial A_1^3}{\partial T_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial A_1^4}{\partial P_1} & \frac{\partial A_1^4}{\partial T_1} & \frac{\partial A_1^4}{\partial r_2} & \frac{\partial A_1^4}{\partial P_2} & \frac{\partial A_1^4}{\partial \ell_2} & \frac{\partial A_1^4}{\partial T_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial A_2^1}{\partial r_2} & \frac{\partial A_2^1}{\partial P_2} & \frac{\partial A_2^1}{\partial \ell_2} & \frac{\partial A_2^1}{\partial T_2} & \frac{\partial A_2^1}{\partial r_3} & \frac{\partial A_2^1}{\partial P_3} & \frac{\partial A_2^1}{\partial \ell_3} & \frac{\partial A_2^1}{\partial T_3} & 0 & 0 \\
0 & 0 & \frac{\partial A_2^2}{\partial r_2} & \frac{\partial A_2^2}{\partial P_2} & \frac{\partial A_2^2}{\partial \ell_2} & \frac{\partial A_2^2}{\partial T_2} & \frac{\partial A_2^2}{\partial r_3} & \frac{\partial A_2^2}{\partial P_3} & \frac{\partial A_2^2}{\partial \ell_3} & \frac{\partial A_2^2}{\partial T_3} & 0 & 0 \\
0 & 0 & \frac{\partial A_2^3}{\partial r_2} & \frac{\partial A_2^3}{\partial P_2} & \frac{\partial A_2^3}{\partial \ell_2} & \frac{\partial A_2^3}{\partial T_2} & \frac{\partial A_2^3}{\partial r_3} & \frac{\partial A_2^3}{\partial P_3} & \frac{\partial A_2^3}{\partial \ell_3} & \frac{\partial A_2^3}{\partial T_3} & 0 & 0 \\
0 & 0 & \frac{\partial A_2^4}{\partial r_2} & \frac{\partial A_2^4}{\partial P_2} & \frac{\partial A_2^4}{\partial \ell_2} & \frac{\partial A_2^4}{\partial T_2} & \frac{\partial A_2^4}{\partial r_3} & \frac{\partial A_2^4}{\partial P_3} & \frac{\partial A_2^4}{\partial \ell_3} & \frac{\partial A_2^4}{\partial T_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial A_3^1}{\partial r_3} & \frac{\partial A_3^1}{\partial P_3} & \frac{\partial A_3^1}{\partial L_3} & \frac{\partial A_3^1}{\partial T_3} & \frac{\partial A_3^1}{\partial r_4} & \frac{\partial A_3^1}{\partial \ell_4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial A_3^2}{\partial r_3} & \frac{\partial A_3^2}{\partial P_3} & \frac{\partial A_3^2}{\partial L_3} & \frac{\partial A_3^2}{\partial T_3} & \frac{\partial A_3^2}{\partial r_4} & \frac{\partial A_3^2}{\partial \ell_4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial A_3^3}{\partial r_3} & \frac{\partial A_3^3}{\partial P_3} & \frac{\partial A_3^3}{\partial L_3} & \frac{\partial A_3^3}{\partial T_3} & \frac{\partial A_3^3}{\partial r_4} & \frac{\partial A_3^3}{\partial \ell_4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial A_3^4}{\partial r_3} & \frac{\partial A_3^4}{\partial P_3} & \frac{\partial A_3^4}{\partial L_3} & \frac{\partial A_3^4}{\partial T_3} & \frac{\partial A_3^4}{\partial r_4} & \frac{\partial A_3^4}{\partial \ell_4}
\end{pmatrix}.$$

So what are these elements of $\mathbf{H}$? They are simply the $\frac{\partial A_j^i}{\partial y_l^k}$. We'll evaluate a couple so that we know what we are doing.

$$\begin{aligned}
\frac{\partial A_1^1}{\partial y_1^2} = \frac{\partial A_1^1}{\partial P_1} &= \frac{\Re(T_1+T_2)}{\pi r_2^2 \mu (P_1+P_2)^2}, \\
\frac{\partial A_1^1}{\partial T_1} &= \frac{-\Re}{\pi r_2^2 \mu (P_1+P_2)}, \\
\frac{\partial A_1^1}{\partial r_2} &= \frac{1}{m_2} + \frac{2\Re(T_1+T_2)}{\pi (P_1+P_2)\mu r_2^3}, \\
\frac{\partial A_1^1}{\partial P_2} &= \frac{\partial A_1^1}{\partial P_1}, \text{ by symmetry}, \\
\frac{\partial A_1^1}{\partial L_2} &= 0, \\
\frac{\partial A_1^1}{\partial T_2} &= \frac{\partial A_1^1}{\partial T_1}, \text{ by symmetry},
\end{aligned}$$

etc. etc. . Of course, when calcuating other derivatives, note that ϵ and κ are functions of P and T.

Anyway, we can now construct $\mathbf{H}$ and $\boldsymbol{\delta}$ and $\mathbf{A}$, w here

$$\boldsymbol{\delta} = (\,\delta P_1 \quad \delta T_1 \quad \delta r_2 \quad \delta P_2 \quad \delta \ell_2 \quad \delta T_2 \quad \delta r_3 \quad \delta P_3 \quad \delta \ell_3 \quad \delta T_3 \quad \delta r_4 \quad \delta \ell_4\,)^T,$$

and

$$A = (\,A_1^1 \quad A_1^2 \quad A_1^3 \quad A_1^4 \quad A_2^1 \quad A_2^2 \quad A_2^3 \quad A_2^4 \quad A_3^1 \quad A_3^2 \quad A_3^3 \quad A_3^4\,)^T.$$

We now have everything we need to solve for the structure of this model star !