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A Family of Perfect Factorisations of Complete Bipartite Graphs

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A 1-factorisation of a graph is perfect if the union of any two of its 1-factors is a Hamiltonian cycle. Let $n=p^2$ for an odd prime p. We construct a family of $(p-1)/2$ non-isomorphic perfect 1-factorisations of $K_{n,n}$. Equivalently, we construct pan-Hamiltonian Latin squares of order n. A Latin square is pan-Hamiltonian if the permutation defined by any row relative to any other row is a single cycle. © 2002 Elsevier Science (USA)

1. INTRODUCTION

A *k-factor* of a graph is a k-regular spanning subgraph. A partitioning of the edges of a graph into k-factors is a *k-factorisation*. A 1-factorisation is *perfect* if the union of any two of its 1-factors is a single (Hamiltonian) cycle. For a full discussion of 1-factorisations see Wallis [11].

The problem of finding perfect 1-factorisations of complete graphs has been studied for a number of years, see the surveys Seah [10] or Wallis

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[11]. There are only two infinite families known, which Laufer [7] attributes to Kotzig. These families are for K_{p+1} and K_{2p} where p is an odd prime. In addition, a perfect 1-factorisation of K_n is known for a number of small values of n, including all even $n \leqslant 50$. There is also interest in perfect 1-factorisations of complete bipartite graphs, and in this paper we construct a new infinite family of such factorisations. We note the following connection.

THEOREM 1.1. *If there is a perfect 1-factorisation of K_{n+1} then $K_{n,n}$ also has a perfect 1-factorisation.*

A perfect 1-factorisation of K_n can exist only when n is even. Slightly less obviously, a perfect 1-factorisation of $K_{n,n}$ can exist only when $n=2$ or n is odd. This result and Theorem 1.1 go at least as far back as Laufer [7] and have been 'discovered' by many people. Note that Theorem 1.1 yields two infinite families of perfect 1-factorisations of complete bipartite graphs. Namely, we have perfect 1-factorisations of $K_{p,p}$ and $K_{2p-1,2p-1}$ for odd primes p. We also have a perfect 1-factorisation of $K_{n,n}$ for all odd $n<50$. It is worth observing that the construction on which Theorem 1.1 is based cannot be reversed to give the converse result. Up to isomorphism there is a unique perfect 1-factorisation of K_{10} but there are 37 of $K_{9,9}$, see Wanless [12].

One reason for studying perfect 1-factorisations of complete bipartite graphs is their connection with pan-Hamiltonian Latin squares. An $n \times n$ matrix M with entries chosen from a set of symbols of cardinality n is *row-Latin* if each row of M contains every symbol once and is *column-Latin* if each column contains every symbol once. A *Latin square* of order n is an $n \times n$ matrix which is both row-Latin and column-Latin. We use $M(i, j)$ to denote the symbol in row i and column j of a matrix M.

For any ordered pair (r_1, r_2) of distinct rows in a row-Latin matrix M, we define a permutation $\sigma[M, r_1, r_2]$ by $M(r_1, c)\,\sigma[M, r_1, r_2] = M(r_2, c)$. Here and throughout we follow the convention of writing permutations on the right, so that $x\pi$ denotes the image of x under the permutation π. For any rows r_1 and r_2 the permutation $\sigma[M, r_1, r_2]$ may be written as a product of disjoint cycles in the standard way. If this product consists of a single cycle for every pair of rows of M, then we say M is *pan-Hamiltonian.*

Given any Latin square of order n you can construct a 1-factorisation of $K_{n,n}$. The vertices of $K_{n,n}$ correspond to the columns and symbols and each row of the Latin square defines a 1-factor in a natural way. If the Latin square happens to be pan-Hamiltonian then the associated 1-factorisation is perfect. Conversely, starting with any perfect 1-factorisation of $K_{n,n}$, you can reverse this construction to obtain a pan-Hamiltonian Latin square of order n. Hence we have the following theorem (which is well known, and appeared in [12]).

THEOREM 1.2. *There exists a pan-Hamiltonian Latin square of order n if and only if $K_{n,n}$ has a perfect 1-factorisation.*

Note that the permutation $\sigma[M, r_1, r_2]$ consists of cycles of lengths $l_1, l_2, ..., l_t$ if and only if the union of the two 1-factors corresponding to rows r_1 and r_2 consists of cycles of lengths $2l_1, 2l_2, ..., 2l_t$ in $K_{n,n}$. Our constructions are phrased in terms of Latin squares (not bipartite graphs) so when we talk about cycles and cycle lengths we shall always be referring to those in the permutations.

In this paper we construct a family of pan-Hamiltonian Latin squares which includes at least one square of order p^2 for all odd primes p. Thus by Theorem 1.2 we also prove the existence of a perfect 1-factorisation of $K_{n,n}$ whenever $n = p^2$. This resolves the existence question for infinitely many new values of n. (Note, for example, that if p is any prime with last digit 7, then p^2 is not of the form q or $2q-1$ for any prime q.)

One source of interest in pan-Hamiltonian Latin squares is that they are N_∞, that is, free of proper subsquares. Such squares are difficult to construct and their existence spectrum is still unknown. See the chapter by Heinrich in [4] for a survey of results. The first constructions of N_∞ squares for a number of odd orders made use of Theorem 1.1. Although N_∞ squares of order p^2 have been constructed, the previous constructions did not produce pan-Hamiltonian Latin squares. Hence our construction gives new N_∞ squares of these orders.

Just as each pair of rows in a Latin square defines a cycle, there are also cycles defined by each pair of columns and by each pair of symbols. We have chosen to define pan-Hamiltonicity in terms of row cycles. We could just as validly have defined it in terms of either column or symbol cycles. Such a definition would be equivalent up to conjugacy of the square. However, there is an interesting, stronger property that a square may have. We say that a Latin square is *atomic* if every conjugate is pan-Hamiltonian. This terminology was introduced in [12], where an infinite family of atomic squares is exhibited. These squares are all of prime orders but separate from the cyclic groups. The author was at the time unaware that Owens and Preece [9] had earlier given just such a family (though the two families are not identical). Note that Owens and Preece asked about the existence of pan-Hamiltonian Latin squares of composite order. Their question had already been answered by the results on perfect 1-factorisations cited above. Wanless [12] points to the existence of an atomic square of order 27, so these are not restricted to prime orders. It is still an open question whether atomic squares exist for non-prime power orders. This paper will not shed any light on this question, as it will become apparent in Section 3 that all the squares we construct have short column and symbol cycles.

It is worth remarking that different terminology has been used to describe the objects under investigation here. Kotzig and Labelle [6] studied perfect 1-factorisations of cubic graphs. They used the term "Hamiltonian decomposition" to describe a perfect 1-factorisation, and said that a graph is "strongly Hamiltonian" if it has a Hamiltonian decomposition. We opt not to use their terminology since it is likely to cause confusion. "Hamiltonian decomposition" is now commonly used to describe a 2-factorisation using Hamiltonian cycles. "Strongly Hamiltonian" has itself been used in several other graph theoretic senses (for examples, see [1] and [2]). One of these is that a graph (or digraph) is strongly Hamiltonian if there is a Hamiltonian path from any vertex to any other. Regrettably, pan-Hamiltonian has also been used, by Lewin [8], in this sense! We stress that our definition of pan-Hamiltonian Latin squares has no relation to Lewin's meaning. It is also worth mentioning that Dénes and Keedwell [3] and Keedwell [5] have studied a weaker property related to pan-Hamiltonicity. They considered Latin squares M for which $\sigma[M, 1, r]$ consists of a single cycle for $r \neq 1$.

We now present our construction for a new family of pan-Hamiltonian Latin squares.

2. THE CONSTRUCTION

Let p be an odd prime, let $\alpha, \beta \in \mathbb{Z}_p$ with $\alpha, \beta \neq 0$, let $r, s, t \in \mathbb{Z}_p \times \mathbb{Z}_p$ and let $L_0(p)$ be the Latin square of order p^2 with symbol $t = r+s$ in row r and column s. Let $L_1(p, \alpha)$ be the Latin square obtained by applying the symbol permutation $(a, y) \mapsto (a, y+\alpha a)$ to each row (a, b) of $L_0(p)$. Note that this symbol permutation is equivalent to the column permutation $(0, d) \mapsto (0, d-\alpha a)$, $d \in \mathbb{Z}_p$. Hence it is clear that $L_1(p, \alpha)$ is a Latin square. Let $L(p, \alpha, \beta)$ be the $p^2 \times p^2$ matrix obtained by applying the symbol permutation $(x, 0) \mapsto (x+\beta b, 0)$ to each row (a, b) of $L_1(p, \alpha)$. Where there can be no confusion, we shall sometimes write L_0, L_1 and L instead of $L_0(p)$, $L_1(p, \alpha)$ and $L(p, \alpha, \beta)$ respectively. A direct rule for determining the entry of $L(p, \alpha, \beta)$ in row $r = (a, b)$ and column $s = (c, d)$ is:

$$L(p, \alpha, \beta)(r, s) = \begin{cases} (a, \alpha a+b+d) & \text{if } c=0 \text{ and } \alpha a+b+d \neq 0, \\ (a+\beta b, 0) & \text{if } c=0 \text{ and } \alpha a+b+d=0, \\ (a+\beta b+c, 0) & \text{if } c \neq 0 \text{ and } b+d=0, \\ (a+c, b+d) & \text{if } c \neq 0 \text{ and } b+d \neq 0. \end{cases} \tag{1}$$

EXAMPLE 2.1. Figure 1 shows the 25 by 25 Latin square $L(5, 1, 3)$, that is, $p=5$, $\alpha=1$, $\beta=3$. To save space we write the symbol (x, y) as xy. Symbols which differ from the direct sum, $L_0(5)$, are shown in bold.

00 01 02 03 04	10 11 12 13 14	20 21 22 23 24	30 31 32 33 34	40 41 42 43 44
01 02 03 04 **30**	11 12 13 14 **40**	21 22 23 24 **00**	31 32 33 34 **10**	41 42 43 44 **20**
02 03 04 **10** 01	12 13 14 **20** 11	22 23 24 **30** 21	32 33 34 **40** 31	42 43 44 **00** 41
03 04 **40** 01 02	13 14 **00** 11 12	23 24 **10** 21 22	33 34 **20** 31 32	43 44 **30** 41 42
04 **20** 01 02 03	14 **30** 11 12 13	24 **40** 21 22 23	34 **00** 31 32 33	44 **10** 41 42 43
11 12 13 14 10	20 21 22 23 24	30 31 32 33 34	40 41 42 43 44	00 01 02 03 04
12 13 14 40 11	21 22 23 24 **00**	31 32 33 34 **10**	41 42 43 44 **20**	01 02 03 04 **30**
13 14 20 11 12	22 23 24 **30** 21	32 33 34 **40** 31	42 43 44 **00** 41	02 03 04 **10** 01
14 00 11 12 13	23 24 **10** 21 22	33 34 **20** 31 32	43 44 **30** 41 42	03 04 **40** 01 02
30 11 12 13 14	24 **40** 21 22 23	34 **00** 31 32 33	44 **10** 41 42 43	04 **20** 01 02 03
22 23 24 20 21	30 31 32 33 34	40 41 42 43 44	00 01 02 03 04	10 11 12 13 14
23 24 00 21 22	31 32 33 34 **10**	41 42 43 44 **20**	01 02 03 04 **30**	11 12 13 14 **40**
24 30 21 22 23	32 33 34 **40** 31	42 43 44 **00** 41	02 03 04 **10** 01	12 13 14 **20** 11
10 21 22 23 24	33 34 **20** 31 32	43 44 **30** 41 42	03 04 **40** 01 02	13 14 **00** 11 12
21 22 23 24 40	34 **00** 31 32 33	44 **10** 41 42 43	04 **20** 01 02 03	14 **30** 11 12 13
33 34 30 31 32	40 41 42 43 44	00 01 02 03 04	10 11 12 13 14	20 21 22 23 24
34 10 31 32 33	41 42 43 44 **20**	01 02 03 04 **30**	11 12 13 14 **40**	21 22 23 24 **00**
40 31 32 33 34	42 43 44 **00** 41	02 03 04 **10** 01	12 13 14 **20** 11	22 23 24 **30** 21
31 32 33 34 20	43 44 **30** 41 42	03 04 **40** 01 02	13 14 **00** 11 12	23 24 **10** 21 22
32 33 34 00 31	44 **10** 41 42 43	04 **20** 01 02 03	14 **30** 11 12 13	24 **40** 21 22 23
44 40 41 42 43	00 01 02 03 04	10 11 12 13 14	20 21 22 23 24	30 31 32 33 34
20 41 42 43 44	01 02 03 04 **30**	11 12 13 14 **40**	21 22 23 24 **00**	31 32 33 34 **10**
41 42 43 44 00	02 03 04 **10** 01	12 13 14 **20** 11	22 23 24 **30** 21	32 33 34 **40** 31
42 43 44 30 41	03 04 **40** 01 02	13 14 **00** 11 12	23 24 **10** 21 22	33 34 **20** 31 32
43 44 10 41 42	04 **20** 01 02 03	14 **30** 11 12 13	24 **40** 21 22 23	34 **00** 31 32 33

FIG. 1. $L(5, 1, 3)$.

We aim to determine when the matrix $L(p, \alpha, \beta)$ defined by (1) is a pan-Hamiltonian Latin square. We begin with two results which will be needed in the proof of the main theorem.

PROPOSITION 2.1. *$L(p, \alpha, \beta)$ is a Latin square if and only if $\alpha\beta \neq 1$.*

Proof. Since L_1 is a Latin square, it is clear that L is row-Latin and, moreover, that L is column-Latin unless for some $x \in \mathbb{Z}_p$, symbol $(x, 0)$ occurs more than once in some column. Suppose $\alpha\beta \neq 1$ and that rows (a_1, b_1) and (a_2, b_2) of column (c, d) both contain the symbol $(x, 0)$. If $c \neq 0$ then $b_1 + d = b_2 + d = 0$ and $a_1 + \beta b_1 + c = a_2 + \beta b_2 + c = x$, which together show that $b_1 = b_2$ and $a_1 = a_2$. So assume that $c = 0$. In this case $\alpha a_1 + b_1 + d = \alpha a_2 + b_2 + d = 0$ and $a_1 + \beta b_1 = a_2 + \beta b_2 = x$. It follows that

$b_1 = b_2 + \alpha a_2 - \alpha a_1$ and consequently $a_1 + \beta(b_2 + \alpha a_2 - \alpha a_1) = a_2 + \beta b_2$, which implies that $(\alpha\beta - 1)(a_2 - a_1) = 0$. Since $\alpha\beta \neq 1$ we conclude that $a_1 = a_2$, from which it also follows that $b_1 = b_2$. Hence L is a Latin square whenever $\alpha\beta \neq 1$. Now, conversely, suppose that $\alpha\beta = 1$. Then symbol $(0, 0)$ occurs in rows $(0, 0)$ and $(1, -\alpha)$ of column $(0, 0)$ and hence L is not a Latin square. ∎

The next result shows that L is pan-Hamiltonian if $\sigma[L, (0, 0), (a, b)]$ consists of a p^2-cycle for all $(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(0, 0)\}$. To do this, it is sufficient to show for all $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}_p \times \mathbb{Z}_p$ that $\sigma[L, (a_1, b_1), (a_2, b_2)] = \phi^{-1}\sigma[L, (0, 0), (a, b)]\,\phi$ for some symbol permutation ϕ and some row $(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p$. For then $\sigma[L, (a_1, b_1), (a_2, b_2)]$ is the permutation obtained by writing $\sigma[L, (0, 0), (a, b)]$ as a product of disjoint cycles and then replacing each symbol (x, y) with $(x, y)\,\phi$.

Proposition 2.2. *Suppose* $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}_p \times \mathbb{Z}_p$. *Then*

$$\sigma[L, (a_1, b_1), (a_2, b_2)] = \phi^{-1}\sigma[L, (0, 0), (a, b)]\,\phi$$

for some symbol permutation ϕ *and some row* $(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p$.

Proof. Let $a = a_2 - a_1$, $b = b_2 - b_1$ and let ϕ be the symbol permutation:

$$\begin{aligned} &(x, y) \mapsto (x + a_1, y) && \text{if} \quad y \neq 0;\\ &(x, y) \mapsto (x + a_1 + \beta b_1, 0) && \text{if} \quad y = 0. \end{aligned}$$

Now, $(x, y)\,\sigma[L, (a_1, b_1), (a_2, b_2)]$ is

$$\begin{array}{lll} (x + a, y + b) & \text{if} & y \neq 0,\ y + b \neq 0,\ x \neq a_1,\\ (x + a, y + b + \alpha a) & \text{if} & y \neq 0,\ y + b + \alpha a \neq 0,\ x = a_1,\\ (x + a + \beta b_2, 0) & \text{if} & y \neq 0,\ y + b = 0,\ x \neq a_1,\\ (x + a + \beta b_2, 0) & \text{if} & y \neq 0,\ y + b + \alpha a = 0,\ x = a_1,\\ (x + a - \beta b_1, b) & \text{if} & y = 0,\ b \neq 0,\ x - \beta b_1 \neq a_1,\\ (x + a - \beta b_1, b + \alpha a) & \text{if} & y = 0,\ b + \alpha a \neq 0,\ x - \beta b_1 = a_1,\\ (x + a, 0) & \text{if} & y = 0,\ b = 0,\ x - \beta b_1 \neq a_1,\\ (x + a + \beta b, 0) & \text{if} & y = 0,\ b + \alpha a = 0,\ x - \beta b_1 = a_1. \end{array} \tag{2}$$

Next we calculate $(x, y)\,\phi^{-1}\sigma[L, (0, 0), (a, b)]\,\phi$. By the definition of ϕ,

$$(x, y)\,\phi^{-1} = \begin{cases} (x - a_1, y) & \text{if} \quad y \neq 0,\\ (x - a_1 - \beta b_1, 0) & \text{if} \quad y = 0. \end{cases}$$

So $(x, y)\,\phi^{-1}\sigma[L, (0,0), (a,b)]$ is

$$\begin{array}{ll}
(x-a_1+a,\ y+b) & \text{if}\quad y+b\neq 0,\ x-a_1\neq 0,\\
(x-a_1+a,\ y+b+\alpha a) & \text{if}\quad y+b+\alpha a\neq 0,\ x-a_1=0,\\
(x-a_1+a+\beta b,\ 0) & \text{if}\quad y+b=0,\ x-a_1\neq 0,\\
(x-a_1+a+\beta b,\ 0) & \text{if}\quad y+b+\alpha a=0,\ x-a_1=0,
\end{array}$$

when $y\neq 0$, whereas for $y=0$ it is

$$\begin{array}{ll}
(x-a_1-\beta b_1+a,\ b) & \text{if}\quad b\neq 0,\ x-a_1-\beta b_1\neq 0,\\
(x-a_1-\beta b_1+a,\ b+\alpha a) & \text{if}\quad b+\alpha a\neq 0,\ x-a_1-\beta b_1=0,\\
(x-a_1-\beta b_1+a,\ 0) & \text{if}\quad b=0,\ x-a_1-\beta b_1\neq 0,\\
(x-a_1-\beta b_1+a+\beta b,\ 0) & \text{if}\quad b+\alpha a=0,\ x-a_1-\beta b_1=0.
\end{array}$$

Finally, applying the permutation ϕ we obtain

$$(x, y)\,\phi^{-1}\sigma[L, (0,0), (a,b)]\,\phi$$
$$=\begin{cases}
(x+a,\ y+b) & \text{if}\quad y\neq 0,\ y+b\neq 0,\ x-a_1\neq 0,\\
(x+a,\ y+b+\alpha a) & \text{if}\quad y\neq 0,\ y+b+\alpha a\neq 0,\ x-a_1=0,\\
(x+a+\beta b+\beta b_1,\ 0) & \text{if}\quad y\neq 0,\ y+b=0,\ x-a_1\neq 0,\\
(x+a+\beta b+\beta b_1,\ 0) & \text{if}\quad y\neq 0,\ y+b+\alpha a=0,\ x-a_1=0,\\
(x+a-\beta b_1,\ b) & \text{if}\quad y=0,\ b\neq 0,\ x-a_1-\beta b_1\neq 0,\\
(x+a-\beta b_1,\ b+\alpha a) & \text{if}\quad y=0,\ b+\alpha a\neq 0,\ x-a_1-\beta b_1=0,\\
(x+a,\ 0) & \text{if}\quad y=0,\ b=0,\ x-a_1-\beta b_1\neq 0,\\
(x+a+\beta b,\ 0) & \text{if}\quad y=0,\ b+\alpha a=0,\ x-a_1-\beta b_1=0,
\end{cases}$$

which equals $(x, y)\,\sigma[L, (a_1, b_1), (a_2, b_2)]$ as calculated in (2). Hence the permutation induced by rows (a_1, b_1) and (a_2, b_2) will have the same cycle structure as the permutation induced by rows $(0, 0)$ and (a, b). ∎

We are now ready to prove our main result.

THEOREM 2.1. *$L(p, \alpha, \beta)$ is a pan-Hamiltonian Latin square if and only if $\alpha\beta$ is a quadratic non-residue in $\mathbb{Z}_p$.*

Proof. First note that L is a Latin square whenever $\alpha\beta$ is a quadratic non-residue in $\mathbb{Z}_p$, by Proposition 2.1. Also, by Proposition 2.2 we need only show that $\sigma[L, (0,0)(a,b)]$ consists of a p^2-cycle for all $(a,b)\in\mathbb{Z}_p\times\mathbb{Z}_p\setminus\{(0,0)\}$.

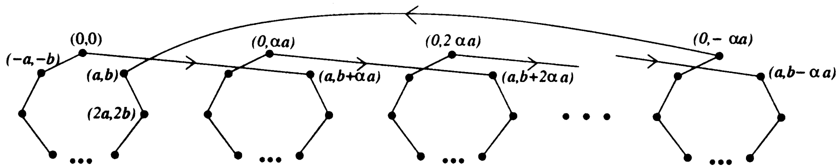

FIG. 2. $\sigma[L_1, (0, 0), (a, b)]$ for $a \neq 0$.

Now, $\sigma[L_0, (0, 0), (a, b)] = \{C_0, C_1, C_2, \ldots, C_{p-1}\}$ where for $i \in \mathbb{Z}_p$, C_i is the cycle

$$((0, i), (a, i+b), (2a, i+2b), \ldots, (-a, i-b)).$$

Given the construction of L_1 from L_0, we have

$$(x, y)\, \sigma[L_1, (0, 0), (a, b)] = \begin{cases} (x, y)\, \sigma[L_0, (0, 0), (a, b)] & \text{if} \quad x \neq 0, \\ (x, y)\, \sigma[L_0, (0, 0), (a, b)] + (0, \alpha a) & \text{if} \quad x = 0. \end{cases}$$

Thus, if $a = 0$ then $\sigma[L_1, (0, 0), (a, b)]$ consists of the p separate cycles $\{C_0, C_1, C_2, \ldots, C_{p-1}\}$ and otherwise it consists of the single p^2-cycle shown in Fig. 2.

By the construction of L from L_1 we know that $(x, y)\, \sigma[L, (0, 0), (a, b)] = (x, y)\, \sigma[L_1, (0, 0), (a, b)]$ unless $(x, y)\, \sigma[L_1, (0, 0), (a, b)] = (z, 0)$ where $z \in \mathbb{Z}_p$, in which case $(x, y)\, \sigma[L, (0, 0), (a, b)] = (x, y)\, \sigma[L_1, (0, 0), (a, b)] + (\beta b, 0)$. If $b = 0$ then $\sigma[L, (0, 0), (a, b)] = \sigma[L_1, (0, 0), (a, b)]$ and hence is a p^2-cycle as shown in Fig. 2.

Now suppose that $a = 0$ (and so $b \neq 0$). It is clear that $\sigma[L_1, (0, 0), (0, b)]$ consists of p p-cycles as shown in Fig. 3.

Thus when $a = 0$, $\sigma[L, (0, 0), (a, b)]$ consists of the p^2-cycle shown in Fig. 4.

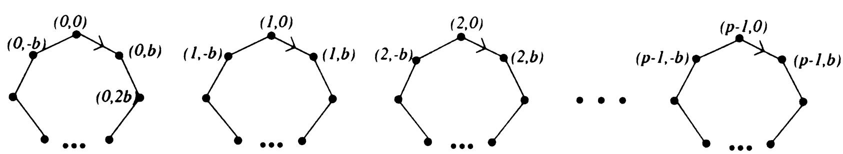

FIG. 3. $\sigma[L_1, (0, 0), (0, b)]$.

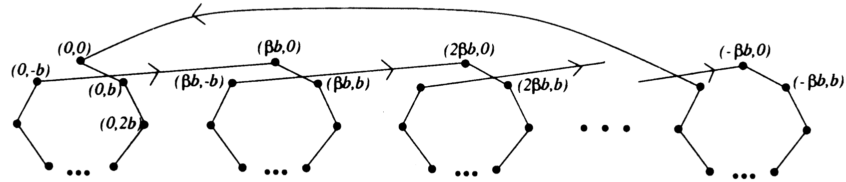

FIG. 4. $\sigma[L, (0,0), (a,b)]$.

Finally, suppose $a \neq 0$ and $b \neq 0$. We have $\sigma[L_1, (0,0), (a,b)]$ (which is given in Fig. 2) and thus need to consider those symbols $(ua, ub + v\alpha a)$ where $ub + v\alpha a = 0$. There are p such symbols given by the p solutions

$$(u, v) = (0,0), (-\alpha ab^{-1}, 1), (-2\alpha ab^{-1}, 2), \ldots, (\alpha ab^{-1}, p-1)$$

to the equation $ub + v\alpha a = 0$. In particular, note that for $i = 0, 1, 2, \ldots, p-1$ there is exactly one such symbol, namely $(-i\alpha a^2 b^{-1}, 0)$, occurring in the "path" $(a, b + i\alpha a), (2a, 2b + i\alpha a), \ldots, (0, i\alpha a)$ of $\sigma[L_1, (0,0), (a,b)]$, see Fig. 2. Thus, we can rewrite $\sigma[L_1, (0,0), (a,b)]$ as in Fig. 5 where those symbols which will have a different image under $\sigma[L, (0,0), (a,b)]$ than under $\sigma[L_1, (0,0), (a,b)]$ are $h_0, h_1, \ldots, h_{p-1}$. (Though unnecessary for the proof, it is straightforward to calculate that the number of symbols on each level is either k or $p+k$ where $k \in \{1, 2, \ldots, p-1\}$ and $k \equiv -\alpha ab^{-1} \bmod p$.)

Now, $h_i \sigma[L, (0,0), (a,b)] = (-(i+1)\alpha a^2 b^{-1} + \beta b, 0)$ and consequently $h_i \sigma[L, (0,0), (a,b)] = g_{i+1+\gamma}$ where $\gamma = -\alpha^{-1}\beta(a^{-1}b)^2$. So, in Fig. 5, the image of each h_i "shifts down γ levels" and, since p is prime, it is clear that a p^2-cycle results unless $\gamma = -1$. That is, unless $-\alpha^{-1}\beta(a^{-1}b)^2 = -1$ which

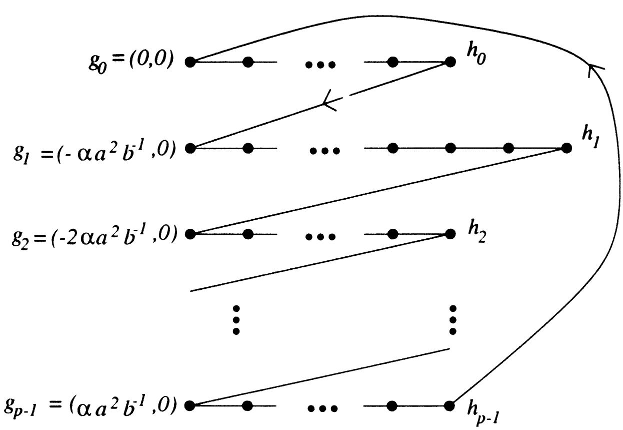

FIG. 5. $\sigma[L_1, (0,0), (a,b)]$ for $a \neq 0$.

is equivalent to $\alpha\beta = (a^{-1}b\beta)^2$. Hence, if $\alpha\beta$ is a quadratic non-residue in $\mathbb{Z}_p$, $L(p, \alpha, \beta)$ is pan-Hamiltonian.

Conversely, suppose that $\alpha\beta$ is a quadratic residue in $\mathbb{Z}_p$, say $\alpha\beta = q^2$. Then $\gamma = -1$ when $a^{-1}b = q\beta^{-1}$ so $\sigma[L, (0, 0), (a, b)]$ will consist of p separate cycles (of lengths k or $p+k$ where $k \in \{1, 2, ..., p-1\}$ and $k \equiv -\alpha ab^{-1} \bmod p$). So $L(p, \alpha, \beta)$ is pan-Hamiltonian if and only if $\alpha\beta$ is a quadratic non-residue in $\mathbb{Z}_p$. ∎

COROLLARY 2.1. *There exists a perfect 1-factorisation of $K_{n,n}$ if $n = p^2$ for some odd prime p.*

Proof. For any prime $p > 2$ there exists α, a quadratic non-residue modulo p. We simply apply Theorem 1.2 to $L(p, \alpha, 1)$, which is a pan-Hamiltonian Latin square of order p^2. ∎

3. ISOMORPHISM

For each prime p we have constructed $\frac{1}{2}(p-1)^2$ perfect 1-factorisations of K_{p^2,p^2}. It is natural to ask how many of these factorisations are essentially different. First, we formally define isomorphism. Let $\mathscr{F} = \{F_1, F_2, ..., F_n\}$ and $\mathscr{G} = \{G_1, G_2, ..., G_n\}$ be two 1-factorisations of $K = K_{n,n}$. Then $\mathscr{F}$ and $\mathscr{G}$ are *isomorphic* if and only if there exists an automorphism ϕ of K such that $\{F_1\phi, F_2\phi, ..., F_n\phi\} = \mathscr{G}$ where $F_i\phi$ denotes the set of all edges $\{x\phi, y\phi\}$ such that $\{x, y\} \in F_i$.

The concept of isomorphism between 1-factorisations is similar to the idea of isotopy between Latin squares. Two Latin squares are *isotopic* if each can be obtained from the other by permuting the rows, columns and symbols. Suppose that L and M are the pan-Hamiltonian Latin squares corresponding to the two perfect 1-factorisations $\mathscr{F}$ and $\mathscr{G}$ of K. Then L and M are isotopic if and only if there is an isomorphism between $\mathscr{F}$ and $\mathscr{G}$ which preserves the colours in a vertex 2-colouring of K. It is also possible to have an isomorphism between $\mathscr{F}$ and $\mathscr{G}$ which reverses the colours in the 2-colouring. In this case L will be isotopic to the so-called $(1, 3, 2)$-conjugate of M, which is the square derived from M by replacing each row by its inverse permutation.

PROPOSITION 3.1. *$L(p, \alpha_1, \beta_1)$ is isotopic to $L(p, \alpha_2, \beta_2)$ if $\alpha_1\beta_1 = \alpha_2\beta_2$.*

Proof. Since isotopy is an equivalence relation it suffices to exhibit an isotopy from $A = L(p, 1, \delta)$ to $B = L(p, \alpha, \delta\alpha^{-1})$ which works for any $\alpha, \delta \in \mathbb{Z}_p \setminus \{0\}$. Let θ be the group automorphism of $\mathbb{Z}_p \times \mathbb{Z}_p$ which sends (u, v) to $(u\alpha^{-1}, v)$. Let ϕ be the isotopy which applies θ to the row, column and symbol indices of a Latin square. If we show that $B = A\phi$ then the required result will follow.

Let $r = (a, b)$ and $s = (c, d)$ be general elements of $\mathbb{Z}_p \times \mathbb{Z}_p$. Then using equation (1) we get

$$A\phi(r\theta, s\theta) = \begin{cases} (a\alpha^{-1}, a+b+d) & \text{if } c=0 \text{ and } a+b+d \neq 0, \\ (\alpha^{-1}a+\delta\alpha^{-1}b, 0) & \text{if } c=0 \text{ and } a+b+d=0, \\ (\alpha^{-1}a+\alpha^{-1}c+\delta\alpha^{-1}b, 0) & \text{if } c\neq 0 \text{ and } b+d=0, \\ (\alpha^{-1}a+\alpha^{-1}c, b+d) & \text{if } c\neq 0 \text{ and } b+d\neq 0. \end{cases}$$

It is now an easy matter to confirm that $B = A\phi$. ∎

Applying Proposition 3.1 we see that $L(3, 1, 2)$ and $L(3, 2, 1)$ are isomorphic (it is easily checked that both are isomorphic to the 6th square in the catalogue given in [12]). Hence our family contains a unique perfect 1-factorisation of $K_{9,9}$ up to isomorphism. However, for $p \geqslant 5$ our family contains non-isomorphic factorisations, as we shall show after proving a preliminary result which is of some independent interest.

PROPOSITION 3.2. *$L(p, \alpha, \beta)$ is isotopic to its $(1, 3, 2)$-conjugate.*

Proof. By Proposition 3.1 it is sufficient for us to exhibit an isotopy from $A = L(p, 1, \beta)$ to the $(1, 3, 2)$-conjugate B of A for each $\beta \in \mathbb{Z}_p \setminus \{0\}$. Let θ_1 and θ_2 be permutations of $\mathbb{Z}_p \times \mathbb{Z}_p$ given by $(u, v)\,\theta_1 = (-v, -\beta^{-1}u)$ and $(u, v)\,\theta_2 = (v, \beta^{-1}u)$. Let ϕ be the isotopy which applies θ_1 to the rows and θ_2 to the columns and symbols of A. We claim that $A\phi = B$. Let (a_1, b_1) and (a_2, b_2) be general elements of $\mathbb{Z}_p \times \mathbb{Z}_p$ and suppose that (a_3, b_3) is the symbol (given by (1)) in row (a_1, b_1) and column (a_2, b_2) of A. It suffices to show that the symbol in row $(a_1, b_1)\,\theta_1 = (-b_1, -\beta^{-1}a_1)$ and column $(a_3, b_3)\,\theta_2 = (b_3, \beta^{-1}a_3)$ of A is $(a_2, b_2)\,\theta_2 = (b_2, \beta^{-1}a_2)$. By (1), there are four cases to consider.

In Case 1, $a_2 = 0$ and $a_1 + b_1 + b_2 \neq 0$, so $(a_3, b_3)\,\theta_2 = (a_1 + b_1 + b_2, \beta^{-1}a_1)$. By (1), the symbol in row $(-b_1, -\beta^{-1}a_1)$ and column $(a_1 + b_1 + b_2, \beta^{-1}a_1)$ is $(-b_1 + \beta(-\beta^{-1}a_1) + a_1 + b_1 + b_2, 0) = (b_2, \beta^{-1}a_2)$.

Case 2 is where $a_2 = a_1 + b_1 + b_2 = 0$, so $(a_3, b_3)\,\theta_2 = (0, \beta^{-1}(a_1 + \beta b_1)) = (0, \beta^{-1}a_1 + b_1)$. The symbol in row $(-b_1, -\beta^{-1}a_1)$ and column $(0, \beta^{-1}a_1 + b_1)$ is $(-b_1 + \beta(-\beta^{-1}a_1), 0) = (-b_1 - a_1, 0) = (b_2, 0) = (b_2, \beta^{-1}a_2)$.

Case 3 is where $a_2 \neq 0$ and $b_1 + b_2 = 0$, so $(a_3, b_3)\,\theta_2 = (0, \beta^{-1}(a_1 + \beta b_1 + a_2))$. The symbol occurring in row $(-b_1, -\beta^{-1}a_1)$ and column $(0, \beta^{-1}(a_1 + \beta b_1 + a_2))$ is $(-b_1, -b_1 - \beta^{-1}a_1 + \beta^{-1}(a_1 + \beta b_1 + a_2)) = (b_2, \beta^{-1}a_2)$.

Case 4 is where $a_2 \neq 0$ and $b_1 + b_2 \neq 0$, so $(a_3, b_3)\,\theta_2 = (b_1 + b_2, \beta^{-1}(a_1 + a_2))$. The symbol occurring in row $(-b_1, -\beta^{-1}a_1)$ and column $(b_1 + b_2, \beta^{-1}(a_1 + a_2))$ is $(-b_1 + b_1 + b_2, -\beta^{-1}a_1 + \beta^{-1}(a_1 + a_2)) = (b_2, \beta^{-1}a_2)$. ▮

Exactly 10 of the 37 perfect 1-factorisations of $K_{9,9}$ have an automorphism which interchanges the two bipartite parts (see [12]), which is equivalent to the symmetry involved in Proposition 3.2. A strictly stronger symmetry is that a Latin square may be isotopic to some square M such that M equals the (1,3,2)-conjugate of M. This stronger symmetry is precisely the condition under which a pan-Hamiltonian Latin square of order n can be derived from a perfect 1-factorisation of K_{n+1} using the process behind Theorem 1.1. Just one of the 37 perfect 1-factorisations of $K_{9,9}$ obeys this condition, and it is not the one derived from $L(3, 1, 2)$. We leave open the possibility that some members of the family we have constructed could yield perfect 1-factorisations of complete graphs, but note that we used a computer to rule out this happening for any $n = p^2 < 1000$.

Our next result, when combined with Theorem 2.1, shows that our family contains $(p-1)/2$ essentially different perfect 1-factorisations for each prime p.

THEOREM 3.1. *$L(p, \alpha_1, \beta_1)$ is never isotopic to $L(p, \alpha_2, \beta_2)$ except when $\alpha_1\beta_1 = \alpha_2\beta_2$.*

Proof. By Proposition 3.1 it is sufficient to treat the case $\alpha_1 = \alpha_2 = 1$. Suppose, contrary to our goal, that there is an isotopy ϕ from $A = L(p, 1, \beta_1)$ to $B = L(p, 1, \beta_2)$ where $1 < \beta_1 < \beta_2 < p$. Since ϕ is an isotopy, it maps each column cycle in A to a column cycle in B of the same length. We describe a pair of columns as being of *type i* (for $i = 0, 1, 2$) if i of the columns in the pair have first coordinate 0. We call a cycle *uninteresting* if it has length congruent to 0 modulo p, otherwise it is *interesting*. Our proof of non-isomorphism will be based on the lengths of interesting cycles in column pairs of type 1. The symbols with second coordinate zero, $\Omega = \mathbb{Z}_p \times \{0\}$, will play a special role.

Both A and B are constructed in two stages, starting from $L_0(p)$ in which every column cycle has length p. Unless a cycle is altered by both stages it will have length either p or p^2, by the same mechanism as operates in Fig. 2. It follows that all cycles in type 0 column pairs are uninteresting, as are all the cycles in type 1 pairs except possibly those involving a symbol in Ω. We look now at the interesting cycles in type 1 column pairs in a square $L(p, 1, \beta)$ where we add the restriction $\beta \neq 2$.

We consider those cycles between columns $(0, c_0)$ and (c_1, c_2) which contain symbols from Ω. Note that $c_1 \neq 0$ since we are interested in type 1 pairs. Let $\mu = c_1(\beta - 2)^{-1}$, let $\lambda = (\beta - 1)(\mu - c_2) - c_0$ and let τ be the index of $(\beta-1)^{-1}$ in $\mathbb{Z}_p$. Denote by $\Gamma_k=\Gamma_k((0,c_0),(c_1,c_2))$ the cycle involving the columns $(0,c_0)$ and (c_1,c_2) which contains the symbol $(\lambda+k,0)$. We now trace Γ_k, starting at the symbol $(\lambda+k,0)$ and ending when the next symbol from Ω is reached. By (1) the symbol $(\lambda+k,0)=((\beta-1)(\mu-c_2)-c_0+k,0)$ in column $(0,c_0)$ lies in row $(-\mu+c_2-c_0-\frac{k}{\beta-1},\mu-c_2+\frac{k}{\beta-1})$. We use (1) to follow the cycle, alternately calculating the next symbol and row. The symbol in column (c_1,c_2) of row $(-\mu+c_2-c_0-\frac{k}{\beta-1},\mu-c_2+\frac{k}{\beta-1})$ is $((\beta-3)\mu+c_2-c_0-\frac{k}{\beta-1},\mu+\frac{k}{\beta-1})$.

Next for $i=2,3,\ldots,\frac{\beta}{\beta-2}+\frac{2k}{c_1(\beta-1)}-1$ we encounter a pair of entries in row

$$\left(((i-1)\beta-(2i-1))\mu+c_2-c_0-\frac{k}{\beta-1},\left(i^2-\frac{i(i-1)}{2}\beta\right)\mu-c_2+\frac{ik}{\beta-1}\right)$$

consisting of the symbol

$$\left(((i-1)\beta-(2i-1))\mu+c_2-c_0-\frac{k}{\beta-1},\right.$$
$$\left.\left((i-1)^2-\frac{(i-1)(i-2)}{2}\beta\right)\mu+\frac{(i-1)k}{\beta-1}\right)$$

in column $(0,c_0)$, and its partner in column (c_1,c_2) which is the symbol

$$\left((i\beta-(2i-1))\mu+c_2-c_0-\frac{k}{\beta-1},\left(i^2-\frac{i(i-1)}{2}\beta\right)\mu+\frac{ik}{\beta-1}\right).$$

This pattern continues until we reach a row with second coordinate $-c_2$, since that is the only way to produce a symbol from Ω in column (c_1,c_2). This occurs when $i=\frac{\beta}{\beta-2}+\frac{2k}{c_1(\beta-1)}$, and the row is $(\mu+c_2-c_0+\frac{k}{\beta-1},-c_2)$. The symbol in column $(0,c_0)$ of this row is $(\mu+c_2-c_0+\frac{k}{\beta-1},\mu+\frac{k}{\beta-1})$, and its companion in column (c_1,c_2) is the symbol

$$\left((\beta-1)(\mu-c_2)-c_0+\frac{k}{\beta-1},0\right)=\left(\lambda+\frac{k}{\beta-1},0\right)\in\Omega.$$

So starting at the symbol $(\lambda+k,0)$, the next symbol we encounter from Ω is $(\lambda+k(\beta-1)^{-1},0)$. We note that Γ_0 contains a unique symbol, $(\lambda,0)$, from Ω and has length $\beta/(\beta-2)$ mod p.

If $k\neq 0$, then Γ_k contains more than one symbol from Ω. These symbols appear in the order

$$(\lambda+k,0),\ \left(\lambda+\frac{k}{\beta-1},0\right),\ \left(\lambda+\frac{k}{(\beta-1)^2},0\right),\ \ldots\left(\lambda+\frac{k}{(\beta-1)^{\tau-1}},0\right),$$

at which point we return to $(\lambda+k,0)$.

Hence all the cycles Γ_k, except for Γ_0, contain the same number τ of symbols from Ω. The length of each cycle Γ_k, $k \neq 0$, is

$$\frac{\tau\beta}{\beta-2}+\frac{2k}{c_1}\left(1+\frac{1}{\beta-1}+\frac{1}{(\beta-1)^2}+\cdots+\frac{1}{(\beta-1)^{\tau-1}}\right)$$

which is congruent to $\tau\beta/(\beta-2) \bmod p$ since $\sum_{i=0}^{\tau-1}(\beta-1)^{-i} \equiv 0 \bmod p$, using the definition of τ. Note that the number of distinct cycles Γ_k is $1+(p-1)/\tau$ and each Γ_k is interesting.

In the case $\beta = 2$, a similar calculation to the above shows that, starting at a symbol $(k, 0)$, there are $2(k+c_0+c_2)/c_1$ symbols not in Ω before the symbol $(k+c_1, 0) \in \Omega$ is encountered. As $c_1 \neq 0$, it follows that all the symbols in Ω occur in the same cycle, Γ_0. Note that all the other column cycles on the same pair of columns are uninteresting, which implies that Γ_0 is also uninteresting.

We are now ready to study how the interesting cycles in A might be mapped by ϕ to the interesting cycles in B. Since type 1 column pairs in B have interesting cycles we conclude that no type 0 column pair from A maps to a type 1 column pair in B. Also, in each of A and B the number of type 2 pairs, $\binom{p}{2}$, is lower than the number of type 1 pairs, $p(p^2-p)$. We conclude that there must be a type 1 pair, π_A, in A which gets mapped by ϕ to a type 1 pair, π_B, in B. However this is impossible, as we shall now argue, because the lengths of interesting cycles in π_A cannot match the lengths of interesting cycles in π_B.

Firstly we rule out the possibility that $\beta_1 = 2$ since then there would be no interesting cycles in π_A, while there are some in π_B. Thus we may assume $2 < \beta_1 < \beta_2 < p$. Let $v_1 \equiv \beta_1/(\beta_1-2) \bmod p$ be in the range $2 \leqslant v_1 < p-1$ (noting that $v_1 = p-1$ would correspond to $\beta_1 = 1$) and let v_1' be v_1 times the index of $(\beta_1-1)^{-1}$ in $\mathbb{Z}_p$. Define v_2 and v_2' similarly, using β_2 in place of β_1. In what follows it will be crucial that $v_1 \neq v_2$, $v_1 \not\equiv v_1'$ and $v_2 \not\equiv v_2' \bmod p$; all of which are consequences of $2 < \beta_1 < \beta_2 < p$.

From our study of the cycles Γ_k, we see that there is a special cycle, $\Gamma_0(\pi_A)$ of length v_1, which contains a unique symbol $(x, 0) \in \Omega$. The other cycles $\Gamma_1(\pi_A), \Gamma_2(\pi_A), \ldots, \Gamma_{p-1}(\pi_A)$ have lengths congruent to $v_1' \bmod p$. The situation in π_B is similar. Given that $v_1 \neq v_2$, the only way the cycle lengths can be matched up is if $\Gamma_1(\pi_A), \Gamma_2(\pi_A), \ldots, \Gamma_{p-1}(\pi_A)$ coincide in a single cycle Γ, and this cycle maps under ϕ to $\Gamma_0(\pi_B)$, meaning that it has length v_2. However Γ must contain the symbol $(i, 0)$ for each $i \in \mathbb{Z}_p \setminus \{x\}$ so it has length at least $p-1$. But $v_2 < p-1$ so in fact ϕ cannot exist and the theorem is proved. ∎

In light of Proposition 3.1, Proposition 3.2 and the discussion at the beginning of this section, Theorem 3.1 has the following consequence.

COROLLARY 3.1. *The 1-factorisation of K_{p^2,p^2} corresponding to $L(p, \alpha_1, \beta_1)$ is isomorphic to the 1-factorisation corresponding to $L(p, \alpha_2, \beta_2)$ if and only if $\alpha_1\beta_1 = \alpha_2\beta_2$.*

4. SUMMARY

We have constructed a family $\{L(p, \alpha, \beta): \alpha, \beta \in \mathbb{Z}_p, \alpha\beta \notin \{0, 1\}\}$ of Latin squares which encode 1-factorisations of K_{p^2,p^2} where p is any odd prime. The factorisations are perfect whenever $\alpha\beta$ is not a quadratic residue in $\mathbb{Z}_p$. Also, the factorisation derived from $L(p, \alpha, \beta)$ is isomorphic to the factorisation derived from $L(p, \alpha', \beta')$ if and only if $\alpha\beta = \alpha'\beta'$. Given these results it is apparent that our family of 1-factorisations is essentially a one parameter family, with the product $\alpha\beta$ being the important quantity. As there are $(p-1)/2$ quadratic non-residues in $\mathbb{Z}_p$, we have the same number of non-isomorphic perfect 1-factorisations of K_{p^2,p^2} in our family.

The significance of our results in terms of Latin squares is that we have constructed $(p-1)/2$ non-isotopic pan-Hamiltonian Latin squares of order p^2. Each of these squares is devoid of subsquares and is isotopic to its (1,3,2)-conjugate.

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