A discrete exact grad-curl-div complex on generic polyhedral meshes

Part II: Analytical properties

Jérôme Droniou
joint work with D. A. Di Pietro

School of Mathematics, Monash University
https://users.monash.edu/~jdroniou/

16th U.S. National Congress on Computational Mechanics

Discrete problem

- Continuous problem: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}(\mathbf{curl}; \Omega) \times \boldsymbol{H}(\mathrm{div}; \Omega)$ s.t.

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\tau} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathbf{curl}; \Omega),$$
$$\int_\Omega \mathbf{curl}\, \boldsymbol{\sigma} \cdot \boldsymbol{v} + \int_\Omega \mathrm{div}\, \boldsymbol{u}\, \mathrm{div}\, \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}; \Omega)$$

- The DDR problem reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} = 0 \qquad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h},$$
$$(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h\, D^k_h \underline{\boldsymbol{v}}_h = \underbrace{\int_\Omega \boldsymbol{f} \cdot \boldsymbol{P}^k_{\mathrm{div},h} \underline{\boldsymbol{v}}_h}_{=: l_h(\underline{\boldsymbol{v}}_h)} \quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}.$$

- Stability follows as in the continuous case using exactness properties of

$$\mathbb{R} \xrightarrow{\underline{I}^k_{\mathrm{grad},h}} \underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{\boldsymbol{G}}^k_h} \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \xrightarrow{\underline{\boldsymbol{C}}^k_h} \underline{\boldsymbol{X}}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^k(\mathcal{T}_h) \xrightarrow{0} \{0\}$$

Plan

Inf-sup property

- The DDR problem reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$
\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} &= 0 &\quad \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}, \\
(\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h \, D^k_h \underline{\boldsymbol{v}}_h &= l_h(\underline{\boldsymbol{v}}_h) &\quad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathrm{div},h}.
\end{aligned}
$$

- Recast: $\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = l_h(\underline{\boldsymbol{v}}_h)$ with

$$
\begin{aligned}
\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = {}& (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} \\
&+ (\underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D^k_h \underline{\boldsymbol{u}}_h D^k_h \underline{\boldsymbol{v}}_h.
\end{aligned}
$$

Inf-sup property

- Recast: $\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = l_h(\underline{\boldsymbol{v}}_h)$ with

$$\begin{aligned}\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = {}& (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} \\ & + (\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{\boldsymbol{u}}_h D_h^k \underline{\boldsymbol{v}}_h.\end{aligned}$$

Theorem (Inf-sup property [Di Pietro and Droniou, 2021b])

$$\sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}} \frac{\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}} \gtrsim \|(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}$$

where

$$\begin{aligned}\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h} = {}& \|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{curl},h} + \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h\|_{\mathrm{div},h} \\ & + \|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h} + \|D_h^k \underline{\boldsymbol{v}}_h\|_{L^2(\Omega)}\end{aligned}$$

and $\|\cdot\|_{\bullet,h}^2 = (\cdot,\cdot)_{\bullet,h}$ *for* $\bullet = \mathbf{curl}, \mathrm{div}$.

Notation: $a \lesssim b$ *for* $a \leq Cb$ *with* C *independent of* h.

Proof of inf-sup property

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{\boldsymbol{u}}_h D_h^k \underline{\boldsymbol{v}}_h.$$

Let

$$\mathcal{S} = \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)} \frac{\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_{1,\mathbf{curl}\times \mathrm{div},h}}.$$

- Make $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) = (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)$:

$$\mathcal{S} \geq \frac{\|\underline{\boldsymbol{\sigma}}_h\|^2_{\mathbf{curl},h} + \|D_h^k \underline{\boldsymbol{u}}_h\|^2_{L^2(\Omega)}}{\|(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)\|_{1,\mathbf{curl}\times \mathrm{div},h}}.$$

Proof of inf-sup property

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{\boldsymbol{u}}_h D_h^k \underline{\boldsymbol{v}}_h.$$

Let

$$\mathcal{S} = \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)} \frac{\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}}.$$

- Make $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) = (\underline{\boldsymbol{0}}, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h)$ and use $\mathrm{Im}\, \underline{\boldsymbol{C}}_h^k \subset \mathrm{Ker}\, D_h^k$:

$$\mathcal{S} \ge \frac{\|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h\|_{\mathrm{div},h}^2}{\|(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}}.$$

Proof of inf-sup property

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} - (\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} + (\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{\boldsymbol{u}}_h D_h^k \underline{\boldsymbol{v}}_h.$$

Let

$$\mathcal{S} = \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)} \frac{\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h))}{\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}}.$$

- Write $\underline{\boldsymbol{u}}_h = \underline{\boldsymbol{u}}_h^\star + \underline{\boldsymbol{u}}_h^\perp \in \mathrm{Ker}\, D_h^k \oplus (\mathrm{Ker}\, D_h^k)^\perp$ and use the

  Poincaré inequality: $\|\underline{\boldsymbol{u}}_h^\perp\|_{\mathrm{div},h} \lesssim \|D_h^k \underline{\boldsymbol{u}}_h^\perp\|_{L^2(\Omega)}$

  together with $D_h^k \underline{\boldsymbol{u}}_h^\perp = D_h^k \underline{\boldsymbol{u}}_h$ to get

$$\|\underline{\boldsymbol{u}}_h^\perp\|_{\mathrm{div},h}^2 \lesssim \|D_h^k \underline{\boldsymbol{u}}_h\|_{L^2(\Omega)}^2 \le \mathcal{S}\, \|(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}.$$

Proof of inf-sup property

$$\mathcal{A}_h((\underline{\sigma}_h, \underline{u}_h), (\underline{\tau}_h, \underline{v}_h)) = (\underline{\sigma}_h, \underline{\tau}_h)_{\mathbf{curl},h} - (\underline{u}_h, \underline{C}_h^k \underline{\tau}_h)_{\mathrm{div},h} + (\underline{C}_h^k \underline{\sigma}_h, \underline{v}_h)_{\mathrm{div},h} + \int_\Omega D_h^k \underline{u}_h D_h^k \underline{v}_h.$$

Let

$$\mathcal{S} = \sup_{(\underline{\tau}_h, \underline{v}_h)} \frac{\mathcal{A}_h((\underline{\sigma}_h, \underline{u}_h), (\underline{\tau}_h, \underline{v}_h))}{\|(\underline{\tau}_h, \underline{v}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}}.$$

- Use $\mathrm{Ker}\, D_h^k \subset \mathrm{Im}\, \underline{C}_h^k$ to find $\underline{\zeta}_h \in (\mathrm{Ker}\, \underline{C}_h^k)^\perp$ s.t. $\underline{C}_h^k \underline{\zeta}_h = \underline{u}_h^\star$, and notice the

  Poincaré inequality: $\|\underline{\zeta}_h\|_{\mathbf{curl},h} \lesssim \|\underline{C}_h^k \underline{\zeta}_h\|_{\mathrm{div},h}$.

  Use $(\underline{\tau}_h, \underline{v}_h) = (-\underline{\zeta}_h, \underline{0})$ to deduce

$$\mathcal{S}\, \|(\underline{\sigma}_h, \underline{u}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h} \geq \|\underline{u}_h^\star\|_{\mathrm{div},h}^2 - \|\underline{u}_h^\perp\|_{\mathrm{div},h} \|\underline{u}_h^\star\|_{\mathrm{div},h} - C\|\underline{\sigma}_h\|_{\mathbf{curl},h} \|\underline{u}_h^\star\|_{\mathrm{div},h}.$$

Poincaré inequalities

Theorem (Poincaré inequality for D_h^k and $\underline{\boldsymbol{C}}_h^k$)

- *The mapping* $D_h^k : (\mathrm{Ker}\, D_h^k)^\perp \to \mathcal{P}^k(\mathcal{T}_h)$ *is an isomorphism, and*

$$\|\underline{\boldsymbol{w}}_h\|_{\mathrm{div},h} \lesssim \|D_h^k \underline{\boldsymbol{w}}_h\|_{L^2(\Omega)} \quad \forall \underline{\boldsymbol{w}}_h \in (\mathrm{Ker}\, D_h^k)^\perp.$$

- *If* Ω *does not enclose any tunnel, then* $\underline{\boldsymbol{C}}_h^k : (\mathrm{Ker}\, \underline{\boldsymbol{C}}_h^k)^\perp \to \mathrm{Ker}\, D_h^k$ *is an isomorphism. If* Ω *is moreover simply connected, then*

$$\|\underline{\boldsymbol{\zeta}}_h\|_{\mathbf{curl},h} \lesssim \|\underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{\zeta}}_h\|_{\mathrm{div},h} \quad \forall \underline{\boldsymbol{\zeta}}_h \in (\mathrm{Ker}\, \underline{\boldsymbol{C}}_h^k)^\perp.$$

- Isomorphisms direct consequence of exactness of the discrete sequence.
- Poincaré for D_h^k: using the continuous equivalent and the commutation property $D_h^k \underline{\boldsymbol{I}}_{\mathrm{div},h}^k = \pi_{\mathcal{P},h}^k \,\mathrm{div}$.
- Poincaré for $\underline{\boldsymbol{C}}_h^k$: relies on the low-order case (sole unknowns on the edges); simple-connectedness probably not required.

Plan

3rd Strang Lemma

- DDR problem: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h) \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h} \times \underline{\boldsymbol{X}}^k_{\mathrm{div},h}$ s.t.

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)) = l_h(\underline{\boldsymbol{v}}).$$

- With $(\boldsymbol{\sigma}, \boldsymbol{u})$ solution to the continuous problem, by inf-sup condition the 3rd Strang Lemma [Di Pietro and Droniou, 2018] yields

$$\|(\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u})\|_{1,\mathbf{curl}\times\mathrm{div},h} \leq \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)} \frac{\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)}{\|(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)\|_{1,\mathbf{curl}\times\mathrm{div},h}}$$

where the consistency error is

$$\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) = l_h(\underline{\boldsymbol{v}}_h) - \mathcal{A}_h((\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}), (\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h)).$$

Consistency analysis [Di Pietro and Droniou, 2021a]

$$
\begin{aligned}
\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{v}}_h) = {} & \int_\Omega \boldsymbol{f} \cdot \boldsymbol{P}^k_{\mathrm{div},h} \underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{\sigma}, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} + (\underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{u}, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} \\
& - (\underline{\boldsymbol{C}}^k_h (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{\sigma}), \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} - \int_\Omega D^k_h (\underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{u}) \, D^k_h \underline{\boldsymbol{v}}_h \\
= {} & \mathcal{E}_{1,h} + \mathcal{E}_{2,h} + \mathcal{E}_{3,h}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{E}_{1,h} &:= \int_\Omega \mathbf{curl}\, \boldsymbol{\sigma} \cdot \boldsymbol{P}^k_{\mathrm{div},h} \underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{C}}^k_h (\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{\sigma}), \underline{\boldsymbol{v}}_h)_{\mathrm{div},h}, \\
\mathcal{E}_{2,h} &:= -\int_\Omega D^k_h (\underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{u}) \, D^k_h \underline{\boldsymbol{v}}_h, \\
\mathcal{E}_{3,h} &:= -(\underline{\boldsymbol{I}}^k_{\mathbf{curl},h} \boldsymbol{\sigma}, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} + (\underline{\boldsymbol{I}}^k_{\mathrm{div},h} \boldsymbol{u}, \underline{\boldsymbol{C}}^k_h \underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h}.
\end{aligned}
$$

Consistency analysis: $\mathcal{E}_{1,h}$

We have

$$\begin{aligned}\mathcal{E}_{1,h} &= \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot \boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}), \underline{\boldsymbol{v}}_h)_{\mathrm{div},h}\\ &= \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot \boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}(\mathbf{curl}\,\boldsymbol{\sigma}), \underline{\boldsymbol{v}}_h)_{\mathrm{div},h}\end{aligned}$$

using:

Theorem (Commutation properties of differential operators)

We have

$$\begin{aligned}\underline{\boldsymbol{C}}^k_h(\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\tau}) &= \underline{\boldsymbol{I}}^k_{\mathrm{div},h}(\mathbf{curl}\,\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau}\in H^2(\Omega)^3,\\ D^k_h(\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}) &= \pi^k_{\mathcal{P},h}(\mathrm{div}\,\boldsymbol{w}) \qquad \forall \boldsymbol{w}\in H^1(\Omega)^3.\end{aligned}$$

- Also for the gradient $\underline{\boldsymbol{G}}^k_h\underline{I}^k_{\mathbf{grad},h} = \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\,\mathbf{grad}$ on $C^1(\overline{\Omega})$.

Consistency analysis: $\mathcal{E}_{1,h}$

Hence

$$\begin{aligned}\mathcal{E}_{1,h} &= \int_\Omega \mathbf{curl}\,\boldsymbol{\sigma}\cdot\boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}(\mathbf{curl}\,\boldsymbol{\sigma}),\underline{\boldsymbol{v}}_h)_{\mathrm{div},h}\\ &\le Ch^{k+1}|\mathbf{curl}\,\boldsymbol{\sigma}|_{H^{k+1}(\Omega)}\|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}\end{aligned}$$

using the consistency of the discrete inner product:

$$\left|(\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w},\underline{\boldsymbol{v}}_h)_{\mathrm{div},h} - \int_\Omega \boldsymbol{w}\cdot\boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{v}}_h\right| \lesssim h^{k+1}|\boldsymbol{w}|_{H^{k+1}(\Omega)}\|\underline{\boldsymbol{v}}_h\|_{\mathrm{div},h}$$

which follows from:

Theorem (Consistency of potential reconstruction and stabilisation)

There is C independent of h s.t., for $\boldsymbol{w}\in H^{k+1}(\Omega)^3$,

$$\|\boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}-\boldsymbol{w}\|_{L^2(\Omega)} + \mathrm{s}_{\mathrm{div},h}(\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w},\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w})^{\frac{1}{2}} \lesssim h^{k+1}|\boldsymbol{w}|_{H^{k+1}(\Omega)}.$$

Consistency analysis: $\mathcal{E}_{1,h}$

Theorem (Consistency of potential reconstruction and stabilisation)

There is C independent of h s.t., for $\boldsymbol{w} \in H^{k+1}(\Omega)^3$,

$$\|\boldsymbol{P}^k_{\mathrm{div},h}\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w} - \boldsymbol{w}\|_{L^2(\Omega)} + \mathrm{s}_{\mathrm{div},h}(\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}, \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w})^{\frac{1}{2}} \lesssim h^{k+1}|\boldsymbol{w}|_{H^{k+1}(\Omega)}.$$

- Comes from local polynomial consistency: $\boldsymbol{P}^k_{\mathrm{div},T}\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}_T = \boldsymbol{w}_T$ and $\mathrm{s}_{\mathrm{div},T}(\underline{\boldsymbol{I}}^k_{\mathrm{div},T}\boldsymbol{w}_T, \cdot) = 0$ if $\boldsymbol{w}_T \in \mathcal{P}^k(T)^3$.
- Also valid for $\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}$, $P^{k+1}_{\mathbf{grad},h}\underline{I}^k_{\mathbf{grad},h}$, $\mathrm{s}_{\mathbf{curl},h}$ and $\mathrm{s}_{\mathbf{grad},h}$.

Consistency analysis: $\mathcal{E}_{2,h}$

Since $\operatorname{div} \boldsymbol{u} = 0$, the commutation property $D_h^k \underline{\boldsymbol{I}}_{\operatorname{div},h}^k = \pi_{\mathcal{P},h}^k \operatorname{div}$ yields

$$\mathcal{E}_{2,h} = -\int_\Omega D_h^k(\underline{\boldsymbol{I}}_{\operatorname{div},h}^k \boldsymbol{u})\, D_h^k \underline{\boldsymbol{v}}_h = -\int_\Omega \pi_{\mathcal{P},h}^k(\operatorname{div} \boldsymbol{u})\, D_h^k \underline{\boldsymbol{v}}_h = 0.$$

In a more general case:

Theorem (Primal consistency of discrete div)

There is C independent of h s.t., for $\boldsymbol{w} \in H^1(\Omega)^3$ s.t. $\operatorname{div} \boldsymbol{w} \in H^{k+1}(\Omega)$,

$$\|D_h^k(\underline{\boldsymbol{I}}_{\operatorname{div},h}^k \boldsymbol{w}) - \operatorname{div} \boldsymbol{w}\|_{L^2(\Omega)} \lesssim h^{k+1} |\operatorname{div} \boldsymbol{w}|_{H^{k+1}(\Omega)}.$$

- Primal consistency also for the other operators: commutation properties and primal consistency of potential reconstructions.

Consistency analysis: $\mathcal{E}_{3,h}$

We have

$$\begin{aligned}
\mathcal{E}_{3,h} &= -(\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} + (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} \\
&= -(\underline{\boldsymbol{I}}^k_{\mathbf{curl},h}(\mathbf{curl}\,\boldsymbol{u}), \underline{\boldsymbol{\tau}}_h)_{\mathbf{curl},h} + (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} \\
&= -\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{\tau}}_h + \mathcal{O}\left(h^{k+1}|\,\mathbf{curl}\,\boldsymbol{u}|_{H^{k+1}(\Omega)}\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{curl},h}\right) \\
&\quad + (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h}
\end{aligned}$$

using the consistency of the discrete inner product $(\cdot,\cdot)_{\mathbf{curl},h}$.

Note: slight change of norm if $k = 0$.

Consistency analysis: $\mathcal{E}_{3,h}$

To estimate

$$\begin{aligned}\mathcal{E}_{3,h} = &-\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot \boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{\tau}}_h + (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{\tau}}_h)_{\mathrm{div},h} \\ &+ \mathcal{O}\left(h^{k+1}|\,\mathbf{curl}\,\boldsymbol{u}|_{H^{k+1}(\Omega)}\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{curl},h}\right)\end{aligned}$$

we need the

Theorem (Adjoint consistency for $\underline{\boldsymbol{C}}^k_h$)

There is C independent of h s.t., for $\boldsymbol{w} \in H^{k+1}(\Omega)$ s.t. $\boldsymbol{w}\times\boldsymbol{n}_{\partial\Omega} = 0$ on $\partial\Omega$, and for $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{\mathbf{curl},h}$,

$$\left|\int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot \boldsymbol{P}^k_{\mathbf{curl},h}\underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{w}, \underline{\boldsymbol{C}}^k_h\underline{\boldsymbol{v}}_h)_{\mathrm{div},h}\right| \lesssim h^{h+1}\left(|\boldsymbol{w}|_{H^{k+1}(\Omega)} + |\boldsymbol{w}|_{H^{k+2}(\Omega)}\right)\|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h}.$$

Consistency analysis: $\mathcal{E}_{3,h}$

Theorem (Adjoint consistency for $\underline{\boldsymbol{C}}_h^k$)

There is C independent of h s.t., for $\boldsymbol{w} \in H^{k+1}(\Omega)$ s.t. $\boldsymbol{w} \times \boldsymbol{n}_{\partial\Omega} = 0$ on $\partial\Omega$, and for $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$,

$$\left| \int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{P}_{\mathbf{curl},h}^k \underline{\boldsymbol{v}}_h - (\underline{\boldsymbol{I}}_{\mathrm{div},h}^k \boldsymbol{w}, \underline{\boldsymbol{C}}_h^k \underline{\boldsymbol{v}}_h)_{\mathrm{div},h} \right| \lesssim h^{h+1} \left(|\boldsymbol{w}|_{H^{k+1}(\Omega)} + |\boldsymbol{w}|_{H^{k+2}(\Omega)} \right) \|\underline{\boldsymbol{v}}_h\|_{\mathbf{curl},h}.$$

- Most challenging estimate (by far).
- Based on suitable liftings of discrete functions in $\underline{\boldsymbol{X}}_{\mathbf{curl},h}^k$, whose design and analysis is based on fine compatibility conditions and regularity properties of div–$\mathbf{curl}$ problems.

Error estimate

Theorem (Error estimate)

Assuming $b_1 = b_2 = 0$*, it holds, with* C *independent of* h*,*

$$\|(\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{I}}^k_{\mathbf{curl},h}\boldsymbol{\sigma}, \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}^k_{\mathrm{div},h}\boldsymbol{u})\|_{1,\mathbf{curl}\times\mathrm{div},h}$$
$$\lesssim h^{k+1}\left(|\,\mathbf{curl}\,\boldsymbol{\sigma}|_{H^{k+1}(\Omega)} + |\boldsymbol{\sigma}|_{H^{k+1}(\Omega)} + |\boldsymbol{u}|_{H^{k+1}(\Omega)} + |\boldsymbol{u}|_{H^{k+2}(\Omega)}\right).$$

Plan

Test case

$\Omega = (0,1)^3$ and exact solution:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = 3\pi \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ 0 \\ -\cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix},$$

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ -2\sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \end{pmatrix}.$$

Implementation in the HArDCore library (C++ open-source):
https://github.com/jdroniou/HArDCore.

Numerical examples

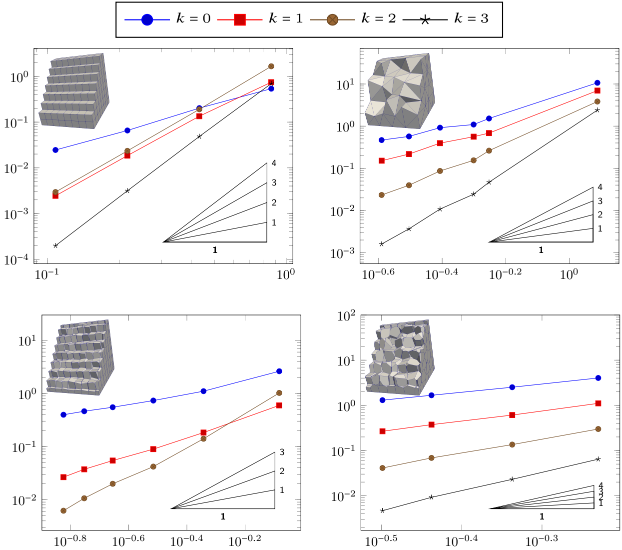

Figure: Energy error versus mesh size h

References I

Di Pietro, D. A. and Droniou, J. (2018).
A third Strang lemma for schemes in fully discrete formulation.
Calcolo, 55(40).

Di Pietro, D. A. and Droniou, J. (2021a).
An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, poincaré inequalities, and consistency.

Di Pietro, D. A. and Droniou, J. (2021b).
An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence.
J. Comput. Phys., 429(109991).

Di Pietro, D. A., Droniou, J., and Rapetti, F. (2020).
Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra.
Math. Models Methods Appl. Sci., 30(9):1809–1855.