High-order methods for linear and non-linear elliptic equations

J. Droniou (Monash University)

Joint work with D. Anderson, D. Di Pietro, R. Eymard, F. Rapetti...

Australian Government
Australian Research Council
Discrete Functional Analysis: bridging pure and numerical mathematics
Plan

1. Hybrid High-Order method
   - An inspiring remark
   - Description of the HHO scheme
   - Miscible flow in porous media

2. Fully Discrete de Rham sequence
   - Principles of discrete exact sequence
   - Fully discrete de Rham sequence
   - Application to magnetostatics

3. High-order schemes for stationary Stefan/PME models
   - Towards a stable numerical approximation
   - High-order approximations
   - Numerical tests
What is “high order”: the case of $P^k$ finite elements
Why go for “high order”: 3D tests with Hybrid High-Order scheme

(a) Energy error vs. nb degrees of freedom

(b) Energy error vs. total running time

$$k = 0$$  $$k = 1$$  $$k = 2$$  $$k = 3$$
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Objectives

Polytopal meshes

Typically: exactly reproduce solutions that are polynomials of degree $k + 1$. 

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Objectives

Polytopal meshes

**Arbitrary order**: choice of an index $k \geq 0$ determining the accuracy of the method

*Typically: exactly reproduce solutions that are polynomials of degree $k + 1$.***
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Where do we put the unknowns?

**Model problem**: Poisson equation.

\[-\Delta u = f \quad \text{in } \Omega\]
\[u = 0 \quad \text{on } \partial\Omega\]

► *Core model in many flows in porous media (including multi-components, multi-phases): oil recovery, CO2 storage, etc.*
Where do we put the unknowns?

**Model problem:** Poisson equation.

Find $u \in H^1_0(\Omega)$ s.t. $(\nabla u, \nabla v)_\Omega = (f, v)_\Omega$ for all $v \in H^1_0(\Omega)$.

▶ $(\cdot, \cdot)_X : L^2$-inner product on $X$, norm denoted by $\|\cdot\|_X$. 

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$\langle \cdot, \cdot \rangle_X : L^2$-inner product on $X$, norm denoted by $\| \cdot \|_X$.

**Computation elliptic projector**: for $T$ a mesh element, $v$ smooth and $q \in P^{k+1}(T)$:

$$(\nabla v, \nabla q)_T = -(v, \Delta q)_T + \sum_{F \in \mathcal{F}_T} (v, \nabla q \cdot n_T)_F.$$

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\[ (\cdot, \cdot)_X : L^2\text{-inner product on } X, \text{ norm denoted by } \| \cdot \|_X. \]

**Computation elliptic projector:** for \( T \) a mesh element, \( v \) smooth and \( q \in P^{k+1}(T) \):

\[ (\nabla v, \nabla q)_T = -(\pi^k_{P,T} v, \Delta q)_T + \sum_{F \in \mathcal{F}_T} (\pi^k_{P,F} v, \nabla q \cdot n_F)_F. \]

\[ \pi^k_{P,Y} : L^2(Y) \to P^k(Y) \text{ orthogonal projector.} \]
Where do we put the unknowns?

**Model problem**: Poisson equation.

\[
\text{Find } u \in H^1_0(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \text{ for all } v \in H^1_0(\Omega).
\]

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**Computation elliptic projector**: for \(T\) a mesh element, \(v\) smooth and \(q \in P^{k+1}(T)\):

\[
(\nabla v, \nabla q)_T = -(\pi^k_{P,T} v, \Delta q)_T + \sum_{F \in \mathcal{F}_T} (\pi^k_{P,F} v, \nabla q \cdot n_T)_F.
\]

▶ \(\pi^k_{P,Y} : L^2(Y) \to P^k(Y)\) orthogonal projector.

The projection \(\Pi_{\nabla P^{k+1}(T)}(\nabla v)\) of \(\nabla v\) on \(\nabla P^{k+1}(T)\) can be computed from \(\pi^k_{P,T} v\) and \((\pi^k_{P,F} v)_F \in \mathcal{F}_T\).
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Which unknowns?

Space of local unknowns:

\[ \mathcal{U}_T^k = \{ \mathbf{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) : v_T \in \mathcal{P}^k(T), v_F \in \mathcal{P}^k(F) \}. \]
Which unknowns?

Space of local unknowns:

\[ U^k_T = \{ v_T = (v_T, (v_F)_{F \in \mathcal{T}}) : v_T \in \mathcal{P}^k(T), \ v_F \in \mathcal{P}^k(F) \} \]

Interpolator: \( I^k_T : H^1(T) \to U^k_T \) such that

\[ I^k_T v = (\pi^k_P, T v, (\pi^k_P, F v)_{F \in \mathcal{T}}) \].
How do we benefit from the hybrid unknowns?

**Discontinuous Galerkin**: polynomial unknowns of degree $k$ in the elements $\mapsto$ method of order $k$. 
How do we benefit from the hybrid unknowns?

**Discontinuous Galerkin**: polynomial unknowns of degree $k$ in the elements $\rightsquigarrow$ method of order $k$.

**HHO**: higher-order potential reconstruction using element and face polynomials $\rightsquigarrow$ method of order $k + 1$. 

\[
\begin{align*}
\nabla p_{k+1}^T v^T & \rightarrow P_{k+1}(T) \\
\n\left(\nabla p_{k+1}^T v^T, w^T\right)_T & = -\left(v^T, \Delta w^T\right)_T + \sum_{F \in F} \left(v_F^T, \nabla w \cdot n^T_F\right)_F \forall w \in P_{k+1}(T),
\end{align*}
\]

\[
\nabla p_{k+1}^T I^k v^T = \Pi \nabla P_{k+1}(T) \left(\nabla v\right) \text{ for all } v \in H_1(T).
\]
How do we benefit from the hybrid unknowns?

**Discontinuous Galerkin**: polynomial unknowns of degree $k$ in the elements $\sim$ method of order $k$.

**HHO**: higher-order potential reconstruction using element and face polynomials $\sim$ method of order $k + 1$.

\[
\begin{align*}
\varphi_{k+1}^T : \underline{U}_T^k & \to \mathcal{P}^{k+1}(T) \quad \text{such that:} \\
(\nabla \varphi_{k+1}^T \underline{v}_T, \nabla w)_T &= - (\underline{v}_T, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (\varphi_F, \nabla w \cdot \mathbf{n}_F)_F \quad \forall w \in \mathcal{P}^{k+1}(T), \\
(\varphi_{k+1}^T \underline{v}_T, 1)_T &= (\underline{v}_T, 1)_T.
\end{align*}
\]
How do we benefit from the hybrid unknowns?

**Discontinuous Galerkin**: polynomial unknowns of degree \( k \) in the elements \( \rightsquigarrow \) method of order \( k \).

**HHO**: higher-order potential reconstruction using element and face polynomials \( \rightsquigarrow \) method of order \( k + 1 \).

\[
\begin{align*}
\mathbf{p}^{k+1}_T : \mathbf{U}^k_T & \rightarrow \mathcal{P}^{k+1}(T) \text{ such that:} \\
(\nabla \mathbf{p}^{k+1}_T \mathbf{v}_T, \nabla \mathbf{w})_T &= - (\mathbf{v}_T, \Delta \mathbf{w})_T + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_F, \nabla \mathbf{w} \cdot \mathbf{n}_{TF})_F \quad \forall \mathbf{w} \in \mathcal{P}^{k+1}(T), \\
(\mathbf{p}^{k+1}_T \mathbf{v}_T, 1)_T &= (\mathbf{v}_T, 1)_T.
\end{align*}
\]

\[
\nabla \mathbf{p}^{k+1}_T |^k_T \mathbf{v} = \Pi_{\nabla \mathcal{P}^{k+1}(T)}(\nabla \mathbf{v}) \quad \text{for all } \mathbf{v} \in H^1(T).
\]
Local bilinear form

\[ a_T : U_T^k \times U_T^k \to \mathbb{R} \text{ such that} \]

\[ a_T(v_T, w_T) = (\nabla p_T^{k+1} v_T, \nabla p_T^{k+1} w_T)_T + s_T(v_T, w_T). \]
Local bilinear form

▷ $a_T : U^k_T \times U^k_T \rightarrow \mathbb{R}$ such that

$$a_T(v_T, w_T) = (\nabla p^{k+1}_T v_T, \nabla p^{k+1}_T w_T)_T + s_T(v_T, w_T).$$

**Stabilisation**: enables bound of unknowns, without degrading the order of exactness.
Local bilinear form

- \( a_T : U_T^k \times U_T^k \rightarrow \mathbb{R} \) such that

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**Stabilisation**: enables bound of unknowns, without degrading the order of exactness.

- **Symmetry & positivity**: \( s_T : U_T^k \times U_T^k \rightarrow \mathbb{R} \) symmetric semi-definite positive.
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a_T(\mathbf{v}_T, \mathbf{w}_T) = (\nabla p_{k+1}^T \mathbf{v}_T, \nabla p_{k+1}^T \mathbf{w}_T)_T + s_T(\mathbf{v}_T, \mathbf{w}_T).
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**Stabilisation**: enables bound of unknowns, without degrading the order of exactness.

- **Symmetry & positivity**: \( s_T : U_T^k \times U_T^k \to \mathbb{R} \) symmetric semi-definite positive.
- **Stability & boundedness**: it holds, for some \( \eta > 0 \),

\[
\eta^{-1} \| \mathbf{v}_T \|_T^2 \leq a_T(\mathbf{v}_T, \mathbf{v}_T) \leq \eta \| \mathbf{v}_T \|_T^2
\]

where the local discrete \( H^1 \)-seminorm is

\[
\| \mathbf{v}_T \|_T^2 = \| \nabla \mathbf{v}_T \|_T^2 + \sum_{F \in \mathcal{F}_T} h_T^{-1} \| \mathbf{v}_F - \mathbf{v}_T \|_F^2.
\]
Local bilinear form

\[ \begin{align*}
\text{a}_T & : U^k_T \times U^k_T \to \mathbb{R} \text{ such that} \\
\text{a}_T(v_T, w_T) & = (\nabla p^{k+1}_T v_T, \nabla p^{k+1}_T w_T)_T + s_T(v_T, w_T).
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- **Symmetry & positivity:** \( s_T : U^k_T \times U^k_T \to \mathbb{R} \) symmetric semi-definite positive.

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\[ \| v_T \|_T^2 = \| \nabla v_T \|_T^2 + \sum_{F \in \mathcal{F}_T} h_T^{-1} \| v_F - v_T \|_F^2. \]

- **Polynomial consistency:** \( s_T(l^k_T w, v_T) = 0 \) for all \( w \in \mathcal{P}^{k+1}(T) \).
Global space (with boundary conditions): patched local spaces.

\[ U_{h,0}^k = \{ \mathbf{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_T}) : v_T \in \mathcal{P}^k(T), \ v_F \in \mathcal{P}^k(F), \ v_F = 0 \text{ if } F \subset \partial \Omega \}. \]
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Global bilinear form: assembled from local ones.

\[ a_h : U_{h,0}^k \times U_{h,0}^k \rightarrow \mathbb{R}, \quad a_h(\mathbf{v}_h, \mathbf{w}_h) = \sum_{T \in T_h} a_T(\mathbf{v}_T, \mathbf{w}_T). \]
HHO scheme

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**Global bilinear form:** assembled from local ones.

\[ a_h : U_{h,0}^k \times U_{h,0}^k \to \mathbb{R}, \quad a_h(v_h, w_h) = \sum_{T \in T_h} a_T(v_T, w_T). \]

**Scheme:**

Find \( u_h \in U_{h,0}^k \) such that, for all \( v_T \in U_{h,0}^k \),

\[ a_h(u_h, v_h) = \sum_{T \in T_h} (f, v_T)_T. \]
Error analysis

- Using a generic framework (3rd Strang lemma) developed for schemes written in fully discrete form (the approximation space is not a space of functions over $\Omega$).

**Energy error estimate:** with $\| \cdot \|_{a,h} = \sqrt{a_h(\cdot, \cdot)}$ norm associated with $a_h$:

$$
\| I^k_h u - u_h \|_{a,h} \leq C h^{k+1} |u|_{H^{k+2}(\Omega)}.
$$
Using a generic framework (3rd Strang lemma) developed for schemes written in fully discrete form (the approximation space is not a space of functions over \( \Omega \)).

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\| I_h^k u - u_h \|_{a,h} \leq Ch^{k+1} |u|_{H^{k+2}(\Omega)}.
\]

**\( L^2 \) error estimate**: with \( (p_h^{k+1} u_h)|_T = p_T^{k+1} u_T \) for all \( T \in T_h \), under elliptic regularity assumption:

\[
\| u - p_h^{k+1} u_h \|_{\Omega} \leq C \begin{cases} 
  h^2 \| f \|_{H^1(\Omega)} & \text{if } k = 0, \\
  h^{k+2} |u|_{H^{k+2}(\Omega)} & \text{if } k \geq 1.
\end{cases}
\]
More about HHO

- This is a **finite volume method**: we can define fluxes that are conservative and satisfy, up to high order volumic term, the balance relation.
More about HHO

► This is a finite volume method: we can define fluxes that are conservative and satisfy, up to high order volumic term, the balance relation.

► Other models with complete error analysis: anisotropic heterogeneous diffusion; degenerate advection–diffusion–equation equations; Stokes & Navier–Stokes (various options for non-linear term); p-Laplace equations; elasticity; Brinkman; etc.
► This is a **finite volume method**: we can define fluxes that are conservative and satisfy, up to high order volumic term, the balance relation.

► Other models with complete error analysis: anisotropic heterogeneous diffusion; degenerate advection–diffusion–equation equations; Stokes & Navier–Stokes (various options for non-linear term); p-Laplace equations; elasticity; Brinkman; etc.

► Variant: polynomial degree in element could be $k \pm 1$.

Leads to links with

- non-conforming $\mathcal{P}^1$ FE;
- Virtual Element Methods;
- Hybridizable Discontinuous Galerkin;
- etc.
More about HHO

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The Hybrid High-Order Method for Polytopal Meshes
Design, Analysis, and Applications

Springer
More about HHO

- Implementation in various libraries, in particular HArD::Core3D library (https://github.com/jdroniou/HArDCore).

  - Open source C++ code for numerical schemes on generic polyhedral meshes.
  - Based on Eigen linear algebra library (http://eigen.tuxfamily.org).
  - Complete and intuitive description of mesh.
  - Routines for handling polynomial spaces (on edges, faces and cells), for quadrature rules, for Gram-like matrices (mass, stiffness), etc.
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Model for enhanced oil recovery

\[
\begin{align*}
\nabla \cdot u &= q^+ - q^- := q \\
\frac{K}{\mu(c)} \nabla p \\
\phi \frac{\partial c}{\partial t} + \nabla \cdot (uc - D(x, u)\nabla c) + q^- c &= q^+
\end{align*}
\]

**Ununknowns**
- \( p(x, t) \) - pressure of the mixture
- \( u(x, t) \) - Darcy velocity
- \( c(x, t) \) - concentration of the injected solvent

**Parameters**
- \( K(x) \) - permeability tensor
- \( \phi(x) \) - porosity

▶ Complemented with no-flow boundary conditions.
$t = 3$ years, various $k$, Cartesian mesh

$k = 0$

$k = 1$

$k = 2$

$k = 3$
$t = 10 \text{ years, various } k, \text{ Cartesian mesh}$
Percentage of oil recovered vs. mesh size, various $k$
Percentage of oil recovered vs. mesh size, various $k$
Computational cost for one time step: time (s) vs. mesh sizes

- Left graph: $k = 0$, $k = 1$, $k = 2$, $k = 3$
- Right graph: $k = 0$, $k = 1$, $k = 2$, $k = 3$
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OMP: open simply connected set in $\mathbb{R}^3$ with connected boundary.

**Gradient:**

$$H^1(\Omega) = \{u \in L^2(\Omega) : \text{grad } u \in L^2(\Omega)^3\},$$

$$\text{grad} : H^1(\Omega) \rightarrow L^2(\Omega)^3.$$

**Curl:**

$$H(\text{curl}; \Omega) = \{u \in L^2(\Omega)^3 : \text{curl } u \in L^2(\Omega)^3\},$$

$$\text{curl} : H(\text{curl}; \Omega) \rightarrow L^2(\Omega)^3.$$

**Divergence:**

$$H(\text{div}; \Omega) = \{u \in L^2(\Omega)^3 : \text{div } u \in L^2(\Omega)\},$$

$$\text{div} : H(\text{div}; \Omega) \rightarrow L^2(\Omega).$$
$i_\Omega : \mathbb{R} \to H^1(\Omega)$ natural embedding.

**Theorem (Exactness of de Rham sequence)**

The following sequence is exact:

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

which means that, if $\mathcal{O}_i$ and $\mathcal{O}_{i+1}$ are two consecutive operators in the sequence, then

$$\text{Im} \mathcal{O}_i = \text{Ker} \mathcal{O}_{i+1}.$$
Why is this exactness important?

\[ \mathbb{R} \xrightarrow{i_{\Omega}} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\}, \]

Stokes problem

\[ \begin{aligned} -\Delta u + \text{grad } p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \quad + \text{BC} \end{aligned} \]

▶ Inf-sup condition: for all \( q \in L^2(\Omega), \)

\[ \sup_{v \in H(\text{div}; \Omega)} \frac{(\text{div } v, q)_{L^2}}{\|v\|_{H(\text{div})}} \geq \beta \|q\|_{L^2}. \]

Proof: Fix \( q \in L^2(\Omega), \) and let \( v \in H(\text{div}; \Omega) \) such that \( \text{div } v = q \ldots \)
Why is this exactness important?

\[ \mathbb{R} \xrightarrow{i_{\Omega}} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0 \rightarrow \{0\}, \]

Magnetostatic problem

\[
\begin{align*}
\sigma - \text{curl } u &= 0 \quad \text{in } \Omega, \\
\text{curl } \sigma &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u \times n &= g \quad \text{on } \partial \Omega.
\end{align*}
\]

▶ Inf-sup condition: for all \((\tau, v) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega),\)

\[
\sup_{(\mu, w) \in H(\text{curl}) \times H(\text{div})} \frac{\mathcal{A}((\tau, v), (\mu, w))}{\| (\mu, w) \|_{H(\text{curl}) \times H(\text{div})}} \geq \beta \| (\mu, v) \|_{H(\text{curl}) \times H(\text{div})}, \quad \text{where}
\]

\[
\mathcal{A}((\tau, v), (\mu, w)) = (\tau, \mu)_{L^2} - (v, \text{curl } \mu)_{L^2} + (w, \text{curl } \tau)_{L^2} + (\text{div } v, \text{div } w)_{L^2}.
\]
Why is this exactness important?

\[ \mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}; \Omega) \xrightarrow{\text{curl}} H(\text{div}; \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{0} \{0\}, \]

Magnetostatic problem

\[
\begin{cases}
\sigma - \text{curl } u = 0 & \text{in } \Omega, \\
\text{curl } \sigma = f & \text{in } \Omega, \\
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▶ Inf-sup condition: for all \((\tau, v) \in H(\text{curl}; \Omega) \times H(\text{div}; \Omega),\)

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\sup_{(\mu, w) \in H(\text{curl}) \times H(\text{div})} \frac{\mathcal{A}((\tau, v), (\mu, w))}{\| (\mu, w) \|_{H(\text{curl}) \times H(\text{div})}} \geq \beta \| (\mu, v) \|_{H(\text{curl}) \times H(\text{div})}, \text{ where}
\[
\mathcal{A}((\tau, v), (\mu, w)) = (\tau, \mu)_{L^2} - (v, \text{curl } \mu)_{L^2} + (w, \text{curl } \tau)_{L^2} + (\text{div } v, \text{div } w)_{L^2}.
\]

Proof: requires two exactness properties in the sequence, to estimate each component of \(v\) on \((\text{Ker div})^\perp\) and \(\text{Ker div}\).
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Objective

- Mimic exact sequence with discrete spaces and operators.
  
  \( \Rightarrow \) *To be used to design stable numerical schemes.*

- Local construction (element by element), as in standard FE.

- Arbitrary order, based on polynomial spaces of degree \( k \geq 0 \).
Local discrete spaces and operators: for $T$ mesh element,

\[
\mathbb{R} \xrightarrow{\mathcal{L}^k_{\text{grad}, T}} \mathbb{X}^k_{\text{grad}, T} \xrightarrow{\mathcal{G}_T^k} \mathbb{X}^k_{\text{curl}, T} \xrightarrow{\mathcal{C}_T^k} \mathbb{X}^k_{\text{div}, T} \xrightarrow{\mathcal{D}_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.
\]
Local discrete spaces and operators: for $T$ mesh element,

$$\mathbb{R} \xrightarrow{I^k_{\text{grad}, T}} X^k_{\text{grad}, T} \xrightarrow{G^k_T} X^k_{\text{curl}, T} \xrightarrow{C^k_T} X^k_{\text{div}, T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$ 

Finite Element approach:
- Finite Element Exterior Calculus (FEEC).
- Requires elements of certain shapes (tetrahedras, hexahedras...) as in usual FE.
- Designed in very generic setting, with exterior derivatives etc.
Local discrete spaces and operators: for $T$ mesh element,

$$\mathbb{R} \xrightarrow{I^k_{\text{grad}, T}} X^k_{\text{grad}, T} \xrightarrow{G^k_T} X^k_{\text{curl}, T} \xrightarrow{C^k_T} X^k_{\text{div}, T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.$$ 

Virtual Element approach:

- Applicable on generic meshes with polyhedral elements.
- Degree decreases by one at each application of differential operator.
- Functions not fully known, only certain moments or values are accessible.
- Exactness not usable in a scheme due to the variational crime in VEM.
Plan

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   - An inspiring remark
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   - Miscible flow in porous media

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Features

- Applicable on polyhedral elements.
- Arbitrary order of exactness.
- Same order of accuracy along the entire sequence.
- Based on explicit spaces and reconstructed differential operators, exactness holding for these objects.
Principles

\[ \mathbb{R} \xrightarrow{I^k_{\text{grad}, T}} X^k_{\text{grad}, T} \xrightarrow{G^k_T} X^k_{\text{curl}, T} \xrightarrow{C^k_T} X^k_{\text{div}, T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}. \]

**Gradient unknowns:** \( q_T = (q_T, (q_F)_F \in \mathcal{F}_T, q_{\partial^2 T}) \).

\( q_T \in \mathcal{P}^{k-1}(T) \)

\( q_F \in \mathcal{P}^{k-1}(F) \)

\( q_{\partial^2 T} \in \mathcal{P}^{k+1}_c(\partial^2 T) \)
\[ \mathbb{R} \xrightarrow{\nabla^k_{grad, T}} X^k_{grad, T} \xrightarrow{\nabla^k_{curl, T}} X^k_{curl, T} \xrightarrow{\nabla^k_{div, T}} X^k_{div, T} \xrightarrow{D^k_T} P^k(T) \to 0 \to \{0\}. \]

**Gradient operator:**

\[ \nabla^k_T q_T = (\nabla^k_T q_T, (\nabla^k_F(q_F, q_\partial^2_T))_F \in F_T, (\nabla_E q_E)_{E \in E_T}). \]

- **\( G_E \):** derivative along edge.
- **\( G^k_F \, (\approx \nabla |_F) \):** reconstruction from face and edge, based on formal IBP (divergence formula),
- **\( G^k_T \, (\approx \nabla) \):** reconstruction based on formal IBP & face potentials (divergence formula).
Principles

\[ \mathbb{R} \xrightarrow{I_{\text{grad}, T}^k} X_{\text{grad}, T}^k \xrightarrow{G_{T}^k} X_{\text{curl}, T}^k \xrightarrow{c_{T}^k} X_{\text{div}, T}^k \xrightarrow{D_{T}^k} P^k(T) \xrightarrow{0} \{0\}. \]

**Curl unknowns:** \( \mathbf{v}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T}, (\mathbf{v}_E)_{E \in \mathcal{E}_T}) \).

\( \mathbf{v}_T \in P^k(T)^3 \)

\( \mathbf{v}_F \in P^k(F)^2 \)

\( \mathbf{v}_E \in P^k(E) \)

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Principles

\[ \mathbb{R} \xrightarrow{I_{\text{grad},T}^k} X_{\text{grad},T}^k \xrightarrow{G_T^k} X_{\text{curl},T}^k \xrightarrow{C_T^k} X_{\text{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}. \]

Curl operator:

\[ C_T^k \mathbf{v}_T = \left( C_T^k \mathbf{v}_T, \left( C_F^k (\mathbf{v}_F, (\mathbf{v}_E)_{E \in \mathcal{E}_F}) \right)_{F \in \mathcal{F}_T} \right). \]

\[ \begin{align*}
\text{▶} \quad C_F^k \approx \text{curl} \cdot \mathbf{n}_F: & \text{ reconstruction from face and edge, based on formal IBP (rot formula in 2D),} \\
\text{▶} \quad C_T^k \approx \text{curl}: & \text{ reconstruction based on formal IBP & face tangential potentials (curl formula).}
\end{align*} \]
Principles

\[ \mathbb{R} \xrightarrow{I^k_{\text{grad}, T}} X^k_{\text{grad}, T} \xrightarrow{G^k_T} X^k_{\text{curl}, T} \xrightarrow{C^k_T} X^k_{\text{div}, T} \xrightarrow{D^k_T} \mathcal{P}^k(T) \xrightarrow{0} \{0\}. \]

**Divergence unknowns:** \( \mathbf{v}_T = (\mathbf{v}_T, (v_F)_{F \in F_T}) \).

\[ \mathbf{v}_T \in \mathcal{P}^k(T)^3 \]

\[ v_F \in \mathcal{P}^k(F) \]
\[
\mathbb{R} \xrightarrow{I_{\text{grad},T}^k} X_{\text{grad},T}^k \xrightarrow{G_T^k} X_{\text{curl},T}^k \xrightarrow{C_T^k} X_{\text{div},T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.
\]

**Divergence operator:**

\[
D_T^k \mathbf{v}_T (\approx \text{div}) \text{ reconstructed in } \mathcal{P}^k(T) \text{ from divergence formula.}
\]

\[
\int_T (D_T^k \mathbf{v}_T)q_T = -\int_T \mathbf{v}_T \cdot \text{grad } q_T + \sum_{F \in \mathcal{F}_T} \omega_T F \int_F v_F q_T \quad \forall q_T \in \mathcal{P}^k(T).
\]
There’s a catch...

\[
\mathbb{R} \xrightarrow{I_{\text{grad}, T}^k} X_{\text{grad}, T}^k \xrightarrow{G_T^k} X_{\text{curl}, T}^k \xrightarrow{C_T^k} X_{\text{div}, T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.
\]

▶ The previous sequence is not exact!
There’s a catch...

\[ \mathbb{R} \xrightarrow{\mathcal{I}_{\text{grad},T}^k} X_{\text{grad},T}^k \xrightarrow{\mathcal{G}_{T}^k} X_{\text{curl},T}^k \xrightarrow{\mathcal{C}_{T}^k} X_{\text{div},T}^k \xrightarrow{\mathcal{D}_{T}^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\} \].

► The previous sequence is not exact!

For \( X = F, T \) of dimension \( d = 2, 3 \) let:

► \( \mathcal{R}^k(X) = \text{curl}(\mathcal{P}^{k+1}(X)^d) \), \( \mathcal{R}^{c,k}(X) \) complement in \( \mathcal{P}^k(X)^d \).

► \( \mathcal{G}^k(X) = \text{grad}(\mathcal{P}^{k+1}(X)^d) \), \( \mathcal{G}^{c,k}(X) \) complement in \( \mathcal{P}^k(X)^d \).
There’s a catch...

\[
\mathbb{R} \xrightarrow{I_{\text{grad}, T}^k} X_{\text{grad}, T}^k \xrightarrow{G_T^k} X_{\text{curl}, T}^k \xrightarrow{C_T^k} X_{\text{div}, T}^k \xrightarrow{D_T^k} \mathcal{P}^k(T) \xrightarrow{0} \{0\}.
\]

▶ The previous sequence is not exact!

For \( X = F \), \( T \) of dimension \( d = 2, 3 \) let:

▶ \( \mathcal{R}^k(X) = \text{curl}(\mathcal{P}^{k+1}(X)^d) \), \( \mathcal{R}^{c,k}(X) \) complement in \( \mathcal{P}^k(X)^d \).

▶ \( \mathcal{G}^k(X) = \text{grad}(\mathcal{P}^{k+1}(X)^d) \), \( \mathcal{G}^{c,k}(X) \) complement in \( \mathcal{P}^k(X)^d \).

**Trimmed spaces**: face/cell gradients and curls have to be projected on trimmed spaces.

- Gradients in \( \mathcal{P}^k(X)^d \) projected on \( \mathcal{R}^{k-1}(X) \oplus \mathcal{R}^{c,k}(X) \).
- Curls in \( \mathcal{P}^k(X)^d \) projected on \( \mathcal{G}^{k-1}(X) \oplus \mathcal{G}^{c,k}(X) \).
Going global

Global spaces/operators: by patching local spaces/operators.

- Additional challenges:
  - Global exactness, especially Ker $D^k_\Omega \subset \text{Im } C^k_\Omega$.
  - Poincaré inequalities (for stability), e.g.
    \[
    \|v_\Omega\|_{X^k_{\text{curl},\Omega}} \leq M\|C^k_\Omega v_\Omega\|_{X^k_{\text{div},\Omega}} \quad \forall v_\Omega \in (X^k_{\text{curl},\Omega})^\perp.
    \]
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Model and exact solution

\[
\begin{align*}
\sigma - \operatorname{curl} \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\operatorname{curl} \sigma &= \mathbf{f} \quad \text{in } \Omega, \\
\operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} \times \mathbf{n} &= \mathbf{g} \quad \text{on } \partial \Omega.
\end{align*}
\]

on \( \Omega = (0, 1)^3 \), with exact solution

\[
\sigma(x) = 3\pi \begin{pmatrix}
\sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \\
0 \\
-\cos(\pi x_1) \cos(\pi x_2) \sin(\pi x_3)
\end{pmatrix},
\]

\[
\mathbf{u}(x) = \begin{pmatrix}
\cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \\
-2 \sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \\
\sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3)
\end{pmatrix}.
\]

▶ All spaces and operators implemented in the HArD::Core3D library.
Convergence graphs in energy norm: cubic cells

$k = 0 \quad k = 1 \quad k = 2 \quad k = 3$

J. Droniou (Monash University)
Convergence graphs in energy norm: tetrahedral cells

- $k = 0$
- $k = 1$
- $k = 2$
- $k = 3$
Convergence graphs in energy norm: Voronoi cells

- $k = 0$
- $k = 1$
- $k = 2$
- $k = 3$

Graph showing convergence in energy norm with different Voronoi cells.
Convergence graphs in energy norm: Voronoi cells 2

$10^{-0.5}$ $10^{-0.4}$ $10^{-0.3}$

$k = 0$ $k = 1$ $k = 2$ $k = 3$
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Stationary version of Stefan/Porous Medium Equation equations:

\[ u - \text{div}(\Lambda \nabla \zeta(u)) = f - \text{div}(F) \text{ in } \Omega, \]
\[ \zeta(u) = 0 \text{ on } \partial\Omega. \]

Non-linearity:

Porous medium: \( \zeta(u) = |u|^{m-1}u \)

Stefan: \( \zeta \) with plateau
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\( \Lambda = \text{Id} \) to simplify.

**Continuous model:** multiply by \( \zeta(u) \), integrate by parts, use \( \zeta(s)s \geq 0 \):

\[
\int_\Omega |\nabla \zeta(u)|^2 \leq \int_\Omega u\zeta(u) + \nabla \zeta(u) \cdot \nabla \zeta(u) = \int_\Omega f\zeta(u) + \int_\Omega F \cdot \nabla \zeta(u).
\]

Use Poincaré inequality on \( \zeta(u) \) and Cauchy–Schwarz inequalities: bound on \( \zeta(u) \), translates into a bound on \( u \).
First challenge: stability...

\[ \Lambda = Id \] to simplify.

**Continuous model**: multiply by \( \zeta(u) \), integrate by parts, use \( \zeta(s)s \geq 0 \):

\[
\int_\Omega |\nabla \zeta(u)|^2 \leq \int_\Omega u\zeta(u) + \nabla \zeta(u) \cdot \nabla \zeta(u) = \int_\Omega f\zeta(u) + \int_\Omega F \cdot \nabla \zeta(u).
\]

Use Poincaré inequality on \( \zeta(u) \) and Cauchy–Schwarz inequalities:

\[
\int_\Omega |\nabla \zeta(u)|^2 \leq C(f, F)
\]

\( \leadsto \) Bound on \( \zeta(u) \), translates into a bound on \( u \).
First challenge: stability...

**Discrete version:** think conforming $\mathbb{P}_1$ finite elements: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$
\int_{\Omega} u_h v_h + \nabla \zeta(u_h) \cdot \nabla v_h = \int_{\Omega} f v_h + \int_{\Omega} F \cdot \nabla v_h.
$$

**Stability:** $v_h = \zeta(u_h)$ not a valid test function, we need to take $v_h = u_h$:

$$
\int_{\Omega} u_h^2 + \zeta'(u_h) |\nabla u_h|^2 \leq \int_{\Omega} f u_h + \int_{\Omega} F \cdot \nabla u_h.
$$

▶ Last term cannot be estimated by left-hand side...
Towards stability

For \( w_h \in V_h \), define \([\zeta(w)]_h\) by nodal values: unique function in \( V_h \) that has the values \( \zeta(w_h(s)) \) at the nodes \( s \) of \( V_h \) (nodes=degrees of freedom).

**Scheme:** find \( u_h \in V_h \) such that, for all \( v_h \in V_h \),

\[
\int_{\Omega} u_h v_h + \nabla[\zeta(u)]_h \cdot \nabla v_h = \int_{\Omega} f v_h + \int_{\Omega} F \cdot \nabla v_h.
\]
Towards stability

For $w_h \in V_h$, define $[\zeta(w)]_h$ by nodal values: unique function in $V_h$ that has the values $\zeta(w_h(s))$ at the nodes $s$ of $V_h$ (*nodes*=*degrees of freedom*).

**Scheme**: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$
\int_{\Omega} u_h v_h + \nabla [\zeta(u)]_h \cdot \nabla v_h = \int_{\Omega} f v_h + \int_{\Omega} F \cdot \nabla v_h.
$$

**Stability**: $v_h = [\zeta(u)]_h$ is a valid test function!

$$
\int_{\Omega} u_h [\zeta(u)]_h + |\nabla [\zeta(u)]_h|^2 \leq \int_{\Omega} f [\zeta(u)]_h + \int_{\Omega} F \cdot \nabla [\zeta(u)]_h.
$$

▶ What to do with the first term? It was $\geq 0$ in the continuous case, but now?
Solution to stability: mass-lumping

\[
\int_{\Omega} u_h[\zeta(u)]_h + |\nabla[\zeta(u)]_h|^2 \leq \int_{\Omega} f[\zeta(u)]_h + \int_{\Omega} F \cdot \nabla[\zeta(u)]_h.
\]

At the nodes, \( u_h(s)[\zeta(u)]_h(s) = u_h(s)\zeta(u_h(s)) \geq 0 \).

- Replace \( u_h \) in the reaction term by a quantity that only uses nodal values.
Solution to stability: mass-lumping

Let \((K_s)_s\) be a partition of \(\Omega\), each \(K_s\) being built “around” \(s\), and set

\[
\Pi_h u_h : \Omega \rightarrow \mathbb{R} \quad (\Pi_h u_h)|_{K_s} = u_h(s) \quad \forall s.
\]

Example for \(P_1\) FE:
Solution to stability: mass-lumping

Mass-lumped scheme: find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$\int_\Omega \Pi_h u_h \Pi_h v_h + \nabla [\zeta(u)]_h \cdot \nabla v_h = \int_\Omega f \Pi_h v_h + \int_\Omega F \cdot \nabla v_h.$$  

Stability: make $v_h = [\zeta(u)]_h$ as before, and use (magic of mass-lumping!)

$$\Pi_h [\zeta(u)]_h = \zeta(\Pi_h u_h)$$

to get

$$\int_\Omega \left( \Pi_h u_h \zeta(\Pi_h u_h) + |\nabla [\zeta(u)]_h|^2 \right) \geq 0 \leq \int_\Omega f [\zeta(u)]_h + \int_\Omega F \cdot \nabla [\zeta(u)]_h.$$
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Mass-lumping and high-order: a conundrum

Mass-lumping = approximate reaction terms by piecewise constant functions

*Order 1 consistency error at best...*

How do we recover a high-order scheme?
Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{ \mathbf{v} = (v_i)_{i \in I} : v_i \in \mathbb{R}, \; v_i = 0 \text{ if } i \in I_{\partial \Omega} \}.$$ 

There is $(x_i)_{i \in I}$ and, for each $K \in \mathcal{T}_h$, $I_K \subset I$ such that $x_i \in K$ if $i \in I_K$. 
Setting

**Discrete space and nodes**: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{ v = (v_i)_{i \in I} : v_i \in \mathbb{R}, \ v_i = 0 \text{ if } i \in I_{\partial \Omega} \}.$$ 

There is $(x_i)_{i \in I}$ and, for each $K \in \mathcal{T}_h$, $I_K \subset I$ such that $x_i \in \overline{K}$ if $i \in I_K$.

**High-order reconstruction**: $\Pi_{hO} : X_h \to \mathbb{P}_k(\mathcal{T}_h)$. For all $v \in X_h$, $K \in \mathcal{T}_h$ and $i \in I_K$, $v_i = (\Pi_{hO} v)_{|K}(x_i)$. 

Mass-lumping: $U_h = (U_i)_{i \in I}$ partition of $\Omega$ and $\Pi_h : X_h \to \mathbb{P}_0(U_h)$ piecewise constant reconstruction.

**High-order gradient reconstruction**: $\nabla \Pi_{hO} : X_h \to L^\infty(\Omega)$. 

**Quadrature**: $Q_h : C(T_h) \to L^\infty(\Omega)$ given by:

$$(Q_h w)_{|K} = \sum_{i \in I_K} w_{|K}(x_i) 1_{U_i \cap K} \forall K \in \mathcal{T}_h.$$
Discrete space and nodes: $T_h$ a mesh of the domain,

$$X_h = \{ \nu = (\nu_i)_{i \in I} : \nu_i \in \mathbb{R}, \nu_i = 0 \text{ if } i \in I_{\partial \Omega} \}.$$ 

There is $(x_i)_{i \in I}$ and, for each $K \in T_h$, $I_K \subset I$ such that $x_i \in \overline{K}$ if $i \in I_K$.

High-order reconstruction: $\Pi^{\text{HO}}_h : X_h \rightarrow P_k(T_h)$. For all $\nu \in X_h$, $K \in T_h$ and $i \in I_K$, $\nu_i = (\Pi^{\text{HO}}_h \nu)_{|K}(x_i)$.

Mass-lumping: $U_h = (U_i)_{i \in I}$ partition of $\Omega$ and $\Pi_h : X_h \rightarrow P_0(U_h)$ piecewise constant reconstruction.

For all $i \in I$ and $K \in T_h$, $U_i \cap K \neq \emptyset$ only if $i \in I_K$. 
Setting

Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{ \mathbf{v} = (v_i)_{i \in I} : v_i \in \mathbb{R}, \; v_i = 0 \text{ if } i \in I_{\partial \Omega} \}.$$ 

There is $(x_i)_{i \in I}$ and, for each $K \in \mathcal{T}_h$, $I_K \subset I$ such that $x_i \in \overline{K}$ if $i \in I_K$.

High-order reconstruction: $\Pi_h^{HO} : X_h \rightarrow \mathbb{P}_k(\mathcal{T}_h)$. For all $\mathbf{v} \in X_h$, $K \in \mathcal{T}_h$ and $i \in I_K$, $v_i = (\Pi_h^{HO} \mathbf{v})|_K(x_i)$.

Mass-lumping: $\mathcal{U}_h = (U_i)_{i \in I}$ partition of $\Omega$ and $\Pi_h : X_h \rightarrow \mathbb{P}_0(\mathcal{U}_h)$ piecewise constant reconstruction. 

For all $i \in I$ and $K \in \mathcal{T}_h$, $U_i \cap K \neq \emptyset$ only if $i \in I_K$.

High-order gradient reconstruction: $\nabla_h^{HO} : X_h \rightarrow L^\infty(\Omega)^d$. 

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Discrete space and nodes: $\mathcal{T}_h$ a mesh of the domain,

$$X_h = \{ \mathbf{v} = (v_i)_{i \in I} : v_i \in \mathbb{R}, \ v_i = 0 \text{ if } i \in I_{\partial \Omega} \}.$$  

There is $(x_i)_{i \in I}$ and, for each $K \in \mathcal{T}_h$, $I_K \subset I$ such that $x_i \in \overline{K}$ if $i \in I_K$.

High-order reconstruction: $\Pi_h^{HO} : X_h \rightarrow \mathbb{P}_k(\mathcal{T}_h)$. For all $\mathbf{v} \in X_h$, $K \in \mathcal{T}_h$ and $i \in I_K$, $v_i = (\Pi_h^{HO} \mathbf{v})|_K(x_i)$.

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For all $i \in I$ and $K \in \mathcal{T}_h$, $U_i \cap K \neq \emptyset$ only if $i \in I_K$.

High-order gradient reconstruction: $\nabla_h^{HO} : X_h \rightarrow L^\infty(\Omega)^d$.

Quadrature: $Q_h : C(\mathcal{T}_h) \rightarrow L^\infty(\Omega)$ given by:

$$(Q_h w)|_K = \sum_{i \in I_K} w_i|_K(x_i) 1_{U_i \cap K} \quad \forall K \in \mathcal{T}_h.$$
Non-linear function of vectors: if $\mathbf{v} = (v_i)_{i \in I} \in X_h$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, we define

$$g(\mathbf{v}) \in X_h \text{ such that } (g(\mathbf{v}))_i = g(v_i) \quad \forall i \in I.$$
Non-linear function of vectors: if \( \mathbf{v} = (v_i)_{i \in I} \in X_h \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \), we define

\[
g(\mathbf{v}) \in X_h \text{ such that } (g(\mathbf{v}))[i] = g(v_i) \quad \forall i \in I.
\]

Scheme:

Find \( u_h \in X_h \) such that, for all \( \mathbf{v}_h \in X_h \),

\[
\int_{\Omega} \Pi_h u_h \Pi_h \mathbf{v}_h + \Lambda \nabla^{\text{HO}}_h \zeta(u_h) \cdot \nabla^{\text{HO}}_h \mathbf{v}_h = \int_{\Omega} Q_h f \Pi_h \mathbf{v}_h.
\]
Assumption on $Q_h$

**Exactness of quadrature:** we assume that $Q_h$ is locally exact of degree $k + \ell$, that is

$$
\int_K q = \int_K Q_h q \left( = \sum_{i \in I_K} |U_i \cap K| q(x_i) \right)
\forall K \in \mathcal{T}_h, \forall q \in \mathbb{P}_{k+\ell}.
$$
Error estimate

- Broken Sobolev space:
  \[ W^{\ell+2,\infty}(\mathcal{T}_h) = \{ w \in L^\infty(\Omega) : w|_K \in W^{\ell+2,\infty}(K) \quad \forall K \in \mathcal{T}_h \}. \]

- Defect of conformity of the (underlying high-order) method: for any \( \psi \in L^2(\Omega)^d \) with \( \text{div} \psi \in L^2(\Omega) \),

\[
W_h^{\text{HO}}(\psi) = \max_{w_h \in X_h \setminus \{0\}} \frac{1}{\|\nabla_h^{\text{HO}} w_h\|_\Omega} \left| \int_\Omega \Pi_h^{\text{HO}} w_h \text{div} \psi q + \nabla_h^{\text{HO}} w_h \cdot \psi \right|
\]
Error estimate

- Broken Sobolev space:
  \[ W^{\ell+2,\infty}(T_h) = \{ w \in L^{\infty}(\Omega) : w|_K \in W^{\ell+2,\infty}(K) \quad \forall K \in T_h \}. \]

- Defect of conformity of the (underlying high-order) method: for any \( \psi \in L^2(\Omega)^d \) with \( \text{div} \, \psi \in L^2(\Omega) \),

\[
W_h^{\text{HO}}(\psi) = \max_{w_h \in X_h \setminus \{0\}} \frac{1}{\| \nabla_h^{\text{HO}} w_h \|_{\Omega}} \left| \int_{\Omega} \Pi_h^{\text{HO}} w_h \, \text{div} \, \psi q + \nabla_h^{\text{HO}} w_h \cdot \psi \right|.
\]

**Theorem (D.-Eymard, 2018)**

Under \( QR_k(\ell) \), if \( u \) and \( f \) belong to \( W^{\ell+2,\infty}(T_h) \) then

\[
\| \nabla_h^{\text{HO}} [I_h \zeta(u) - \zeta(u_h)] \|_{\Omega} \leq CW_{h,\text{HO}}(\Lambda \nabla \zeta(u)) + C \| \nabla_h^{\text{HO}} I_h \zeta(u) - \nabla \zeta(u) \|_{\Omega} + Ch^{\ell+2},
\]

with \( I_h \zeta(u) = (\zeta(u)(x_i))_{i \in I} \) interpolate of \( \zeta(u) \) on \( X_h \).
Theorem (D.-Eymard, 2018)

Under $QR_k(\ell)$, if $u$ and $f$ belong to $W^{\ell+2,\infty}(T_h)$ then

\[ \| \nabla_h^{\text{HO}} [I_h \zeta(u) - \zeta(u_h)] \|_\Omega \leq CW_h^{\text{HO}}(\Lambda \nabla \zeta'(u)) + C \| \nabla_h^{\text{HO}} I_h \zeta(u) - \nabla \zeta(u) \|_\Omega + Ch^{\ell+2}, \]

with $I_h \zeta(u) = (\zeta(u)(x_i))_{i \in I}$ interpolate of $\zeta(u)$ on $X_h$.

▶ Real limiting factor is $h^{\ell+2}$, dictated by $QR_k(\ell)$ (and regularity of $u$ and $f$).
1 Hybrid High-Order method
   - An inspiring remark
   - Description of the HHO scheme
   - Miscible flow in porous media

2 Fully Discrete de Rham sequence
   - Principles of discrete exact sequence
   - Fully discrete de Rham sequence
   - Application to magnetostatics

3 High-order schemes for stationary Stefan/PME models
   - Towards a stable numerical approximation
   - High-order approximations
   - Numerical tests
### Quadrature rules (dimension 1)

| Name          | \((x_i)_{i \in I_K}\) | \((\frac{|U_i \cap K|}{|K|})_{i \in I_K}\) | DOE | Illustration |
|---------------|------------------------|-----------------------------------------------|-----|--------------|
| Trapezoidal   | \((a, b)\)             | \((\frac{1}{2}, \frac{1}{2})\)               | 1   |              |
| Simpson       | \((a, \frac{a+b}{2}, b)\) | \((\frac{1}{6}, \frac{2}{3}, \frac{1}{6})\) | 3   |              |
| Equi6         | \((a, \frac{2a+b}{3}, \frac{a+2b}{3}, b)\) | \((\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6})\) | 1   |              |
| Equi8         | \((a, \frac{2a+b}{3}, \frac{a+2b}{3}, b)\) | \((\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})\) | 3   |              |
| Gauss–Lobatto | \((a, \frac{5+\sqrt{5}}{10} a + \frac{5-\sqrt{5}}{10} b, \frac{5-\sqrt{5}}{10} a + \frac{5+\sqrt{5}}{10} b, b)\) | \((\frac{1}{12}, \frac{5}{12}, \frac{5}{12}, \frac{1}{12})\) | 5   |              |

**Table:** Examples of quadrature rules in dimension \(d = 1\) for \(K = (a, b)\). DOE stands for degree of exactness (corresponds to \(k + \ell\)).
### Gradient discretisations for $\mathbb{P}_k$ finite elements

<table>
<thead>
<tr>
<th>Name</th>
<th>Degree $k$</th>
<th>Quadrature rule</th>
<th>$\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{D}_1^g(0)$</td>
<td>1</td>
<td>Trapezoidal</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{D}_2^g(1)$</td>
<td>2</td>
<td>Simpson</td>
<td>1</td>
</tr>
<tr>
<td>$\mathcal{D}_3^g(−)$</td>
<td>3</td>
<td>Equi6</td>
<td>−</td>
</tr>
<tr>
<td>$\mathcal{D}_3^g(0)$</td>
<td>3</td>
<td>Equi8</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{D}_3^g(2)$</td>
<td>3</td>
<td>Gauss–Lobatto</td>
<td>2</td>
</tr>
</tbody>
</table>

**Table:** Mass-lumped GDs for $\mathbb{P}_k$ Finite Element in dimension $d = 1$. These methods satisfy $QR_k(\ell)$ with the corresponding $k, \ell$. $g = u$ for uniform meshes, $g = r$ for random meshes.
We provide $(C, \alpha)$ such that

$$\|\nabla_h^{HO} (I_h \zeta(u) - \zeta(u_h))\|_\Omega \approx C \text{Card}(I)^{-\alpha/d}.$$

$\alpha \sim \text{rate of convergence in meshsize}.$
**Linear model:** $\zeta(u) = u$

**Test R:** regular exact solution $u(x) = x(1-x)e^x$.

**Results**

<table>
<thead>
<tr>
<th></th>
<th>$D_1^u(0)$</th>
<th>$D_1^r(0)$</th>
<th>$D_2^u(1)$</th>
<th>$D_2^r(1)$</th>
<th>$D_3^u(-)$</th>
<th>$D_3^r(-)$</th>
<th>$D_3^u(0)$</th>
<th>$D_3^r(0)$</th>
<th>$D_3^u(2)$</th>
<th>$D_3^r(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>0.44</td>
<td>0.31</td>
<td>0.14</td>
<td>0.13</td>
<td>0.15</td>
<td>0.15</td>
<td>0.2</td>
<td>0.2</td>
<td>0.0002</td>
<td>0.00024</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2</td>
<td>1.9</td>
<td>3</td>
<td>2.98</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1.99</td>
<td>2.95</td>
<td>2.97</td>
</tr>
<tr>
<td>$\min(k, \ell + 2)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Porous medium equation: $\zeta(u) = \max(u, 0)^2$

**Test P1**: exact solution with $s^{3/2}$ singularity – piecewise smooth, singularity not aligned with meshes.

![Graph of the test function](image)

**Results**

<table>
<thead>
<tr>
<th></th>
<th>$D_1^u(0)$</th>
<th>$D_1^i(0)$</th>
<th>$D_2^u(1)$</th>
<th>$D_2^i(1)$</th>
<th>$D_3^u(–)$</th>
<th>$D_3^i(–)$</th>
<th>$D_3^u(0)$</th>
<th>$D_3^i(0)$</th>
<th>$D_3^u(2)$</th>
<th>$D_3^i(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>12</td>
<td>12</td>
<td>4.3</td>
<td>16</td>
<td>0.41</td>
<td>0.42</td>
<td>2.7</td>
<td>2.3</td>
<td>1.2</td>
<td>0.22</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2</td>
<td>1.98</td>
<td>2.45</td>
<td>2.69</td>
<td>1.03</td>
<td>1.03</td>
<td>1.99</td>
<td>1.95</td>
<td>2.42</td>
<td>1.98</td>
</tr>
<tr>
<td>$\min(k, \ell + 2)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>–</td>
<td>–</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Porous medium equation: $\zeta(u) = \max(u, 0)^2$

**Test P2**: exact solution $u(x) = \max(x - 1.5, 0)^2/12$ corresponding to $f = 0$.

Results

<table>
<thead>
<tr>
<th></th>
<th>$D_1^u(0)$</th>
<th>$D_1^l(0)$</th>
<th>$D_2^u(1)$</th>
<th>$D_2^l(1)$</th>
<th>$D_3^u(-)$</th>
<th>$D_3^l(-)$</th>
<th>$D_3^u(0)$</th>
<th>$D_3^l(0)$</th>
<th>$D_3^u(2)$</th>
<th>$D_3^l(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>0.19</td>
<td>0.38</td>
<td>0.17</td>
<td>0.18</td>
<td>0.14</td>
<td>0.15</td>
<td>0.24</td>
<td>0.24</td>
<td>0.27</td>
<td>0.0014</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2</td>
<td>1.97</td>
<td>2.99</td>
<td>2.98</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1.99</td>
<td>3.1</td>
<td>3.46</td>
</tr>
<tr>
<td>$\min(k, \ell + 2)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>–</td>
<td>–</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

J. Droniou (Monash University)
Nonlinearity: $\zeta(u) = 0$ if $0 \leq u \leq 1$, slope 1 otherwise.
**Stefan model**

**Test S1**: \( f(x) = 3(0.5 - |0.5 - x|) \), exact solution (computable up to parameters that are numerically evaluated):

![Graph showing the function \( f(x) \) with values at various points]

### Results

<table>
<thead>
<tr>
<th></th>
<th>( D_i^u(0) )</th>
<th>( D_i^f(0) )</th>
<th>( D_2^u(1) )</th>
<th>( D_2^f(1) )</th>
<th>( D_3^u(\cdot) )</th>
<th>( D_3^f(\cdot) )</th>
<th>( D_3^u(0) )</th>
<th>( D_3^f(0) )</th>
<th>( D_3^u(2) )</th>
<th>( D_3^f(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C</strong></td>
<td>12</td>
<td>3.2</td>
<td>0.62</td>
<td>1</td>
<td>0.37</td>
<td>0.38</td>
<td>0.72</td>
<td>0.28</td>
<td>0.36</td>
<td>0.39</td>
</tr>
<tr>
<td><strong>( \alpha )</strong></td>
<td>1.87</td>
<td>1.66</td>
<td>1.54</td>
<td>1.65</td>
<td>1.03</td>
<td>1.05</td>
<td>1.61</td>
<td>1.53</td>
<td>1.58</td>
<td>1.59</td>
</tr>
<tr>
<td><strong>min(( k, \ell + 2 ))</strong></td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>–</td>
<td>–</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
What happens with $D^g_3(0)$ and $D^g_3(2)$? No improvement over low order?
► Due to lack of regularity of $\zeta(u)$, only belongs to $H^2$ as $(\zeta(u))'' = u - f$ is discontinuous.
► Recover $O(h^2)$ convergence if error calculated far from discontinuity; but not $O(h^3)$ even for $D^g_3(2)$. 

J. Droniou (Monash University)
Degree $k = 3$, various quadrature rules, for PME and Stefan ($f = 0$).

<table>
<thead>
<tr>
<th></th>
<th>$D_3^u(-)$</th>
<th>$D_3^r(-)$</th>
<th>$D_3^u(0)$</th>
<th>$D_3^r(0)$</th>
<th>$D_3^u(2)$</th>
<th>$D_3^r(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>0.15</td>
<td>0.17</td>
<td>0.22</td>
<td>0.22</td>
<td>0.023</td>
<td>0.011</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.01</td>
<td>1.02</td>
<td>2</td>
<td>1.99</td>
<td>3.25</td>
<td>3.01</td>
</tr>
<tr>
<td>Test P1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>0.42</td>
<td>0.42</td>
<td>2.9</td>
<td>2.9</td>
<td>1.4</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.03</td>
<td>1.03</td>
<td>1.98</td>
<td>1.97</td>
<td>2.39</td>
<td>2.32</td>
</tr>
<tr>
<td>Test P2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>0.15</td>
<td>0.15</td>
<td>0.27</td>
<td>0.28</td>
<td>0.039</td>
<td>0.019</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.01</td>
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<td>2</td>
<td>3.42</td>
<td>3.08</td>
</tr>
<tr>
<td>Test S2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>0.09</td>
<td>0.085</td>
<td>0.082</td>
<td>0.074</td>
<td>0.054</td>
<td>0.058</td>
</tr>
<tr>
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<td>1.01</td>
<td>1</td>
<td>1.5</td>
<td>1.57</td>
<td>1.49</td>
<td>1.58</td>
</tr>
<tr>
<td>$\min(k, \ell + 2)$</td>
<td>–</td>
<td>–</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Conclusion

Hybrid High-Order method

► High-order method for diffusion problems on polytopal meshes.
► Uses element and face unknowns to reconstruct higher-order potential.
► Optimal rates of convergence, for many models of practical interest.
► Finite volume method, provides appropriate fluxes for coupling with transport.
Conclusion

**Discrete de Rham sequence**

- Preserve exactness property at discrete level: essential in some applications.
- Fully computable (purely polynomial) spaces and operators.
- In its infancy, lot of work remains to be done...
Conclusion

**High-order schemes for Stefan/PME**

- Stefan/PME requires mass lumping.
- Higher order schemes still possible, provided a key quadrature rule is respected.
- Convergence benefits from high order, unless restricted by regularity of solution (even then, local improvement is possible).
Main books/papers:


Thanks.