The mathematics behind sudoku

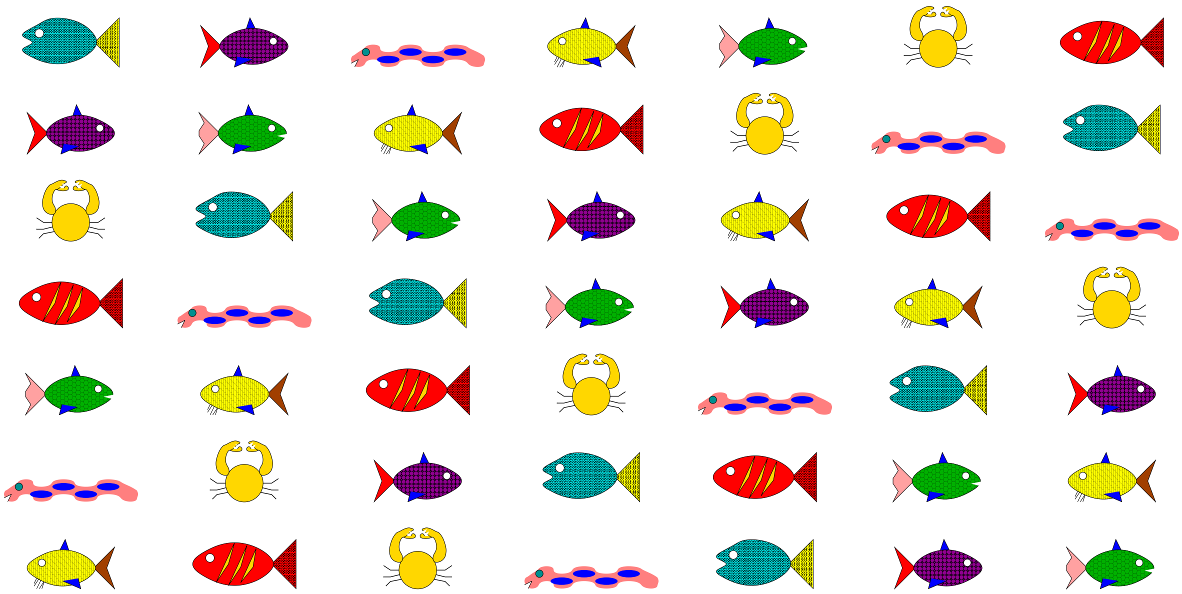
5	6	2	7	9	4	1	8	3
4	3	8	1	5	2	7	6	9
9	1	7	6	3	8	5	2	4
6	2	9	3	1	7	8	4	5
3	7	4	5	8	9	2	1	6
8	5	1	4	2	6	9	3	7
2	8	5	9	4	3	6	7	1
1	4	6	8	7	5	3	9	2
7	9	3	2	6	1	4	5	8

Latin squares

A *latin square* is a matrix in which each symbol occurs exactly once in each row and once in each column.

e.g.
$$\begin{array}{|cccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$
is a latin square.

A fishy latin square of order 7



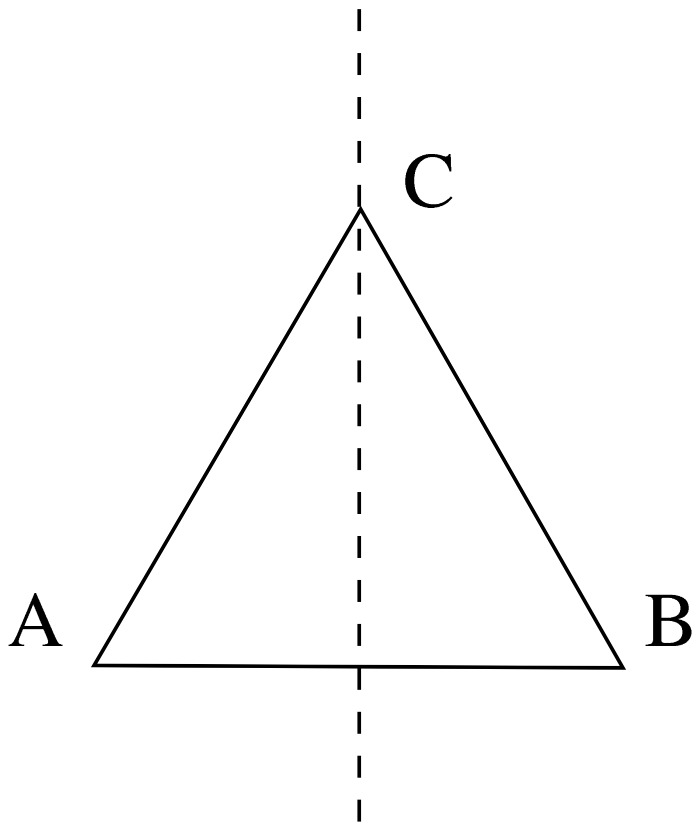
Sudoku

5	6	2	7	9	4	1	8	3
4	3	8	1	5	2	7	6	9
9	1	7	6	3	8	5	2	4
6	2	9	3	1	7	8	4	5
3	7	4	5	8	9	2	1	6
8	5	1	4	2	6	9	3	7
2	8	5	9	4	3	6	7	1
1	4	6	8	7	5	3	9	2
7	9	3	2	6	1	4	5	8

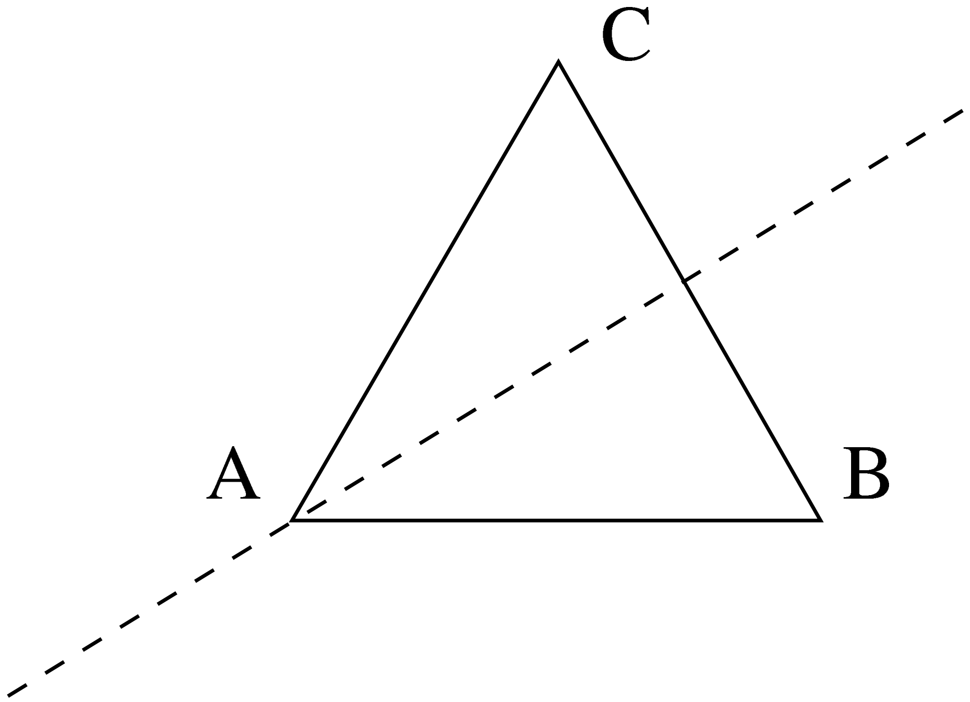
2010 World Cup, Group H

	Honduras	Chile	Spain	Switzerland
Honduras	–	16 June	21 June	25 June
Chile	16 June	–	25 June	21 June
Spain	21 June	25 June	–	16 June
Switzerland	25 June	21 June	16 June	–

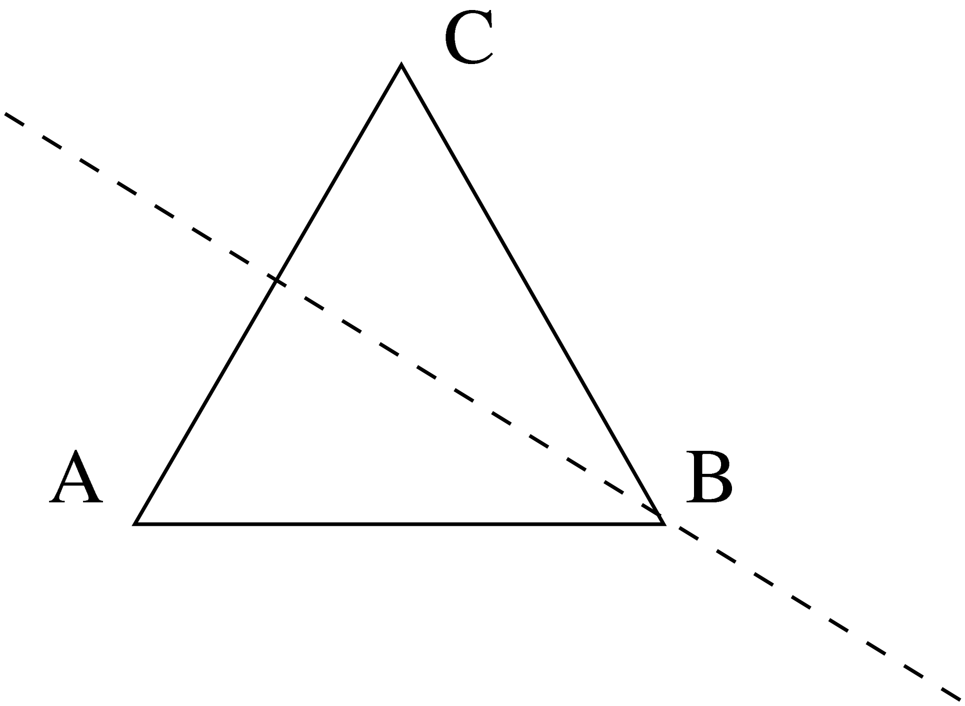
Symmetry of a triangle



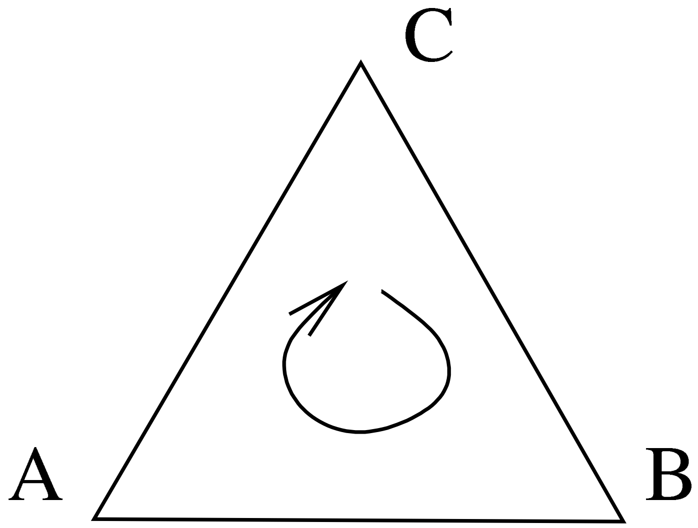
Symmetry of a triangle



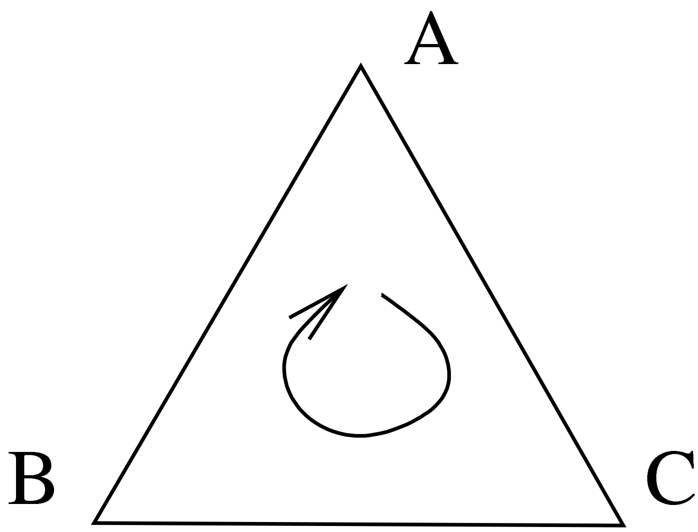
Symmetry of a triangle



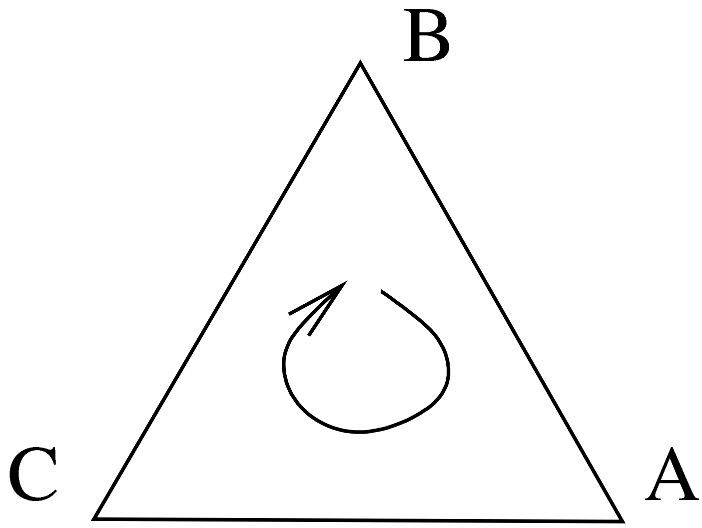
Symmetry of a triangle



Symmetry of a triangle



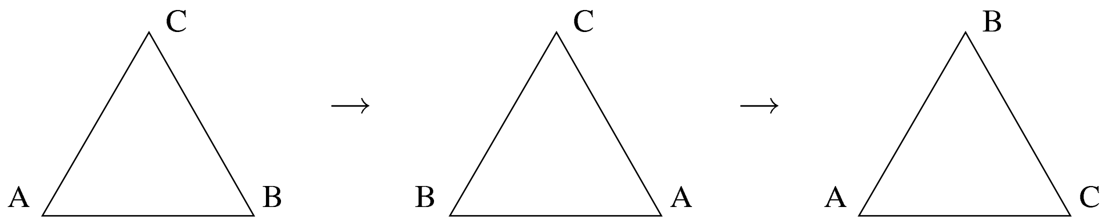
Symmetry of a triangle



Combining operations

All up, there are 6 different operations. 3 reflections ($|$, $/$, $\backslash$) and 3 rotations (0°, 120°, 240°).

If I combine any two of these the net effect is equivalent to one single operation.



For example, if I reflect in $|$ followed by rotation by 120°, the overall effect is reflection in $/$.

The combinations

Now if I construct a table of all combinations...

0°	120°	240°	$\vert$	$/$	$\backslash$
120°	240°	0°	$\backslash$	$\vert$	$/$
240°	0°	120°	$/$	$\backslash$	$\vert$
$\vert$	$/$	$\backslash$	0°	120°	240°
$/$	$\backslash$	$\vert$	240°	0°	120°
$\backslash$	$\vert$	$/$	120°	240°	0°

Then the table is a latin square!

Multiplication

Suppose I define a new kind of multiplication where I multiply numbers in the usual way except I divide the answer by 5 and just keep the remainder.

Example: $3 \times 4 = 12 = 2 \times 5 + 2$ so with my new rule $3 \times 4 = 2$.

If I construct a table as before, I find:

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

And again, we find a latin square.

How many latin squares are there?

Let L_n be the number of latin squares of size n.

There is no known useful formula for L_n.

Discoverers of the known values of L_n

$n \leq 5$	Euler (1779)
$n = 6$	Clausen (1842)
$n = 7$	Sade (1948)
$n = 8$	Wells (1967)
$n = 9$	Bammel and Rothstein (1975)
$n = 10$	McKay and Rogoyski (1995)
$n = 11$	McKay and Wanless (2005)

The number of latin squares of size 11

$$L_{11} = 776966836171770144107444346734230682311065600000$$

Known values of R_n

A latin square is *reduced* if its first row and column are in natural order.

R_n is the number of reduced latin squares.

n	R_n
2	1
3	1
4	4
5	56
6	9408
7	16942080
8	535281401856
9	377597570964258816
10	7580721483160132811489280
11	5363937773277371298119673540771840

Factorisations of R_n

n	R_n
2	1
3	1
4	2^2
5	$2^3 \times 7$
6	$2^6 \times 3 \times 7^2$
7	$2^{10} \times 3 \times 5 \times 1103$
8	$2^{17} \times 3 \times 1361291$
9	$2^{21} \times 3^2 \times 5231 \times 3824477$
10	$2^{28} \times 3^2 \times 5 \times 31 \times 37 \times 547135293937$
11	$2^{35} \times 3^4 \times 5 \times 2801 \times 2206499 \times 62368028479$

Unsolved puzzle: Why are these numbers divisible by such a high power of 2?

An unsolved Sudoku puzzle

·	·	·	·	·	·	·	1	·
4	·	·	·	·	·	·	·	·
·	2	·	·	·	·	·	·	·
·	·	·	·	5	·	4	·	7
·	·	8	·	·	·	3	·	·
·	·	1	·	9	·	·	·	·
3	·	·	4	·	·	2	·	·
·	5	·	1	·	·	·	·	·
·	·	·	8	·	6	·	·	·

The above suduko has only 17 clues.

Nobody knows if it is possible to have fewer!!!

More general problem: How few entries can you specify in a latin square and still ensure that there is only one way to complete it?

The 16 card trick

Take the picture cards (aces, kings, queens & jacks) from a standard pack and arrange them in a 4×4 array so that each row and column contains one card of each suit and one card of each rank.

There are 6912 ways to do the puzzle,
but 20922789881088 ways to fail to do it.
Here's one

$\spadesuit A$	$\heartsuit K$	$\diamondsuit J$	$\clubsuit Q$
$\heartsuit Q$	$\spadesuit J$	$\clubsuit K$	$\diamondsuit A$
$\clubsuit J$	$\diamondsuit Q$	$\heartsuit A$	$\spadesuit K$
$\diamondsuit K$	$\clubsuit A$	$\spadesuit Q$	$\heartsuit J$

Each solution is the superposition of two latin squares

♠	♡	♢	♣
♡	♠	♣	♢
♣	♢	♡	♠
♢	♣	♠	♡

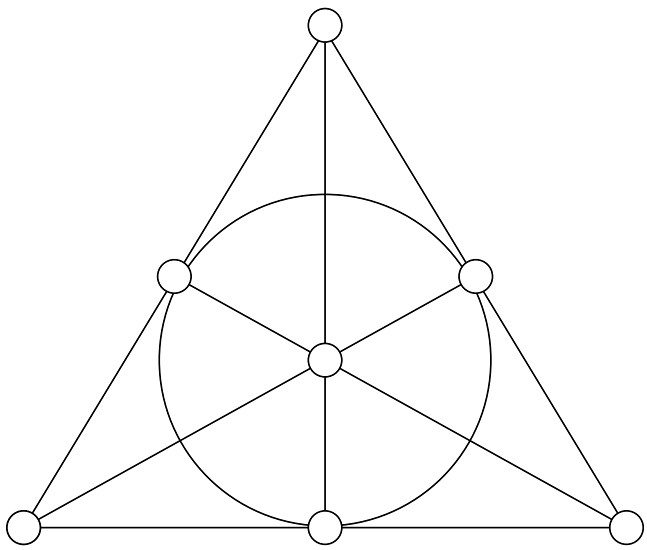
A	K	J	Q
Q	J	K	A
J	Q	A	K
K	A	Q	J

These squares have a special property – they are called *orthogonal mates*.

When we overlay them each ordered pair of symbols occurs once.

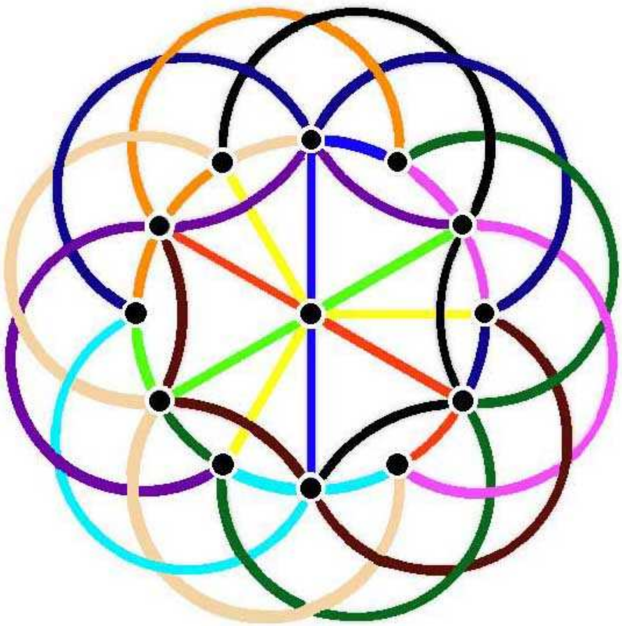
Uses of orthogonal latin squares

Using orthogonal latin squares you can build designs for statistical experiments, codes for communication, ...and projective planes.



7 lines and 7 points
Every pair of lines meets at a point.
Every pair of points is on a line.

Another projective plane



Magic squares

0	2	1	3
1	3	0	2
3	1	2	0
2	0	3	1

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

0	9	6	15
7	14	1	8
13	4	11	2
10	3	12	5

Tudoku?

5I			1A			8D		
	B				9D		4C	
1		7D		6C				9A
F		9	H	7	B	6		E
6		B	2		5	I		1
D		1	F	3	C	9		B
9C				1H		3E		6
	8H		9E				I	
		4G			6I			2H

Solution to tudoku

5I	9F	2C	1A	4B	7H	8D	6E	3G
8A	6B	3H	5G	2E	9D	1F	4C	7I
1G	4E	7D	8I	6C	3F	2B	5H	9A
2F	5C	9I	4H	7A	1B	6G	3D	8E
6H	3A	8B	2D	9G	5E	4I	7F	1C
4D	7G	1E	6F	3I	8C	9H	2A	5B
9C	2I	5F	7B	1H	4A	3E	8G	6D
3B	8H	6A	9E	5D	2G	7C	1I	4F
7E	1D	4G	3C	8F	6I	5A	9B	2H

For some puzzles to try, or a copy of this talk, visit:
http://users.monash.edu.au/∼iwanless/openday.html