Classification of some 3-subgroups of the finite groups of Lie type E_6

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Abstract

We consider the finite exceptional group of Lie type $G = E_6^\varepsilon(q)$ (universal version) with $3 \mid q - \varepsilon$, where $E_6^{+1}(q) = E_6(q)$ and $E_6^{-1}(q) = {}^2E_6(q)$. We classify, up to conjugacy, all maximal-proper 3-local subgroups of G, that is, all 3-local $M < G$ which are maximal with respect to inclusion among all proper subgroups of G which are 3-local. To this end, we also determine, up to conjugacy, all elementary-abelian 3-subgroups containing $Z(G)$, all extraspecial subgroups containing $Z(G)$, and all cyclic groups of order 9 containing $Z(G)$. These classifications are an important first step towards a classification of the 3-radical subgroups of G, which play a crucial role in many open conjectures in modular representation theory.

Errata for published version https://doi.org/10.1016/j.jpaa.2018.02.018; the corrected text is in red; current corrections are:

- In Table II, the groups E'_{15} and E''_{15} were missing.
- References to [5] have been corrected.

Keywords: maximal local subgroups, finite groups of Lie type, exceptional type

1. Introduction

This paper is the sequel to a series of papers which investigates the p-radical subgroups of the finite exceptional groups of Lie type, see [2, 3] for two recent studies. A subgroup $R \leq G$ is p-radical if it is the largest normal p-subgroup in $N_G(R)$, that is, $R = O_p(N_G(R))$. Radical subgroups play an important role in many of the central open conjectures in modular representation theory, for example, in the inductive versions of the Dade, McKay, or Alperin-Weight Conjectures. To keep this exposition short and to avoid repetition, we refer to [2, 3] for a more detailed discussion of recent progress, applications, and many references.

In [3] we have classified all radical 3-subgroups of $G = E_6^\varepsilon(q)$ with $3 \mid q + \varepsilon$. Our approach to that classification was to first determine, up to conjugacy, all elementary abelian 3-subgroups of G, and, subsequently, all maximal 3-local subgroups $M < G$. (Recall that $M < G$ is **maximal p-local** if M is maximal with respect to inclusion among all p-local subgroups of G.) Then we determined the 3-radical

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[1] An was supported by the Marsden Fund (of New Zealand), via award number UOA 1626.

[2] Dietrich was supported by an ARC DECRA (Australia), project DE140100088; he thanks the University of Auckland for their hospitality during research visits in 2016 and 2017.

[3] Huang was supported by the Ministry of Science and Technology (project MOST 106-2115-M-006-012), Taiwan.

subgroups of each such M, and eventually considered their G-fusion. The aim of the present paper is to consider $G = E_6^{\varepsilon}(q)$ with $3 \mid q - \varepsilon$. In this case, G has a center of order 3, hence the only maximal 3-local subgroup of G is $Z = Z(G)$. Thus, this case requires a modified approach. Analogous to our work in [2], we proceed as follows: We say that $M < G$ is **maximal-proper p-local** if M is p-local and maximal with respect to inclusion among all proper subgroups of G which are p-local. Clearly, if $O_p(G) = 1$, then the maximal-proper p-local subgroups are exactly the maximal p-local subgroups. If $R \leq G$ is p-radical and $O_p(G) < R$, then $N_G(R)$ is p-local and $N_G(R) \leq N_G(C)$ for every characteristic subgroup $C \leq R$. In particular, $N_G(R)$ is contained in some maximal-proper p-local $M \leq G$, so that $N_G(R) = N_M(R)$ and R is p-radical in M . Hence, every radical p-subgroup of G is radical in some maximal-proper p-local subgroup of G. The main results of this paper are summarised in Theorem 1.1. This theorem is a first important step towards a classification of radical 3-subgroups of G; the latter appears in [4].

Theorem 1.1. *Let $G = E_6^{\varepsilon}(q)$ with $3 \mid q - \varepsilon$; let $Z = Z(G)$ be the center of G. Up to conjugacy in G, the classification of all ...*

a) *... elementary abelian 3-subgroups $E \leq G$ with $Z < E$ is given in Proposition 3.4.*

b) *... cyclic subgroups $E \leq G$ of order 9 with $Z < E$ is given in Proposition 4.1.*

c) *... extraspecial 3-subgroups $E \leq G$ with $Z < E$ is given in Proposition 5.5.*

d) *... maximal-proper 3-local subgroups $M < G$ is given in Theorem 6.1.*

We note that Cohen et al. [8] classified local maximal subgroups of exceptional groups of Lie type. However, not every maximal-proper p-local subgroup is local maximal, and, in recent studies, the details obtained in the classification of maximal-proper p-local subgroups have been proven to be very useful for the determination of the radical subgroups.

2. Notation and known results

Our notation for finite simple groups and group extensions is as in [2, 3] and follows [9, 16]. If not indicated by brackets, then group extensions $A.B.C$ are read from the left, that is, $A.B.C = (A.B).C$. If n, m are positive integers, then n^m denotes the direct product of m copies of cyclic groups of order n. This notation is ambiguous if n is written as a power itself; there are only a few cases where this occurs, but the meaning should always follow from the context. Recall the notation $\mathrm{SL}_n^{\varepsilon}(q)$ and $\mathrm{GL}_n^{\varepsilon}(q)$: if $\varepsilon = 1$, then these are the special linear and general linear groups of degree n over the field $\mathbb{F}_q$ with q elements; if $\varepsilon = -1$, then these are the corresponding special unitary and unitary group, respectively, defined over $\mathbb{F}_{q^2}$.

For a prime p and an integer $n \neq 0$ we denote by n_p the largest p-power dividing n. Let H be a finite group. We denote by $O_p(H)$ the largest normal p-subgroup of H, and, if H is a finite p-group, then $\Omega_1(H)$ is the subgroup generated by elements of order p. If $A, B \leq H$, then we write $A \leq_H B$ whenever there exists $x \in H$ with $A^x \leq B$. Analogously, $A =_H B$ and $y \in_H B$ with $y \in H$ are defined. If $K \leq Z(A) \cap Z(B)$, then $A \circ_K B$ is the central product of A and B over K. We denote by $\mathcal{R}_p(H)$ the set of all p-radical subgroups of H and write $\mathrm{Out}_H(A) = N_H(A)/AC_H(A)$.

Let $\overline{G}$ be a simple algebraic group, defined over an algebraically closed field $\overline{F}$ of positive characteristic p. All encountered algebraic subgroups of $\overline{G}$ are closed, and all homomorphisms we encounter between algebraic groups are morphisms of varieties. We denote by $\overline{G}^{\circ}$ the connected component of the identity element. Let $\overline{T}$ be a fixed maximal torus of $\overline{G}$, and define the Weyl group of $\overline{G}$ as $W = N_{\overline{G}}(\overline{T})/\overline{T}$; this does not depend on the choice of $\overline{T}$ since all maximal tori in a linear algebraic group are conjugate [18, Corollary 6.5]. For a positive integer m, let $\overline{T}_m \leq \overline{G}$ be a torus of rank m, if it exists. By a *Steinberg morphism* of $\overline{G}$ we mean an endomorphism σ whose fixed-point set, denoted $C_{\overline{G}}(\sigma)$ or $(\overline{G})^{\sigma}$, is finite. If $\overline{G}$ is defined over $\mathbb{F}_q$, then the q-power map $\overline{F} \to \overline{F}$ induces a Steinberg morphism on $\overline{G}$, which we call a *(standard) Frobenius morphism*. Since $\overline{G}$ is simple, every endomorphism of $\overline{G}$ is either an automorphism of algebraic groups or a Steinberg morphism, and the latter occurs if and only if some power of the endomorphism is a Frobenius morphism (cf. [18, Theorem 21.5]). Let E be an elementary abelian subgroup of $\overline{G}$ consisting of semisimple elements. Using [14, (2.13)(iii)], we can assume that E is contained in the normaliser $N_{\overline{G}}(\overline{T})$ of some maximal torus $\overline{T}$ of $\overline{G}$. So E is **toral** if $E \leq \overline{T}$, and **non-toral** if E has nontrivial image in $W = N_{\overline{G}}(\overline{T})/\overline{T}$, that is, if $1 \neq E\overline{T}/\overline{T} \leq W$.

2.1. Local structure, from algebraic groups to finite groups.

We recall a few important results from the forthcoming paper [5].

Proposition 2.1. ([5, Proposition 5.1]) *Let $\overline{G}$ be a simple algebraic group, with maximal torus $\overline{T}$ and Weyl group W. If $A, B \leq \overline{T}$ are finite subgroups, then the following hold.*

a) *If $A = B^g$ with $g \in \overline{G}$, then $g = vc$ for some $v \in N_{\overline{G}}(\overline{T})$ and $c \in C_{\overline{G}}(A)^\circ$; in particular, A and B are conjugate in $N_{\overline{G}}(\overline{T})$.*

b) *We can decompose $N_{\overline{G}}(A) \cong C_{\overline{G}}(A)^\circ.(C_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ).(N_{\overline{G}}(A)/C_{\overline{G}}(A))$, with isomorphisms*

$$C_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ \cong C_W(A)/W(C_{\overline{G}}(A)^\circ) \quad \text{and} \quad N_{\overline{G}}(A)/C_{\overline{G}}(A) \cong N_W(A)/C_W(A),$$

where $W(C_{\overline{G}}(A)^\circ)$ is the Weyl group of the reductive group $C_{\overline{G}}(A)^\circ$.

Remark 2.2. In [5], based on Proposition 2.1, an algorithm is described to classify, up to conjugacy, all toral elementary abelian subgroups of $\overline{G}$; this algorithm is implement for the computer algebra system Magma [7] and also allows us to compute $C_{\overline{G}}(E)^\circ$, $C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ$, and $N_{\overline{G}}(E)/C_{\overline{G}}(E)$ for each such toral E. If $\overline{G}$ is simply-connected and exceptional, then, based on the classification of maximal non-toral elementary abelian subgroups of $\overline{G}$ described in [14], the paper [5] also classifies the non-toral elementary abelian subgroups of $\overline{G}$, up to conjugacy.

Proposition 2.3. ([5, Propositions 4.1 & 4.3 & 4.4]) *If $A \leq (\overline{G})^\sigma$ is an abelian subgroup of order coprime to the characteristic of $\overline{F}$, then $N_{\overline{G}}(A)^\circ = C_{\overline{G}}(A)^\circ$ and the following hold.*

a) *There is a 1–1 correspondence between the $(\overline{G})^\sigma$-classes of subgroups of $(\overline{G})^\sigma$ which are $\overline{G}$-conjugate to A, and the σ-classes in $N_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ$ contained in $C_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ$; here $w, y \in C_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ$ are σ-conjugate if $w = xy\sigma(x)^{-1}$ for some $x \in N_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ$. More precisely, the σ-class of $w \in C_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ$ corresponds to the $(\overline{G})^\sigma$-class of subgroups with representative $A_w = {}^gA$, where $g \in \overline{G}$ is chosen with $g^{-1}\sigma(g)C_{\overline{G}}(A)^\circ = w$.*

b) *Let $A_w \leq (\overline{G})^\sigma$ be the $\overline{G}$-conjugate of A as in a). If $\dot{w} \in C_{\overline{G}}(A)$ is any lift of w, then*

$$(C_{\overline{G}}(A_w)^\circ)^\sigma \cong (C_{\overline{G}}(A)^\circ)^{\dot{w}\sigma}$$

where

$$(C_{\overline{G}}(A)^\circ)^{\dot{w}\sigma} = \{c \in C_{\overline{G}}(A)^\circ \mid c = \dot{w}\sigma(c)\dot{w}^{-1}\}.$$

Furthermore, $(C_{\overline{G}}(A)^\circ)^{\dot{w}\sigma}$ is independent of the choice of lift $\dot{w}$. In particular, if w acts as an inner automorphism of $C_{\overline{G}}(A)^\circ$ then $(C_{\overline{G}}(A_w)^\circ)^\sigma \cong (C_{\overline{G}}(A)^\circ)^\sigma$.

c) *If A_w is as in a) and $w\sigma$ is identified with the map $x \mapsto w\sigma(x)w^{-1}$, then*

$$(N_{\overline{G}}(A_w))^\sigma/(C_{\overline{G}}(A_w)^\circ)^\sigma \cong (N_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ)^{w\sigma}, \tag{2.1}$$

$$C_{\overline{G}}(A_w)^\sigma/(C_{\overline{G}}(A_w)^\circ)^\sigma \cong (C_{\overline{G}}(A)/C_{\overline{G}}(A)^\circ)^{w\sigma}. \tag{2.2}$$

2.2. Maximal-proper p-local subgroups

In the following, let G be a finite group and p a prime. A subgroup $M \leq G$ is maximal-proper p-local if M is a proper p-local subgroup (that is, $M = N_G(P) < G$ for some p-subgroup $P \leq G$) and M is not properly contained in any proper p-local subgroup of G. We need some results on maximal-proper p-local subgroups; for convenience, we recall these results here. This section is a summary of [2, Section 3]. We start by recalling that every maximal p-local $M < G$ has the form $M = N_G(E)$ with E in

$$\mathcal{ER}_p(G) = \{E \leq G \mid 1 \neq E = \Omega_1(Z(O_p(N_G(E))))\}.$$

Lemma 2.4. ([2, Lemma 3.1]) *Let $E \in \mathcal{ER}_p(G)$ and $R = O_p(N_G(E))$. Then $N_G(E)$ is maximal p-local if and only if $N_G(E) = N_G(Y)$ for every nontrivial elementary abelian p-subgroup Y of $\Omega_1(R)$ which is normal in $N_G(E)$; in particular, if R is abelian, then $Y \leq E$.*

In the following two lemmas let $G = Z.K$ be a central extension of $Z = p$ by a finite group $K \neq 1$ with $O_p(K) = 1$. Note that if $M < G$ is p-local, say $M = N_G(E)$ for a p-subgroup E, then $Z < O_p(M)$: clearly, $Z \leq O_p(M)$; if $Z = O_p(M)$, then $E = Z$, a contradiction to $M \neq G$. If $Z \leq E \leq G$, then $N_G(E) \to N_K(E/Z)$, $g \mapsto gZ$, is surjective with kernel $Z = Z \cap N_G(E)$: if $hZ \in N_K(E/Z)$, then $E^h Z/Z = E/Z$, and $Z \leq E$ proves that $E^h = E$; we have therefore shown that $N_G(E)/Z = N_K(E/Z)$.

Lemma 2.5. ([2, Lemma 3.2]) *If $G = Z.K$ is as before, then the following hold.*

a) *The group $M < G$ is maximal-proper p-local if and only if $Z \leq M$ and $M/Z \leq K$ is maximal-proper p-local. In this case, $M/Z = N_K(Q/Z)$ and $M = N_G(Q)$ where $Q = O_p(M)$ and $Q/Z = O_p(M/Z)$.*

b) *Let $Z < E < G$ such that $E/Z \in \mathcal{ER}_p(K)$ and $O_p(N_K(E/Z))$ is abelian. Then $M = N_G(E)$ is maximal-proper p-local if and only if $N_G(E) \not\leq N_G(F)$ for all $Z < F < E$ with $F/Z \in \mathcal{ER}_p(K)$.*

Lemma 2.6. ([2, Lemma 3.3]) *If $G = Z.K$ is as before, then the following hold.*

a) *Let $M < G$ be maximal-proper p-local. If $Z < E < G$ is defined by $E/Z = \Omega_1(Z(O_p(M/Z)))$, then $M = N_G(E)$ and $E/Z \in \mathcal{ER}_p(K)$. Also, $M = N_G(Y)$ for some $Z < Y \leq E$ such that one of the following holds:*

   (1) *$Y = \Omega_1(Z(E))$ elementary abelian; if $O_p(M)$ is abelian, then $Y = \Omega_1(O_p(M)) \in \mathcal{ER}_p(G)$,*

   (2) *$Y = Z(\Omega_1(E))$ elementary abelian, p odd, and E extraspecial with $Z = Z(E)$ and exponent p^2,*

   (3) *$Y = Z(E)$ cyclic of order p^2 with $Z = \Omega_1(Y)$,*

   (4) *$Y = E$ extraspecial with $Z = Z(Y)$; if p is odd, then Y has exponent p.*

b) *If $E \in \mathcal{R}_p(G)$ with $Z < E$ is extraspecial and $N_K(E/Z) \not\leq N_K(X/Z)$ for every $Z < X < E$ with $X/Z \in \mathcal{ER}_p(K)$, then $N_G(E)$ is maximal-proper p-local.*

c) *If $Z < E \leq G$ is cyclic of order p^2 and $O_p(N_G(E))$ is cyclic, then $N_G(E)$ is maximal-proper p-local.*

3. Elementary abelian 3-subgroups of $G = E_6^\varepsilon(q)$

Throughout, let $G = E_6^\varepsilon(q)$ with $\varepsilon \in \{\pm 1\}$ such that $3 \mid q - \varepsilon$. Let $T = (q-\varepsilon)^6$ be a maximal torus of G and, as before, write $Z = Z(G) = \langle z \rangle$. We classify, up to conjugacy, elementary abelian 3-subgroups which contain Z. The first subsection considers subgroups of type 3^2; the second investigates the action of the Weyl group of G on $V = \Omega_3(O_3(T))$. The remaining subsection then complete the classification of the elementary abelian groups of order dividing 3^6. We determine which of these subgroups yield maximal-proper 3-local subgroups.

3.1. Projective type

By [16, Table 4.7.3A], the group $G = E_6^{\varepsilon}(q)$ contains three subgroups of order 3, called A, B, and C, with generators z_{A}, z_{B}, and z_{C}, respectively, such that if $y \in G \setminus Z$ has order 3, then there is $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ such that, up to conjugacy in $\mathrm{Inndiag}(G) = G/Z.3$, the elements y and z_X induce the same element in $\mathrm{Inn}(G)$, that is, $y \in \langle z_X, z\rangle$. The local structure is as follows; the notation is explained below:

$$\begin{aligned} C_G(A) &= ((q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{\varepsilon}(q)).x_{\mathrm{A}}, & N_G(A) &= C_G(A).\gamma_{\mathrm{A}},\\ C_G(B) &= ((q-\varepsilon)^2 \circ_{(2_\varepsilon)^2} \mathrm{Spin}_8^{+}(q)).\langle x_{\mathrm{B}}, x'_{\mathrm{B}}\rangle, & N_G(B) &= C_G(B).\gamma_{\mathrm{B}},\\ C_G(C) &= (\mathrm{SL}_3^{\varepsilon}(q))^3/D.x_{\mathrm{C}}, & N_G(C) &= C_G(C).\gamma_{\mathrm{C}}, \end{aligned}$$

where $D = \langle (h,h,h)\rangle$ with $\langle h\rangle = Z(\mathrm{SL}_3^{\varepsilon}(q))$, and x_X and γ_X act as follows:

$\gamma_{\mathrm{A}} = \iota : 1$	ι acts as $x \mapsto x^{-1}$ on $q-\varepsilon$
$\gamma_{\mathrm{B}} = \leftrightarrow : \gamma$	γ acts as a order 2 graph automorphism on $\mathrm{Spin}_8^{+}(q)$, and $\leftrightarrow$ interchanges the two factors of $(q-\varepsilon)^2$
$\gamma_{\mathrm{C}} = \leftrightarrow : \gamma$	γ acts as the order 2 graph automorphism on one $\mathrm{SL}_3^{\varepsilon}(q)$, so γ is inverse-transpose; $\leftrightarrow$ swaps the other two factors,
$x_{\mathrm{A}} = 1 : 2_\varepsilon$	if q is odd, then x_{A} acts as $\mathrm{diag}(1,1,1,1,1,\lambda) \in \mathrm{GL}_6^{\varepsilon}(q)$ on $\mathrm{SL}_6^{\varepsilon}(q)$ with $\lambda \in \mathbb{F}_{q^2}$ a non-square element; if q is even, then $x_{\mathrm{A}} = 1$
$x_{\mathrm{B}} = 1 : 2_\varepsilon$ $x'_{\mathrm{B}} = 1 : 2_\varepsilon$	if q is even, then $x_{\mathrm{B}} = 1 = x'_{\mathrm{B}}$; if q is odd, then $\langle x_{\mathrm{B}}, x'_{\mathrm{B}}\rangle$ acts as $\mathrm{Outdiag}(\mathrm{Spin}_8^{+}(q)) = 2^2$; more precisely, x_{B} is induced by an element of $\mathrm{SO}_8^{+}(q) \setminus \Omega_8^{+}(q)$ with $\Omega_8^{+}(q) = \mathrm{Spin}_8^{+}(q)/Z$, x'_{B} is induced by a $\mathbb{F}_q$-linear conformal endomorphism of the underlying space of $\mathrm{Spin}_8^{+}(q)$, corresponding to a non-square multiplier, cf. [12, p. 124]
$x_{\mathrm{C}} = s_1 : s_2 : s_3$	each s_i acts as $o_i = \mathrm{diag}(1,1,\tau) \in \mathrm{GL}_3^{\varepsilon}(q)$, where $\tau \in \mathbb{F}_{q^2}$ is an element of maximal 3-power order 3^a; define $\omega = \tau^{3^{a-1}}$

A few comments are in order. According to the comment on [16, p. 209], each z_X is $\mathrm{Inndiag}(G)$-conjugate to its inverse; if $g \in \mathrm{Inndiag}(G)$ with $z_X^g = z_X^{-1}$, then also $z_X^{g^3} = z_X^{-1}$ with $g^3 \in \mathrm{Inn}(G)$. We note that [16, Table 4.7.3A] considers the normaliser of $3X$ only in the adjoint group G/Z, so it first follows that $N_{G/Z}(\langle z_X Z\rangle)/C_{G/Z}(z_X Z) = 2$. Together with the $3X$-pureness of $3X$, we deduce that $\mathrm{Out}_G(3X) = 2$.

The structure of D in $C_G(C)$ follows from [16, Table 4.7.3A], which shows that $N_G(\langle Z, z_{\mathrm{C}}\rangle)$ contains an element Δ acting as $(1,2,3)$ on the three factors of $O^{r'}(C_G(C))$; this implies that

$$O^{r'}(C_G(C)) = (\mathrm{SL}_3^{\varepsilon}(q) \times \mathrm{SL}_3^{\varepsilon}(q) \times \mathrm{SL}_3^{\varepsilon}(q))/D.$$

Note that x_{A} does not necessarily have order 2; we only know that $x_{\mathrm{A}}^2 \in (q-\varepsilon) \circ_{2_\varepsilon} \mathrm{SL}_6^{\varepsilon}(q)$; similarly for $x_{\mathrm{B}}, x'_{\mathrm{B}}, x_{\mathrm{C}}$ and γ_X. For example, we have $x_{\mathrm{C}}^3 \in O^{r'}(C_G(C))$, and each $o_i^3 = \mathrm{diag}(1,1,\tau^3)$ acts as the inner automorphism $\mathrm{diag}(\tau^{-1}, \tau^{-1}, \tau^2)$, where $\tau \in \mathbb{F}_{q^2}$ is defined as above.

For $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, the groups $\langle 3X, z\rangle$ are toral subgroups (cf. [14, (2.13) (vi)]), so we can assume that each $\langle 3X, z\rangle \le T$. On the other hand, Table I below implies that there are, up to $N_G(T)$-conjugacy, exactly three subgroups $3^2 \le T$ containing Z. This proves that, up to conjugacy, G has three subgroups of order 3, namely 3A, 3B, and 3C. We use this fact in the following definition.

Definition 3.1. If $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, then $Y = \Omega_1(O_3(C_G(X))) = \langle z, z_X\rangle = 3^2$; if $E = 3^2 =_G Y$, then E has **(projective) type** $3\overline{X}$, and we write $E = 3\overline{X}$. If $E \le G$ is an elementary abelian 3-subgroup containing Z, then we write $E = 3\overline{\mathrm{A}}_u\overline{\mathrm{B}}_v\overline{\mathrm{C}}_w$ if E contains exactly u, v, and w subgroups of type $3\overline{\mathrm{A}}$, $3\overline{\mathrm{B}}$, and $3\overline{\mathrm{C}}$, respectively.

3.2. Weyl group action

As before, let $T = (q-\varepsilon)^6$ be a maximal torus of G with Weyl group $W = N_G(T)/T$, and define

$$V = \Omega_1(O_3(T)) = 3^6.$$

Every maximal torus of G isomorphic to $(q-\varepsilon)^6$ is conjugate to T, see [10, p. 903]. We may suppose $V \leq \overline{T}^{\sigma}$, and a direct computation shows $C_{\overline{G}}(V) = \overline{T}$, hence $C_G(V) = C_G(T) = T$. This implies $N_G(V) \leq N_G(T)$, thus $N_G(V) = N_G(T)$ and $\mathrm{Out}_G(V) = \mathrm{Out}_G(T) = W \leq \mathrm{Aut}(V) = \mathrm{GL}_6(3)$. Recall that

$$W = W(E_6) \cong \mathrm{Aut}(\mathrm{PSp}_4(3)) \cong \mathrm{PSp}_4(3).2 \cong \mathrm{SO}_6^-(2) \cong \mathrm{SO}_5(3).$$

We consider $W \leq \mathrm{Aut}(V) = \mathrm{GL}_6(3) = H$; note that $W \leq C_H(z)$ where $z \in V$ generates Z. A direct computation shows that $C_H(z)$ contains three H-classes of subgroups isomorphic to W. Also, up to conjugacy, W is the unique subgroup of $C_H(z)$ such that V contains exactly three W-orbits of planes containing z, denoted P_1, P_2 and P_3, with $C_W(P_1) \cong W(A_5) = \mathrm{S}_6$, $C_W(P_2) \cong W(D_4) = 2_+^{1+4}.\mathrm{S}_3$, and $C_W(P_3) \cong W(3A_2) = (\mathrm{S}_3)^3$. By Section 3.1, these centraliser conditions are sufficient to identify W as a subgroup of H.

We use the notation of Section 3.1: if $L \leq V$ is a subspace with $Z \leq L$, then L has *projective type* $3\overline{\mathrm{A}}_u\overline{\mathrm{B}}_v\overline{\mathrm{C}}_w$ if L contains exactly u, v, and w planes of type $3\overline{\mathrm{A}}$, $3\overline{\mathrm{B}}$, and $3\overline{\mathrm{C}}$, respectively.

Lemma 3.2. *Consider $W \leq \mathrm{GL}_6(3)$ as constructed above, with natural W-module $V = 3^6$.*

a) *There are 17 W-orbits of subspaces $L \leq V$ with $Z < L$; Table I lists representatives of these subspaces, their projective type, $C_W(L)$, and $\mathrm{Out}_W(L) = W(L)/C_W(L)$.*

b) *The group $L = 3\overline{\mathrm{A}}_6\overline{\mathrm{B}}_3\overline{\mathrm{C}}_4$ has, up to $N_W(L)$-conjugacy, a unique subgroup $Y = 3X$ for each $X \in \{\overline{\mathrm{A}}, \overline{\mathrm{B}}, \overline{\mathrm{C}}\}$; in each case $N_{N_W(L)}(Y) < N_W(L)$. Up to $N_W(L)$-conjugacy, L also has a unique subgroup $R = 3\overline{\mathrm{A}}_2\overline{\mathrm{B}}_2$, and $N_{N_W(L)}(R) < N_W(L)$.*

c) *The group $L = 3\overline{\mathrm{A}}_2\overline{\mathrm{B}}_2$ has, up to $N_W(L)$-conjugacy, a unique $Y = 3X$ for each $X \in \{\overline{\mathrm{A}}, \overline{\mathrm{B}}\}$; in each case $N_{N_W(L)}(Y) < N_W(L)$.*

d) *If $a \in 3\overline{\mathrm{A}} \setminus Z$ and $b \in 3\overline{\mathrm{B}} \setminus Z$, then $\langle z, ab\rangle$ is W-conjugate to $3\overline{\mathrm{C}}$.*

PROOF. This follows from an explicit computation using the computer algebra system Magma [7]. □

3.3. Elementary abelian subgroups of G containing Z

We now complete the classification of subgroups $E \leq G$ with $E = 3^i$ and $Z < E$, up to conjugacy. We start with the following result which we will use frequently.

Lemma 3.3. *Let $\overline{G}$ be a simply-connected algebraic group of rank n with Frobenius map σ. Let $\overline{\mathrm{C}} = C_{\overline{G}}(y)$, where $y \in (\overline{G})^{\sigma}$ is semisimple of parabolic type as defined in [16, Definition 4.1.8(A)], of order dividing $(q-\varepsilon)^n$. Then $\overline{\mathrm{C}} = \overline{\mathrm{C}}^{\circ} = \overline{S}_1\overline{L}$ where $\overline{S}_1 = Z(\overline{\mathrm{C}})$ is a torus of $\overline{\mathrm{C}}$ and $\overline{L} = [\overline{\mathrm{C}}, \overline{\mathrm{C}}]$ is semisimple. Suppose $C_{(\overline{G})^{\sigma}}(y) = (\overline{S}_1)^{\sigma} \circ_Q ((\overline{L})^{\sigma}.R)$ for some Q and R, where $(\overline{S}_1)^{\sigma} = (q-\varepsilon)^t$, and $(\overline{L})^{\sigma}$ is semisimple, containing a maximal torus $(\overline{S}_2)^{\sigma} = (q-\varepsilon)^{n-t}$. Then $(q-\varepsilon)^n \leq C_{(\overline{G})^{\sigma}}(y)$ is a maximal torus of $C_{(\overline{G})^{\sigma}}(y)$.*

PROOF. We can assume that $y \in \overline{S}$ where $\overline{S}$ is a maximal σ-stable torus of $\overline{G}$ with $(\overline{S})^{\sigma} = (q-\varepsilon)^n$. Now clearly $\overline{S} \leq C_{\overline{G}}(y)$ and so $(q-\varepsilon)^n = \overline{S}^{\sigma} \leq C_{G^{\sigma}}(y)$, as claimed. □

Let $(\overline{G}, \sigma)$ be a σ-setup of $G = E_6^{\varepsilon}(q)$ (cf. [16, Definition 2.2.1]). Proposition 2.3 shows that the classification of the elementary abelian p-subgroups Y of G up to G-conjugacy can be deduced from the

L	proj. type	$C_W(L)$	$\mathrm{Out}_W(L)$
3^2	$3\overline{\mathrm{A}}$	S_6	2
3^2	$3\overline{\mathrm{B}}$	$2_+^{1+4}.\mathrm{S}_3$	S_3
3^2	$3\overline{\mathrm{C}}$	$\mathrm{S}_3\times\mathrm{S}_3\times\mathrm{S}_3$	S_3
3^3	$(3\overline{\mathrm{C}}^2)_1$	3	$3^2.\mathrm{GL}_2(3)$
3^3	$3\overline{\mathrm{B}}_3\overline{\mathrm{C}}_1$	S_3	$\mathrm{S}_3\times\mathrm{S}_3$
3^3	$3\overline{\mathrm{A}}_3\overline{\mathrm{C}}_1$	$\mathrm{S}_3\times\mathrm{S}_3$	D_{12}
3^3	$3\overline{\mathrm{A}}_1\overline{\mathrm{B}}_1\overline{\mathrm{C}}_2$	2^3	D_{12}
3^3	$3\overline{\mathrm{A}}_2\overline{\mathrm{B}}_2$	S_4	D_8
3^4	$3\overline{\mathrm{B}}_9\overline{\mathrm{C}}_4$	1	$3_+^{1+2}.\mathrm{GL}_2(3)$
3^4	$3\overline{\mathrm{A}}_6\overline{\mathrm{B}}_6\overline{\mathrm{C}}_1$	S_3	$(\mathrm{S}_3\times\mathrm{S}_3):2$
3^4	$3\overline{\mathrm{A}}_3\overline{\mathrm{B}}_3\overline{\mathrm{C}}_7$	1	$\mathrm{S}_3\times\mathrm{S}_3\times\mathrm{S}_3$
3^4	$3\overline{\mathrm{A}}_3\overline{\mathrm{B}}_6\overline{\mathrm{C}}_4$	2	$2\times\mathrm{S}_4$
3^4	$3\overline{\mathrm{A}}_6\overline{\mathrm{B}}_3\overline{\mathrm{C}}_4$	2^2	$2\times\mathrm{S}_4$
3^5	$3\overline{\mathrm{A}}_9\overline{\mathrm{B}}_{18}\overline{\mathrm{C}}_{13}$	1	$(\mathrm{S}_3\times\mathrm{S}_3\times\mathrm{S}_3).\mathrm{S}_3$
3^5	$3\overline{\mathrm{A}}_{12}\overline{\mathrm{B}}_{12}\overline{\mathrm{C}}_{16}$	1	$2_+^{1+4}.(\mathrm{S}_3\times\mathrm{S}_3)$
3^5	$3\overline{\mathrm{A}}_{15}\overline{\mathrm{B}}_{15}\overline{\mathrm{C}}_{10}$	2	S_6
3^6	$3\overline{\mathrm{A}}_{36}\overline{\mathrm{B}}_{45}\overline{\mathrm{C}}_{40}$	1	W

Table I: W-orbits of subspaces of $V=\Omega_1(T)=3^6$ containing $Z=Z(G)$.

classification of the elementary abelian p-subgroups E of G up to $\overline{G}$-conjugacy: then each Y has the form $Y=E_w$ for some $w\in C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ$, and the local structure is determined as $N_G(Y)=N_{\overline{G}}(Y)^\sigma$ and $C_G(Y)=C_{\overline{G}}(Y)^\sigma$. Moreover, as mentioned in Remark 2.2, the toral elementary abelian p-subgroups of G, up to $\overline{G}$-conjugacy, can be classified directly using Magma; the non-toral elementary abelian p-subgroups of G, up to $\overline{G}$-conjugacy, are given by [5]. Recall that we write $A=_{\overline{G}}B$ if A and B are $\overline{G}$-conjugate.

Proposition 3.4. *Let $G=E_6^\varepsilon(q)$ with $3\mid q-\varepsilon$. Let $T=(q-\varepsilon)^6\leq G$ be a maximal torus with Weyl group $W=N_G(T)/T$. Up to conjugacy, G has 20 elementary abelian 3-subgroups E which contain $Z=Z(G)$. Their projective type and local structure are listed in Table II. The third column contains the centraliser of E in $\overline{G}$, where $(\overline{G},\sigma)$ is a σ-setup of G.*

PROOF. The elementary abelian 3-subgroups of $\overline{G}$, up to $\overline{G}$-conjugacy, can be determined as described in see Remark 2.2. This computation yields that, up to $\overline{G}$-conjugacy, G contains 17 toral elementary abelian 3-subgroups E with $Z<E$; representatives for these groups are $\{E_1,\ldots,E_{20}\}\setminus\{E_8,E_{15},E_{16}\}$ as given in Table II. This computation also tells us the component group $C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ$ and the structure of the torus $\overline{S}_1=Z(C_{\overline{G}}(E)^\circ)$ and the semisimple part $\overline{L}=[C_{\overline{G}}(E)^\circ,C_{\overline{G}}(E)^\circ]$ of the centraliser. All centralisers are connected, except for $E=E_7$ in which case $C_{\overline{G}}(E_7)=\overline{T}.3$. To determine $C_{\overline{G}}(E)$ for the other groups, it remains to describe the structure of their central product $C_{\overline{G}}(E)^\circ=\overline{S}\circ_Q\overline{L}$. For E_1, E_2, and E_3 this information is given in [16, Table 4.7.1]; for E_7 the computation directly shows that $C_{\overline{G}}(E_7)=\overline{T}.3$. Now consider $Y=E_i$ with $i\in\{4,5,6\}$; we can assume that $Y=\langle E_1,x\rangle$ for some $x\in C_{\overline{G}}(E)=\overline{T}_1\circ_{2^*}\mathrm{SL}_6$. In particular, $x=x_1x_2$ with $x_1\in\overline{T}_1$ and $x_2\in\mathrm{SL}_6$, so that $C_{\overline{G}}(Y)=\overline{T}_1\circ_{2^*}C_{\mathrm{SL}_6}(x_2)$ and we may suppose $Y=\langle E,x\rangle$ with $x\in\mathrm{SL}_6$, that is, we can suppose

E		proj. type	$C_{\overline{G}}(E)$	$C_G(E)$	$N_G(E)$
E_1	3^2	$3\overline{\mathrm{A}}$	$\overline{T}_1 \circ_{2^*} \mathrm{SL}_6$	$(q-\varepsilon) \circ_{2_\varepsilon} (\mathrm{SL}_6^\varepsilon(q).2_\varepsilon)$	$\boldsymbol{C_G(E).2}$
E_2	3^2	$3\overline{\mathrm{B}}$	$\overline{T}_2 \circ_{(2^*)^2} \mathrm{Spin}_8$	$(q-\varepsilon)^2 \circ_{(2_\varepsilon)^2} (\mathrm{Spin}_8^+(q).(2_\varepsilon)^2)$	$\boldsymbol{C_G(E).\mathrm{S}_3}$
E_3	3^2	$3\overline{\mathrm{C}}$	$(\mathrm{SL}_3)^3/D$	$((\mathrm{SL}_3^\varepsilon(q))^3/D).3$	$\boldsymbol{C_G(E).\mathrm{S}_3}$
E_4	3^3	$3\overline{\mathrm{A}}_3\overline{\mathrm{C}}_1$	$\overline{T}_2 \circ_3 (\mathrm{SL}_3)^2$	$(q-\varepsilon)^2 \circ_3 ((\mathrm{SL}_3^\varepsilon(q))^2.3)$	$C_G(E).D_{12}$
E_5	3^3	$3\overline{\mathrm{A}}_2\overline{\mathrm{B}}_2$	$\overline{T}_3 \circ_{4^*} \mathrm{SL}_4$	$(q-\varepsilon)^3 \circ_{4_\varepsilon} (\mathrm{SL}_4^\varepsilon(q).4_\varepsilon)$	$\boldsymbol{C_G(E).D_8}$
E_6	3^3	$3\overline{\mathrm{A}}_1\overline{\mathrm{B}}_1\overline{\mathrm{C}}_2$	$\overline{T}_3 \circ_{(2^*)^3} (\mathrm{SL}_2)^3$	$(q-\varepsilon)^3 \circ_{(2_\varepsilon)^3} ((\mathrm{SL}_2(q))^3.(2_\varepsilon)^3)$	$C_G(E).D_{12}$
E_7	3^3	$(3\overline{\mathrm{C}}^2)_1$	$\overline{T}.3$	$(q-\varepsilon)^6.3$	$C_G(E).3^2.\mathrm{GL}_2(3)$
E_8	3^3	$(3\overline{\mathrm{C}}^2)_2$	$\overline{T}.3$	$(q^2+\varepsilon q+1)^3.3$	$\boldsymbol{C_G(E).3^2.\mathrm{SL}_2(3)}$
E_9	3^3	$3\overline{\mathrm{B}}_3\overline{\mathrm{C}}_1$	$\overline{T}_4 \circ_3 \mathrm{SL}_3$	$(q-\varepsilon)^4 \circ_3 (\mathrm{SL}_3^\varepsilon(q).3)$	$C_G(E).(\mathrm{S}_3)^2$
E_{10}	3^4	$3\overline{\mathrm{A}}_6\overline{\mathrm{B}}_6\overline{\mathrm{C}}_1$	$\overline{T}_4 \circ_3 \mathrm{SL}_3$	$(q-\varepsilon)^4 \circ_3 (\mathrm{SL}_3^\varepsilon(q).3)$	$C_G(E).((\mathrm{S}_3)^2{:}2)$
E_{11}	3^4	$3\overline{\mathrm{A}}_6\overline{\mathrm{B}}_3\overline{\mathrm{C}}_4$	$\overline{T}_4 \circ_{(2^*)^2} (\mathrm{SL}_2)^2$	$(q-\varepsilon)^4 \circ_{(2_\varepsilon)^2} ((\mathrm{SL}_2(q))^2.(2_\varepsilon)^2)$	$\boldsymbol{C_G(E).(2\times\mathrm{S}_4)}$
E_{12}	3^4	$3\overline{\mathrm{A}}_3\overline{\mathrm{B}}_6\overline{\mathrm{C}}_4$	$\overline{T}_5 \circ_{2^*} \mathrm{SL}_2$	$(q-\varepsilon)^5 \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$	$C_G(E).(2\times\mathrm{S}_4)$
E_{13}	3^4	$3\overline{\mathrm{A}}_3\overline{\mathrm{B}}_3\overline{\mathrm{C}}_7$	$\overline{T}$	$(q-\varepsilon)^6$	$C_G(E).(\mathrm{S}_3)^3$
E_{14}	3^4	$3\overline{\mathrm{B}}_9\overline{\mathrm{C}}_4$	$\overline{T}$	$(q-\varepsilon)^6$	$C_G(E).3_+^{1+2}.\mathrm{GL}_2(3)$
E_{15}	3^4	$(3\overline{\mathrm{C}}^3)_1$	3^4	3^4	$\boldsymbol{C_G(E).3^3.\mathrm{SL}_3(3)}$
E'_{15}	3^4	$(3\overline{\mathrm{C}}^3)_1$	3^4	3^4	$\boldsymbol{C_G(E).3^3.\mathrm{SL}_3(3)}$
E''_{15}	3^4	$(3\overline{\mathrm{C}}^3)_1$	3^4	3^4	$\boldsymbol{C_G(E).3^3.\mathrm{SL}_3(3)}$
E_{16}	3^4	$(3\overline{\mathrm{C}}^3)_2$	3^4	3^4	$C_G(E).3^4.\mathrm{SL}_2(3)$
E_{17}	3^5	$3\overline{\mathrm{A}}_9\overline{\mathrm{B}}_{18}\overline{\mathrm{C}}_{13}$	$\overline{T}$	$(q-\varepsilon)^6$	$C_G(E).(\mathrm{S}_3)^3.\mathrm{S}_3$
E_{18}	3^5	$3\overline{\mathrm{A}}_{15}\overline{\mathrm{B}}_{15}\overline{\mathrm{C}}_{10}$	$\overline{T}_5 \circ_{2^*} \mathrm{SL}_2$	$(q-\varepsilon)^5 \circ_{2_\varepsilon} (\mathrm{SL}_2(q).2_\varepsilon)$	$\boldsymbol{C_G(E).\mathrm{S}_6}$
E_{19}	3^5	$3\overline{\mathrm{A}}_{12}\overline{\mathrm{B}}_{12}\overline{\mathrm{C}}_{16}$	$\overline{T}$	$(q-\varepsilon)^6$	$C_G(E).2_+^{1+4}.(\mathrm{S}_3)^2$
E_{20}	3^6	$3\overline{\mathrm{A}}_{36}\overline{\mathrm{B}}_{45}\overline{\mathrm{C}}_{40}$	$\overline{T}$	$(q-\varepsilon)^6$	$\boldsymbol{C_G(E).W}$

Table II: Elementary abelian 3-subgroups of $G = E_6^\varepsilon(q)$ properly containing $Z = Z(G)$ with $3 \mid q-\varepsilon$ and $n_\varepsilon = \gcd(n, q-\varepsilon)$ and $n^* = n$ or 1 according as q is odd or even.

that $x \in \{\mathrm{diag}(\omega I_3, \omega^{-1} I_3), \mathrm{diag}(\omega, \omega^{-1}, I_4), \mathrm{diag}(\omega I_2, \omega^{-1} I_2, I_2)\}$, and therefore $C_{\mathrm{SL}_6}(x)$ is one of $\overline{T}_1 \circ_3 (\mathrm{SL}_3)^2$, $\overline{T}_2 \circ_{4^*} \mathrm{SL}_4$, and $\overline{T}_2 \circ_{(2^*)^2} (\mathrm{SL}_2)^3$. This allows us to determine $C_{\overline{G}}(E_i)$ for $i \in \{4,5,6\}$. Similarly, for $C_{\overline{G}}(E_9) = \overline{T}_4\mathrm{SL}_3$ we may suppose $E_9 = \langle E_3, x\rangle$ for some $x \in (\mathrm{SL}_3)^3/D$ of the form $x = \mathrm{diag}(\omega, \omega^{-1}, 1, \omega, \omega^{-1}, 1, I_3)D$; this determines $C_{\overline{G}}(E_9)$. The other centralisers $C_{\overline{G}}(E)$ are calculated similarly; note that work is only necessary for those centralisers which are computed to be a central product. The projective type for each toral elementary abelian 3-subgroup E and the structure of $\mathrm{Out}_G(E)$ can be obtained by another direct computation, together with the results listed in Table I. If $E \neq_G E_7$ is toral, then $C_{\overline{G}}(E)$ is connected, and Proposition 2.3 implies that G contains a unique G-conjugacy class of subgroups which are $\overline{G}$-conjugate to E. If $E = E_7$, then $C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ = 3$ and $\mathrm{Out}_{\overline{G}}(E) = 3^2.\mathrm{GL}_2(3) = \mathrm{Out}_W(E)$, so $\mathrm{Out}_{\overline{G}}(E) = \mathrm{Out}_G(E)$, which has two σ-classes in $C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ = 3$; here σ-classes are conjugacy classes since $W = V/O_2(V)$ (with V the extended Weyl group) is centralised by σ, as shown by a direct computation. Thus Proposition 2.3 yields that G contains exactly two classes of subgroups which are $\overline{G}$-conjugate to E_7, with representatives E_7 and E_8. This completes the classification, up to G-conjugacy, of the toral elementary abelian 3-subgroups of G and their centralisers in $\overline{G}$.

As shown in [5], up to $\overline{G}$-conjugacy, G has a two non-toral elementary abelian subgroups and only one of these classes contains Z, cf. [14, (11.13)]; this class has representative $E = E_{15} = 3^4$ with $C_{\overline{G}}(E) = E$ and $N_{\overline{G}}(E) = 3^4.3^3.\mathrm{SL}_3(3)$. We argue that $N_{\overline{G}}(E) = N_G(E)$: It follows from [17, Section 2] that the finite group $E_6^{\varepsilon}(p)$, with $p \equiv \varepsilon \bmod 3$ and $p \geq 5$ has a subgroup $3^{3+3}.\mathrm{SL}_3(3)$. For $p = 2$ a direct computation shows that $E_6^{-}(2)$ has a subgroup 3^4 with normaliser $3^4.3^3.\mathrm{SL}_3(3)$. It follows that the corresponding simply connected group in characteristic not 3 has a subgroup 3^4 whose normaliser contains $3^4.3^3.\mathrm{SL}_3(3)$. Indeed, this is the normaliser: if a subgroup $E = 3^4$ has a normaliser in $\overline{G}$ that contains $E.3^3.\mathrm{SL}_3(3)$, then $C_{\overline{G}}(E) = E$ and $\mathrm{Out}_{\overline{G}}(E) \geq 3^3.\mathrm{SL}_3(3)$. In particular, E is non-toral, and [5, Table 4] shows that $E =_G E_{15}$, and then $N_{\overline{G}}(E) = E.3^3.\mathrm{SL}_3(3)$.

The discussion in [5, Section 3.1] shows that the existence of this finite group is independent of the characteristic as long as it is different to 3. Thus, we can assume that this finite group also exists in G, that is, E satisfies $N_G(E) = N_{\overline{G}}(E) = 3^4.3^3.\mathrm{SL}_3(3)$. Note that $C_{\overline{G}}(E)^\circ = 1$ and $C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ = 3^4$. Since $N_{\overline{G}}(E) = N_G(E)$, the σ-classes of $\mathrm{Out}_{\overline{G}}(E)$ in $C_{\overline{G}}(E)/C_{\overline{G}}(E)^\circ$ are the conjugacy classes of $3^4.3^3.\mathrm{SL}_3(3)$ in 3^4. It follows from Proposition 2.3 that G contains four G-conjugacy classes of subgroups which are $\overline{G}$-conjugate to E, with representatives E_{15}, E'_{15}, E''_{15}, and E_{16} corresponding to $1, z, z^2, w \in 3^4$, where z generates the centre of $\overline{G}$ and $w \in 3^4 \setminus \langle z \rangle$; moreover, $C_G(E_{16}) = C_G(E_{15}) = C_G(E'_{15}) = C_G(E''_{15}) = 3^4$ and $\mathrm{Out}_G(E_{16}) = C_{\mathrm{Out}_G(E_{15})}(w) = 3^4.\mathrm{SL}_2(3)$. In particular, it follows from Proposition 2.3c) that $N_G(E_{15}) = N_G(E'_{15}) = N_G(E''_{15})$. This completes the discussion of the non-toral elementary abelian 3-subgroups of G, and their local structure.

The structure of $\mathrm{Out}_G(E)$ for toral $E \neq_G E_8$ follows from Table I. The structure of $C_G(E)$ for toral $E \neq_G E_8$ follows from Lemma 3.3: By [16, Theorem 4.2.2(a-c)], we have $C_G(E) = SL$, where $L = L_1 \cdots L_s$ and S is an abelian r'-group inducing inner-diagonal on each L_i. On the other hand, we know that $C_{\overline{G}}(E) = \overline{S} \circ_Q \overline{L}$ with $\overline{S}^\sigma = (q-\varepsilon)^t$ and $Q^\sigma = Q$. Lemma 3.3 shows that we can assume $T \leq C_G(E)$. We may suppose $\overline{S}^\sigma \leq T$ and $T \cap L = T \cap \overline{L}^\sigma = (q-\varepsilon)^{n-t}$, so $T = ((q-\varepsilon)^t \circ_Q (q-\varepsilon)^{n-t}).Q$. Note that if $E \neq E_4$, then $Q = \mathrm{Outdiag}(L)$ and S induces only inner-diagonal automorphisms on each Lie component L_i of L. Together with $T = ((q-\varepsilon)^t \circ_Q (q-\varepsilon)^{n-t}).Q \leq C_G(E)$, we deduce that $C_{\overline{G}}(E)^\sigma = (\overline{S}^\sigma \circ_Q \overline{L}^\sigma).Q$. If $E = E_4$, then the structure of $C_G(E)$ follows from $C_{\mathrm{SL}_6^\varepsilon(q)}(x) = (q-\varepsilon) \circ_3 ((\mathrm{SL}_3^\varepsilon(q))^2.3)$.

It remains to determine $C_G(E_8)$ and $\mathrm{Out}_G(E_8)$. We can assume that $E_7 = \langle E_3, x \rangle$, so that $C_{\overline{G}}(E_7) = \overline{T}.3$ and $C_{\overline{G}}(E_3) = H/D$, where $H = (\mathrm{SL}_3)^3 = H_1 \times H_2 \times H_3$ with $H_i = \mathrm{SL}_3$ and $D = \langle z_1 z_2 z_3 \rangle$ with each $z_i \in Z(H_i) \setminus \{1\}$. Let $X_i = \langle x_i, y_i \rangle \leq H_i$ such that $X_i \cong 3_+^{1+2}$ and $[x_i, y_i] = z_i$, with $C_{H_i}(x_i) = \overline{T}_2$ and y_i a permutation matrix. We can choose $x = x_1 x_2 x_3 \in H$, so that $y = y_1 y_2 y_3 \in C_{\overline{G}}(E_7) \setminus \overline{T}$ and E_8 corresponds to y under the correspondence given in Proposition 2.3. Since $(H/D)^\sigma = (\mathrm{SL}_3^\varepsilon(q))^3/D.3$, we have $(H_i)^\sigma = \mathrm{SL}_3^\varepsilon(q)$ and each $y_i \in \mathrm{SL}_3^\varepsilon(q)$; in particular, $[y, \sigma] = 1$. The σ-conjugacy class of y_i corresponds to a maximal torus T_{y_i} of $\mathrm{SL}_3^\varepsilon(q)$ and $T_{y_i} = (\overline{T}_{\mathrm{SL}_3})^{y_i\sigma}$, where $\overline{T}_{\mathrm{SL}_3}$ is a σ-stable maximal torus of SL_3. Now Proposition 2.3 shows that $(C_{\overline{G}}(E_8)^\circ)^\sigma = (\overline{T})^{y\sigma} = (q^2 + \varepsilon q + 1)^3$ and $(C_{\overline{G}}(E_8)/C_{\overline{G}}(E_8)^\circ)^\sigma = (C_{\overline{G}}(E_7)/C_{\overline{G}}(E_7)^\circ)^{y\sigma} \cong 3$, so $C_G(E_8) = (q^2 + \varepsilon q + 1)^3.y$. We can write $N_{\overline{G}}(E_7)/C_{\overline{G}}(E_7)^\circ = 3.3^2.\mathrm{GL}_2(3) = 3_+^{1+2}.\mathrm{GL}_2(3) \leq W$ with $Z(3_+^{1+2}) = \langle y \rangle$; since W is fixed under σ and $[y, \sigma] = 1$, Proposition 2.3 and a direct computation in W yield $(N_{\overline{G}}(E_8)/C_{\overline{G}}(E_8)^\circ)^\sigma = (3_+^{1+2}.\mathrm{GL}_2(3))^{y\sigma} = C_{3_+^{1+2}.\mathrm{GL}_2(3)}(y) = 3_+^{1+2}.\mathrm{SL}_2(3)$, so $\mathrm{Out}_G(E_8) = 3^3.\mathrm{SL}_2(3)$. □

Corollary 3.5. *The maximal-proper 3-local subgroups of G among the normalisers listed in Table II are the groups $N_G(E_i)$ with $i \in \{1,2,3,5,8,11,15,18,20\}$ if $q \neq 2$ and with $i \in \{1,2,3,5,15,20\}$ if $q = 2$.*

PROOF. Lemmas 2.4 and 2.5 show that $N_G(E_i)$ with $i \in \{1,2,3,20\}$ is maximal-proper 3-local. Recall that if E has projective type $3\overline{\mathrm{A}}_u\overline{\mathrm{B}}_v\overline{\mathrm{C}}_w$ for some u, v, w, and $w = 1$, then E has a unique subgroup of type $3\overline{\mathrm{C}}$, and therefore $N_G(E) \leq_G N_G(3\overline{\mathrm{C}})$; the analogous statement holds if $u = 1$ or $v = 1$, which proves that $N_G(E_i)$ is not maximal-proper 3-local if $i \in \{4,6,9,10\}$. Each group $E_i \neq 3^6$ with $C_G(E_i) = (q-\varepsilon)^6$ satisfies $N_G(E_i) \leq N_G(C_G(E_i)) = N_G(T) = N_G(E_{20})$, hence is not maximal-proper 3-local; this holds for $i \in \{13, 14, 17, 19\}$. If $N_G(E_5)$ is not maximal-proper 3-local, then Lemmas 2.4 and 2.5 show that there is $i \in \{1, 2, 3\}$ with $Z < E_i < E_5$ such that $N_G(E_5) \leq N_G(E_i)$ and $N_G(E_i)$ is maximal-proper 3-local. Thus $N_{N_G(T)}(E_5) = N_G(E_5) \cap N_G(T) \leq N_G(E_i) \cap N_G(T) = N_{N_G(T)}(E_i)$ and, since $T \leq N_{N_G(T)}(E_5)$, we deduce that $N_W(E_5) \leq N_W(E_i)$ and $N_{N_W(E_5)}(E_i) = N_W(E_i)$, contradicting Lemma 3.2c); this proves that $N_G(E_5)$ is maximal-proper 3-local. Table I shows that $N_W(E_7) = 3.3^2.\mathrm{GL}_2(3)$; together with $N_G(E_7) = C_G(E).3^2.\mathrm{GL}_2(3)$ and $C_G(E_7) = T.3$, this implies that $N_G(E_7) \leq N_G(T)$, and $N_G(E_7)$ is not maximal-proper 3-local. The structure of $C_G(E_{12})$ and $C_G(E_{18})$ imply that we can assume $E_{12} < E_{18}$ and $N_G(E_{12}) < N_G(E_{18})$, hence $N_G(E_{12})$ is not maximal-proper 3-local. The group $\mathrm{Out}_G(E_{18}) = \mathrm{S}_6$ acts irreducibly on $E_{18}/Z = 3^4$, hence, if $q \neq 2$, then $N_G(E_{18})$ is maximal-proper 3-local by Lemmas 2.4 and 2.5. If $q = 2$, then $N_G(E_{18}) = (3^5 \times \mathrm{S}_3).\mathrm{S}_6$, so $N_G(E_{18}) \leq N_G(E_{20}) = T.W$. If $q = 2$, then E_8 is non-toral with $C_G(E_8) = 3^4$, hence $C_G(E_8) = E_i$ for $i \in \{15, 16\}$ and so $N_G(E_8) \leq_G N_G(E_i)$; in particular, $N_G(E_8)$ is not maximal-proper 3-local if $q = 2$. If $q \neq 2$, then $q^2 + \varepsilon q + 1 \neq 3$ and $E_8/Z \in \mathcal{ER}_3(G/Z)$. Suppose $N_G(E_8)$ is not maximal-proper 3-local, then, by Lemmas 2.4 and 2.5, we may suppose $N_G(E_8) \leq N_G(E_3) = (H/D).\mathrm{S}_3$, where $H = (\mathrm{SL}_3^\varepsilon(q))^3$. In particular, $\mathrm{Out}_G(E_8) = 3^2.\mathrm{SL}_2(3) \leq \mathrm{Out}_H(E_8).\mathrm{S}_3$ and $\mathrm{Out}_H(E_8)$ is nonabelian; but $\mathrm{Out}_H(E_8) \leq (\mathrm{Out}_{\mathrm{SL}_3^\varepsilon(q)}(q^2 + \varepsilon q + 1))^3 = 3^3$, which is a contradiction. Thus $N_G(E_8)$ is maximal-proper 3-local if $q \neq 2$. If $q \neq 2$ and $N_G(E_{11})$ is not maximal-proper 3-local, then Lemmas 2.4 and 2.5 imply that $N_G(E_{11}) \leq_G N_G(E_j)$ for some maximal-proper 3-local $N_G(E_j)$ with $Z < E_j < E_{11}$. The previous classification implies that E_j is toral and $E_j \in \{E_1, E_2, E_3, E_5\}$; note that $j = 8$ is not possible since $C_G(E_{11}) \not\leq_G C_G(E_8)$. In particular, $T \leq N_{N_G(T)}(E_{11}) \leq N_{N_G(T)}(E_j)$, and so $N_W(E_{11}) \leq N_W(E_j)$ and $N_W(E_{11}) = N_{N_W(E_j)}(E_{11})$, contradicting Lemma 3.2b); this proves that $N_G(E_{11})$ is maximal-proper 3-local if $q \neq 2$. If $q = 2$, then $N_G(E_{11}) = (3^4 \times \mathrm{S}_3^2).(2 \times \mathrm{S}_4)$, so $N_G(E_{11}) \leq N_G(E_{20}) = T.W$. If $i \in \{15, 16\}$, then $N_G(E_i)$ is independent of q, so we can construct it in any explicit version of $G = E_6^\varepsilon(q)$. A direct computation shows that $O_3(N_G(E_{16})) = 3^{2+6}$, hence $Z(O_3(N_G(E_{16}))) = 3^2 =_G E_j$ for some $j \in \{1, 2, 3\}$, and so $N_G(E_{16})$ is not maximal-proper 3-local. If $N_G(E_{15})$ is not maximal-proper 3-local, then, by Lemmas 2.4 and 2.5, we may suppose that $N_G(E_{15}) \leq N_G(E_j)$ for some maximal-proper 3-local $N_G(E_j)$ with $Z < E_j < E_{15}$, in particular, $j \in \{1, 2, 3, 5, 8\}$. Since $N_G(E_{15})$ has a composition factor $\mathrm{SL}_3(3)$, we deduce that $j \in \{1, 2, 3, 5\}$. A direct computation shows that $N_G(E_{15})$ is perfect, so $N_G(E_{15}) \leq N_G(E_j)$ implies that $N_G(E_{15}) \leq C_G(E_j)$. But then $\mathrm{SL}_3(3)$ centralises the 2- or 3-dimensional subspace E_j of the 4-dimensional space E_{15}, which is impossible. This contradiction proves that $N_G(E_{15})$ is maximal-proper 3-local. □

4. Cyclic subgroups of G of order 9

In view of Lemma 2.6, we are interested in those cyclic subgroups of G of order 9 which contain the center Z of G. Recall that $a \geq 1$ is defined by $3^a \mid (q - \varepsilon)$ and $3^{a+1} \nmid (q - \varepsilon)$.

Proposition 4.1. *Let $E \leq G$ be a cyclic subgroup of order 9 with $Z < E$. Then $a \geq 2$ and, up to G-conjugacy, there are two such subgroups E_{21} and E_{22} with*

$$C_G(E_{21}) = (q - \varepsilon) \circ_{4_\varepsilon} \mathrm{Spin}_{10}^+(q).4_\varepsilon, \qquad N_G(E_{21}) = C_G(E_{21})$$
$$C_G(E_{22}) = (q - \varepsilon) \circ_{10_\varepsilon} (\mathrm{SL}_2(q) \times \mathrm{SL}_5^\varepsilon(q)).10_\varepsilon, \qquad N_G(E_{22}) = C_G(E_{22}).$$

Both groups $N_G(E_{21})$ and $N_G(E_{22})$ are maximal-proper 3-local.

Proof. It follows from [16, Table 4.7.3A] that there exists an element $y \in G$ of order 9 with $y^3 \in Z$ if and only if $a \geq 2$; in this case, up to G-conjugacy, there are exactly two such elements z_{D} and z_{E}. Let $E_{21} = \langle z_{\mathrm{D}} \rangle$ and $E_{22} = \langle z_{\mathrm{E}} \rangle$. The local structure of E_{21} and E_{22} follows from [16, Table 4.7.3A] and Lemma 3.3. Since both $Z(C_G(E_{21}))$ and $Z(C_G(E_{22}))$ are cyclic, it follows from Lemma 2.6c) that $C_G(E_{21})$ and $C_G(E_{22})$ are maximal-proper 3-local subgroups of G (if $a \geq 2$). □

5. Extraspecial 3-subgroups in $G = E_6^\varepsilon(q)$

We now consider the extraspecial 3-subgroups in G, containing Z; we start with two preliminary sections.

5.1. *Radical subgroups in* $\mathrm{SL}_6^\varepsilon(q)$ *of symplectic type*

Recall that a p-group has **symplectic type** if every characteristic subgroup is cyclic. If p is odd, then a p-group Y of symplectic type is a central product of the cyclic subgroup $Z(Y)$ and $E = p_\pm^{1+2\gamma}$ for some $\gamma \geq 0$, see [15, Theorem 5.4.9]. In this section we classify radical subgroups in $\mathrm{SL}_6^\varepsilon(q)$ of symplectic type; these results will be useful later. We write

$$L = \mathrm{SL}_6^\varepsilon(q) \leq K = \mathrm{GL}_6^\varepsilon(q) = \mathrm{GL}^\varepsilon(V)$$

where V is a 6-dimensional linear (unitary) space.

Proposition 5.1. *If $R \in \mathcal{R}_3(L)$ is of symplectic type, then*

$$R \in_L \{O_3(L) = 3,\ K_1 = 3_+^{1+2}\} \quad \textit{if } a = 1, \textit{ and}$$
$$R \in_L \{O_3(L) = 3,\ K_1 = 3_+^{1+2},\ K_2 = 3_+^{1+2},\ K_3 = 3_+^{1+2},\ R_1 = 3^a,\ R_2 = 3^a\} \quad \textit{if } a \geq 2,$$

where $K_i \neq_L K_j$ for $i \neq j$. If $R \cong 3_+^{1+2}$, then $q \geq 4$ and $C_L(R) = 3 \times \mathrm{SL}_2(q)$; moreover, $\mathrm{Out}_L(R) = Q_8$ if $a = 1$ and $\mathrm{Out}_L(R) = \mathrm{SL}_2(3)$ if $a \geq 2$. In both cases, the order 2 outer-diagonal automorphism of L centralises each radical $R \cong 3_+^{1+2}$ of L. Moreover,

$$\begin{aligned} C_L(R_1) &= N_L(R_1) = (q-\varepsilon) \circ_{4^*} ((\mathrm{SL}_4^\varepsilon(q) \times \mathrm{SL}_2^\varepsilon(q)).4^*) \\ C_L(R_2) &= N_L(R_2) = (q-\varepsilon) \circ_{5^*} (\mathrm{SL}_5^\varepsilon(q).5^*). \end{aligned}$$

PROOF. Let $R \in \mathcal{R}_3(L) \setminus \{O_3(Z(L))\}$ be of symplectic type, so that $R = XE$ where $X = Z(R)$ is cyclic of order 3^β and $E = 3_\pm^{1+2\gamma}$. By Maschke's Theorem, the space V is a semisimple R-module. Since the generator of X is semisimple in K, we have

$$C_K(X) = \prod\nolimits_{i=1}^{t} \mathrm{GL}_{m_i}^{\varepsilon_i}(q^{\alpha_i}) \tag{5.1}$$

with $\sum_{i=1}^t m_i\alpha_i = 6$; we refer to [13, Proposition (1A)] for the precise conditions on the parameters ε_i, m_i, and α_i. Let $U_i \leq V$ be the underlying space of $\mathrm{GL}_{m_i}^{\varepsilon_i}(q^{\alpha_i})$, with GF($q$)-dimension $m_i\alpha_i$, so that $V = U_1 \oplus \ldots \oplus U_t$. Since $R \leq C_K(X)$, each U_i is an R-module and an E-module. Since R is radical in L, we have

$$O_3(Z(C_K(X))) \cap L \leq X,$$

and $O_3(C_L(R)) = Z(R) = X$. Since X is cyclic, it follows that $t \leq 2$.

We now first consider the case that $\gamma \geq 1$. Recall that every faithful irreducible E-module over GF(q^2) has dimension 3^γ, see [15, Theorem 5.5.5], so $\gamma = 1$ and we may suppose $m_1 \geq 3$. If $t = 1$, then $m_1\alpha_1 = 6$, and it follows that $C_K(X) = K$ or $C_K(X) = \mathrm{GL}_3^\varepsilon(q^2)$. In the former case, $X \leq Z(L)$, that is, $X = 3$. In the latter case, $X \leq Z(\mathrm{GL}_3^\varepsilon(q^2))$, so $|X|$ divides 3^a and so X is diagonalisable over GF(q). But then [13, Proposition (1A)] shows that $C_K(X) \not\cong \mathrm{GL}_3^\varepsilon(q^2)$. Thus, if $t = 1$, then $X = 3 \leq Z(L)$. If $t = 2$, then $\alpha_1 = 1$ and $3 \leq m_1 < 6$. If $m_1 \neq 3$, then U_1 is not an irreducible E-module, and therefore $U_1 = W_1 \oplus W_2$ for some faithful irreducible E-module W_1 of dimension 3 and some E-module W_2 of dimension $m_1 - 3$. It follows that $C_K(R) = C_{\mathrm{GL}(W_1)}(E) \times C_{\mathrm{GL}(W_2)}(E) \times C_{\mathrm{GL}(U_2)}(E)$, which implies that $O_3(C_L(R))$ is non-cyclic, a contradiction to what we have established above. This contradiction shows that $m_1 = 3$, and therefore $m_2\alpha_2 = 3$. Now either $C_K(X) = (\mathrm{GL}_3^\varepsilon(q))^2$ or $C_K(X) = \mathrm{GL}_3^\varepsilon(q) \times \mathrm{GL}_1^\varepsilon(q^3)$, and in both cases $O_3(Z(C_K(X))) \cap L$ is not cyclic, which is a contradiction to $O_3(Z(C_K(X))) \cap L \leq X$. Thus we have proved that $t = 1$ and $X = 3 \leq Z(L)$. In particular, $R = E = 3_{\pm}^{1+2}$ since $\gamma = 1$. Suppose, for a contradiction, that $R = E = 3_{-}^{1+2} = \langle x, y\rangle$ with $|x| = 9$ and $x^3 = z \in Z(L)$, so that $y \in N_G(\langle x\rangle)\setminus C_G(x)$. This is impossible by Section 4, which proves that $R = E = 3_{+}^{1+2}$.

Let $V_1 \leq V$ be a faithful irreducible R-submodule, hence $\dim V_1 = 3$. Since V is semisimple, there exists an R-submodule $V_2 \leq V$ with $V = V_1 \oplus V_2$. For $i = 1, 2$ let $L_i^{\varepsilon} = \mathrm{SL}^{\varepsilon}(V_i) \leq K_i^{\varepsilon} = \mathrm{GL}^{\varepsilon}(V_i)$. If $V_1 \not\cong V_2$ as R-modules, then, since V_1 is an irreducible R-module, a short direct calculation shows that $C_K(R) \leq C_{\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)}(R)$. This yields $O_3(Z(L_1^{\varepsilon})) \times O_3(Z(L_2^{\varepsilon})) \leq Z(R)$ since R is radical. But $O_3(Z(L_1^{\varepsilon})) \times O_3(Z(L_2^{\varepsilon})) = 3^2 \leq Z(R)$ is impossible, thus we must have $V_1 \cong V_2$ as R-modules. In particular, we can assume that $R = \{I_2 \otimes g \mid g \in 3_{+}^{1+2}\}$ for some $3_{+}^{1+2} \leq L_1^{\varepsilon}$. Schur's Lemma [15, Theorem 3.5.3] implies that $C_{\mathrm{GL}(V_i)}(R) = Z(\mathrm{GL}(V_i)) = q - \varepsilon$. Thus, $C_K(R) = \mathrm{GL}_2^{\varepsilon}(q) \otimes I_3$ with $Z(C_K(R)) = Z(K)$; note that $g \otimes I_3 \in C_K(R)$ with $g \in \mathrm{GL}_2^{\varepsilon}(q)$ has determinant $\det(g)^3$, which implies that

$$C_L(R) = 3 \times \mathrm{SL}_2(q);$$

the direct factor 3 is generated by $I_3 \otimes \mathrm{diag}(y, 1)$ where $y \in \mathrm{GF}(q^2)$ is an element of order 3. The structure of $C_K(R)$ and $C_L(R)$ also proves the last assertion, namely, that the order 2 outer-diagonal automorphism of L centralises R.

In this paragraph we freely use [11, (3B) & (3C)], [1, (1B) & (1C)], and [3, Lemma 5.3]. Recall that $\tau \in \mathrm{GF}(q^2)$ is an element of maximal 3-power order 3^a. If $a = 1$, then $3_{+}^{1+2} \leq L_1^{\varepsilon}$ is a 3-Sylow subgroup, $\mathrm{Out}_{L_1^{\varepsilon}}(3_{+}^{1+2}) = Q_8$, and $\mathrm{Out}_{K_1^{\varepsilon}}(3_{+}^{1+2}) = \mathrm{SL}_2(3) = \mathrm{Sp}_2(3)$. Since $\mathrm{SL}_2(3) = 3 \ltimes Q_8$, we can assume that $n = \mathrm{diag}(1, 1, \tau) \in K_1^{\varepsilon}$ normalises 3_{+}^{1+2}; note that n has order 3 since we assume $a = 1$. Let $N = \{I_2 \otimes g \mid g \in N_{K_1^{\varepsilon}}(3_{+}^{1+2})\}$. A direct computation shows that the subgroup $\mathrm{Out}^0(3_{+}^{1+2})$ of $\mathrm{Out}(3_{+}^{1+2})$ consisting of all elements centralising $Z(3_{+}^{1+2})$ satisfies $\mathrm{Out}^0(3_{+}^{1+2}) = \mathrm{SL}_2(3)$; this implies that $N_K(R) = C_K(R)RN$. Since no 3-element $I_2 \otimes \mathrm{diag}(1, 1, \alpha) \in N$ lies in L, it follows that $\mathrm{Out}_L(R) = Q_8$, as claimed. Now consider the case $a \geq 2$. Up to L_1^{ε}-conjugacy, there are three radical subgroups $Y_i = 3_{+}^{1+2} \leq L_1^{\varepsilon}$ for $i \in \{1, 2, 3\}$, and each $\mathrm{Out}_{L^{\varepsilon}}(Y_i) = \mathrm{SL}_2(3) = \mathrm{Out}_{K^{\varepsilon}}(Y_i)$; we can assume that $\mathrm{diag}(1, 1, \tau)$ permutes Y_1, Y_2, Y_3 cyclically. It follows that, up to K-conjugacy, there is a unique radical subgroup $R = 3_{+}^{1+2} \leq K$ with $C_K(R) = \mathrm{GL}_2^{\varepsilon}(q) \otimes I_3$ and $N_K(R) = C_K(R)RN$, where $N = \{I_2 \otimes g \mid g \in N_{K_1^{\varepsilon}}(3_{+}^{1+2})\}$. As in the previous case, we deduce that $\mathrm{Out}_L(R) = \mathrm{SL}_2(3)$. Note that if $q = 2$, then R is not radical; recall that $\mathrm{SL}_2(2) \cong \mathrm{S}_3$. Thus, if there is a radical $R = E$, then $q \geq 4$.

Finally, suppose $\gamma = 0$, so that $R = X = 3^{\beta}$ is cyclic. First let $\beta > a$. In this case, since 3^a is the largest 3-power dividing $q - \varepsilon$, it is straightforward to deduce from (5.1) that $\beta = a + 1$ and $\alpha_1 = 3$ or $\alpha_1 = 6$; note that $t \leq 2$ follows as before. In particular, X lies in a maximal torus S of K which has a direct factor $q^3 - \varepsilon$ or $q^6 - \varepsilon$. But $X \cap Z(K) = O_3(Z(L)) = 3$, so X cannot be a subgroup of L containing Z. This contradiction proves that $\beta \leq a$. In this case, X is diagonalisable in L. If $t = 1$, then $C_K(X) = K$ and $R = O_3(Z(L)) = 3$. If $t = 2$, then $C_K(X) = \mathrm{GL}_{m_1}^{\varepsilon}(q) \times \mathrm{GL}_{6-m_1}^{\varepsilon}(q)$ for some $m_1 \in \{1, \ldots, 5\}$. Since X is cyclic, $\gcd(m_1, 3) = 1$, and so we can assume that $m_1 = 1$ or $m_1 = 2$, that is, X is generated by $y = \mathrm{diag}(w, w, w, w, w, w^{-5})$ or $y = \mathrm{diag}(w, w, w, w, w^{-2}, w^{-2})$ for some $w \in \mathrm{GF}(q^2)$ of order 3^a. Note that if $a = 1$, then $|w| = 3$ and $y = wI_6 \in Z(L)$ and $C_K(X) = K$, which is impossible; thus $a \geq 2$ in this case. Section 4 implies that $C_K(R_i) = N_L(R_i)$ for $i = 1, 2$. □

Corollary 5.2. *Let* $L = \mathrm{SL}_6^{\varepsilon}(q)$. *If* $Q = 3_{+}^{1+2} \leq L$ *satisfies* $O^{r'}(C_L(Q)) = \mathrm{SL}_2(q)$, *then, up to* L-*conjugacy, there is one such subgroup if* $a = 1$, *and three such subgroups if* $a \geq 2$. *In addition, if* $a = 1$, *then* $\mathrm{Out}_G(Q) \cong Q_8$, *and* $\mathrm{Out}_G(Q) = \mathrm{SL}_2(3)$ *otherwise. If* $q \geq 4$, *then* Q *is radical.*

PROOF. The claim follows the lines of the proof of Proposition 5.1; we sketch the main steps. First decompose $V = V_1 \oplus V_2$ where V_1 is a faithful and irreducible Q-module. If $V_1 \not\cong V_2$ as Q-modules, then $C_K(Q) = C_{\mathrm{GL}^{\varepsilon}(V_1)}(Q) \times C_{\mathrm{GL}^{\varepsilon}(V_2)}(Q)$ and $O^{r'}(C_L(Q)) \neq \mathrm{SL}_2(q)$, contradiction our assumption. Thus, $V_1 \cong V_2$ as Q-modules, and we can assume that $Q = \{I_2 \otimes g \mid g \in 3_{+}^{1+2}\}$. It follows that

$C_L(Q) = 3 \times \mathrm{SL}_2(q)$, and so $C_K(Q) = \mathrm{GL}_2^\varepsilon(q)$. In particular, $Z(Q) \le Z(L)$, and one can show that $\mathrm{Out}_L(Q) = Q_8$ or $\mathrm{Out}_L(Q) = \mathrm{SL}_2(3)$, that is, Q is radical. Now the claim follows as before. The proof of $\mathrm{Out}_G(Q)$ is analogous to the argument given in the proof of Proposition 5.1. □

Lemma 5.3. *Let $A = \mathrm{GL}_3^\varepsilon(q)$, $B = \mathrm{SL}_3^\varepsilon(q)$, and $C = Z(B) = 3$. Up to conjugacy, A has a unique subgroup $E = 3_+^{1+2}$ with $C \le E$; we have $E \le B$. If $a = 1$, then E is also unique up to conjugacy in B. If $a \ge 2$, then, up to conjugacy, B has three subgroups $E = 3_+^{1+2}$ containing C. In all cases, $\mathrm{Out}_B(E)$ has type Q_8 if $a = 1$, and type $\mathrm{SL}_2(3)$ if $a \ge 2$.*

PROOF. Let $E = 3_+^{1+2} = \langle g, h\rangle \le A$. Note that $[g,h]$ generates C, and so h permutes the eigenspaces of g. Thus, up to B-conjugacy, we can assume that $g = \mathrm{diag}(1,\omega,\omega^2)$ and $h = \mathrm{diag}(r_1,r_2,r_3)\sigma$ with $\sigma = (1,2,3)$. Note that $r_1r_2r_3 = 1$ since $|h| = 3$, which shows that $E \le B$. Up to A-conjugacy, acting with $u \in C_A(g) = (q-\varepsilon)^3$, we may suppose $h = \pi$; this shows that E is unique up to A-conjugacy. A similar argument shows that, up to B-conjugacy, there are three subgroups $E = 3_+^{1+2}$ with $C \le E$. For the normaliser structure see [3, Lemma 5.3]. □

5.2. *Another preliminary lemma*

Recall that $\omega \in \mathrm{GF}(q)$ is an element of order 3 and $Z = Z(G) = \langle z\rangle$.

Lemma 5.4. *Write $C = C_G(3\overline{\mathrm{A}}) = (q-\varepsilon) \circ_{2_\varepsilon} (L.2_\varepsilon)$ with $L = O^{r'}(C) = \mathrm{SL}_6^\varepsilon(q)$ and $O_3(Z(L)) = Z$.*

a) *If $u_1 = \mathrm{diag}(\omega I_3, \omega^{-1} I_3)$, $u_2 = \mathrm{diag}(\omega, \omega^{-1}, I_4)$, and $u_3 = \mathrm{diag}(I_2, \omega I_2, \omega^{-1} I_2)$, then*

$$\langle z, u_1\rangle =_G 3\overline{\mathrm{C}}, \quad \langle z, u_2\rangle =_G 3\overline{\mathrm{A}}, \quad \langle z, u_3\rangle =_G 3\overline{\mathrm{B}}.$$

b) *Let $P = 3_+^{1+2} \le L$ with $Z(P) = Z$ such that $O^{r'}(C_L(P)) = \mathrm{SL}_2(q)$. Let $Q = \langle z_{\mathrm{A}}, P\rangle \le C$ and $E = 3_+^{1+2} \le Q$ with $Z(E) = Z$. Then $Z < E \cap P$ and $E \cap 3\mathrm{B} \ne \emptyset$. If $E \cap 3\mathrm{C} = \emptyset$, then $E = P$.*

PROOF. a) Let $T_L = (q-\varepsilon)^5$ be a maximal torus of L containing each u_i. We may suppose that $C_G(Z(C) \circ_{2_\varepsilon} T_L) = T$. If $V = \Omega_1(O_3(T))$, then $W = W(E_6) = \mathrm{SO}_6^-(2) = \mathrm{SO}_5(3)$ acts faithfully on V and centralises Z; recall that the action of W on V is given as in the proof of Table I. Let $U = V/Z$; it is shown in [6, p. 71] that W acts on U as group $\mathrm{SO}_5(3)$. Take $y_{\mathrm{A}}Z \in U = 3^5$ such that $C_W(y_{\mathrm{A}}Z) = \mathrm{S}_6$, cf. Table I. Let U_a be the orthogonal complement of $\langle y_{\mathrm{A}}Z\rangle$ in U. A direct computation shows that we can choose $w_1, w_2, w_3 \in U_a$ such that $C_{\mathrm{S}_6}(w_1) = \mathrm{S}_3 \times \mathrm{S}_3$, $C_{\mathrm{S}_6}(w_2) = \mathrm{S}_4$, and $C_{\mathrm{S}_6}(w_3) = 2^3$; moreover, $C_W(w_1) = (\mathrm{S}_3)^3$, $C_W(w_2) = \mathrm{S}_6$, and $C_W(w_3) = 2_+^{1+4}.\mathrm{S}_3$. Thus $\langle w_1\rangle =_W 3\overline{\mathrm{C}}/Z$, $\langle w_2\rangle =_W 3\overline{\mathrm{A}}/Z$, and $\langle w_3\rangle =_W 3\overline{\mathrm{B}}/Z$. Note that each $\langle z, u_i\rangle/Z$ is W-conjugate to $\langle w_j\rangle$ for some j. Since $N_L(T_L)/T_L = \mathrm{S}_6$, the well-known structure of $C_{\mathrm{S}_6}(u_i)$ implies the claim.

b) Write $E = \langle g, h\rangle$ and $g = z_{\mathrm{A}}^\ell g_1$ and $h = z_{\mathrm{A}}^k h_1$ for some $\ell, k \in \{0,1,2\}$ and $g_1, h_1 \in P$. Note that $[g,h] = [g_1,h_1] \in Z$ since $g_1, h_1 \in C_G(z_{\mathrm{A}})$. Thus, if $\ell = 0$, then $\langle g, z\rangle \le P \cap E$, hence $Z < P \cap E$; similarly, if $k = 0$, then $\langle h, z\rangle \le E \cap P$ and $Z < E \cap P$. Let $\ell, k \in \{1,2\}$ in the following. Replacing g and h by g^{-1} and h^{-1}, if necessary, we may suppose that $g = z_{\mathrm{A}} g_1$ and $h = z_{\mathrm{A}} h_1$, hence $[g,h] = [g_1,h_1] \in Z$ and $g^{-1}h = g_1^{-1}h_1 \in (P \cap E) \setminus Z$, so $Z < P \cap E$. By Corollary 5.2, the group $P \in \mathcal{R}_3(L)$ is radical and of symplectic type. Now Proposition 5.1 shows that $\mathrm{Out}_L(P) = Q_8$ or $\mathrm{Out}_L(P) = \mathrm{SL}_2(3)$, depending on whether $a = 1$ or $a \ge 2$, respectively; in particular, all the non-central elements of P are L-conjugate. Write $P = \langle u, v\rangle$; as shown in the proof of Proposition 5.1, we may suppose $u = I_2 \otimes u_1$ and $v = I_2 \otimes v_1$ for some $u_1, v_1 \in L^\varepsilon = \mathrm{SL}_3^\varepsilon(q)$ with $\langle u_1, v_1\rangle = 3_+^{1+2}$. In particular, u_1 is L^ε-conjugate to $\mathrm{diag}(1,\omega,\omega^{-1})$, and a) proves that $\langle z, u\rangle =_G 3\overline{\mathrm{B}}$. In particular, $\langle z, w\rangle =_G 3\overline{\mathrm{B}}$ for every $w \in P \setminus Z$, so $E \cap 3\mathrm{B} \ne \emptyset$.

To prove the last claim of the lemma, let $E \cap 3\mathrm{C} = \emptyset$ and suppose, for a contradiction, that $E \ne P$; without loss of generality, $g \in E \setminus P$, say $g = z_{\mathrm{A}} g_1 \in E$ for some $g_1 \in P$. If $g_1 \in Z(P) = Z$, then $z_{\mathrm{A}} \in E$ and hence $\langle z, z_{\mathrm{A}}\rangle \le Z(E)$, which is impossible. Thus, $g_1 \notin Z(P)$ and so $\langle z, g_1\rangle =_G 3\overline{\mathrm{B}}$ as shown above; in particular, we can suppose that $g_1 = z_{\mathrm{B}}$ since all non-central elements of P are conjugate. It follows now from Lemma 3.2d) that $\langle z, g\rangle =_G 3\overline{\mathrm{C}}$. This contradicts our assumption $E \cap 3\mathrm{C} = \emptyset$, and thus $E = P$. □

5.3. Extraspecial 3*-subgroups*

The next proposition is the main result of this section and considers the extraspecial subgroups of $G = E_6^{\varepsilon}(q)$ which contain the center $Z = Z(G) = \langle z\rangle$. Throughout the proof, we use the following notation. We write

$$C_G(3\overline{\mathrm{C}}) = L.x_{\mathrm{C}} \quad \text{and} \quad L = (L_1 \times L_2 \times L_3)/D$$

with each $L_i = L^{\varepsilon} = \mathrm{SL}_3^{\varepsilon}(q)$ and $Z(L_{\varepsilon}) = \langle d\rangle$, and $D = \langle (d, d, d)\rangle \leq L_1 \times L_2 \times L_3$. Note that

$$N_G(3\overline{\mathrm{C}}) = C_G(3\overline{\mathrm{C}}).\mathrm{S}_3 = C_G(3\overline{\mathrm{C}}).\langle \gamma_{\mathrm{C}}, \xi\rangle,$$

where x_{C} acts as an order 3 outer-diagonal automorphism on each L_i, and ξ permutes the three factors L_i cyclically, cf. [16, Table 4.7.3A]. As before, let 3^a be the largest 3-power dividing $q - \varepsilon$. Part a) of the next proposition is a preliminary result which will be established in the course of proving parts b+c).

Proposition 5.5. *If $E \leq G$ is of type $3_+^{1+2\gamma}$ with $\gamma \geq 1$ and $Z(E) = Z(G)$, then $\gamma = 1$ and either $3\mathrm{B} \cap E \neq \emptyset$ or $3\mathrm{C} \cap E \neq \emptyset$. Moreover, the following hold.*

a) *In $N_G(3\overline{\mathrm{C}}) = C_G(3\overline{\mathrm{C}}).\langle \gamma_{\mathrm{C}}, \xi\rangle$, we can suppose that $|x_{\mathrm{C}}| = 3$, $[x_{\mathrm{C}}, \xi] = 1$, and $x_{\mathrm{C}}^{\xi} = x_{\mathrm{C}}^{-1}$.*

b) *Suppose $3\mathrm{C} \cap E = \emptyset$, so that $\langle z, y\rangle =_G 3\overline{\mathrm{B}}$ for every $y \in E \setminus Z(E)$. If $a = 1$, then E is unique up to conjugacy. If $a \geq 2$, then there exist three such groups, up to conjugacy. We have $C_G(E) = 3 \times G_2(q)$, and $N_G(E) = (3_+^{1+2} \times G_2(q)).Q_8$ for $a = 1$ and $N_G(E) = (3_+^{1+2} \times G_2(q)).\mathrm{SL}_2(3)$ for $a \geq 2$. The group $N_G(E)$ is maximal-proper 3-local.*

c) *Suppose $3\mathrm{C} \cap E \neq \emptyset$. In this case, $3\mathrm{B} \cap E \neq \emptyset$ and we can suppose $z_{\mathrm{B}} \in E$ and so $E \leq N_G(3\overline{\mathrm{B}})$. If $a = 1$, then E is unique up to conjugacy. If $a \geq 2$, then there exist three such groups, up to conjugacy. We have $C_G(E) = 3 \times (\mathrm{PSL}_3^{\varepsilon}(q).3)$, and $N_G(E) = (3_+^{1+2} \times \mathrm{PSL}_3^{\varepsilon}(q).3).2 = N_{N_G(3\overline{\mathrm{B}})}(E)$ for $a = 1$ and $N_G(E) = (3_+^{1+2} \times \mathrm{PSL}_3^{\varepsilon}(q).3).6 = N_{N_G(3\overline{\mathrm{B}})}(E)$ if $a \geq 2$.*

d) *The group $N_G(E)$ is maximal-proper 3-local if and only if $E \cap 3\mathrm{C} = \emptyset$.*

Proof. If $E = 3_+^{1+2\gamma} \leq G$ with $\gamma > 1$, then $E = U_1 \circ_3 U_2$ with $U_1 = 3_+^{1+2}$ and $U_2 = 3_+^{1+2(\gamma-1)}$. Thus, as a first step, we consider $\gamma = 1$. If $E = 3_+^{1+2} \leq G$ satisfies $Z(E) = \langle z\rangle = Z(G)$, then $E = \langle x, y\rangle$ with $[x, y] = z$. Writing $U = \langle z, x\rangle$, we have $E \leq N_G(U)$ and $y \notin C_G(U)$; since $N_G(3\mathrm{A}) = C_G(3\overline{\mathrm{A}}).2$, we must have $U =_G 3\overline{\mathrm{B}}$ or $U =_G 3\overline{\mathrm{C}}$. We now proceed in several steps.

(1) We first construct all $E = \langle x, y\rangle = 3_+^{1+2}$ containing z_{B}. Suppose $z_{\mathrm{B}} \in E \leq N_G(3\overline{\mathrm{B}}) = C_G(3\overline{\mathrm{B}}).\mathrm{S}_3$, where $C_G(3\overline{\mathrm{B}}) = (q-\varepsilon)^2 \circ_{(2^*)^2} \mathrm{Spin}_8^+(q).(2^*)^2$. It follows from our computation in Section 3.2 together with [16, Table 4.7.3A] that

$$N_G(3\overline{\mathrm{B}}) = C_G(3\overline{\mathrm{B}}).\langle \gamma_{\mathrm{B}}', \gamma_{\mathrm{B}}\rangle,$$

where $\gamma_{\mathrm{B}}' = \mu{:}\gamma$ acts as $(u, v)^{\mu} = (v, (uv)^{-1})$ on the two factors of $(q-\varepsilon)^2$, and γ acts as a graph automorphism of order 3 on $\mathrm{Spin}_8^+(q)$. Note that $z \in C_{(q-\varepsilon)^2}(\gamma_B')$, so $z = (b, b)$ for some $b \in \mathrm{GF}(q^2)$ of order 3. By [16, Table 4.7.3A] again, $\mathrm{Spin}_8^+(q)$ has exactly two graph automorphisms γ_1 and γ_2, and they satisfy

$$C_{\mathrm{Spin}_8^+(q)}(\gamma_1) = G_2(q) \quad \text{and} \quad C_{\mathrm{Spin}_8^+(q)}(\gamma_2) = \mathrm{PSL_3}^{\varepsilon}(q).3,$$

where the outside 3 of $\mathrm{PSL}_3^{\varepsilon}(q).3$ induces an outer-diagonal automorphism of order 3 on $\mathrm{PSL}_3^{\varepsilon}(q)$; note that [16, Definition 2.5.13] implies that γ_2 is induced by $t\gamma_1$ for some $t \in \mathrm{Spin}_8^+(q)$; in particular, we can assume that $\gamma_i = t_i\gamma'_{\mathrm{B}}$ for some $t_i \in \mathrm{Spin}_8^+(q)$. For $i = 1, 2$ let $Y_i = \langle x, y_i\rangle$ with $y_i = \mu{:}\gamma_i$, so that each $Y_i \cong 3_+^{1+2}$ with $Z(Y_i) = Z(G)$, and

$$C_G(Y_1) = 3 \times G_2(q) \quad \text{or} \quad C_G(Y_2) = 3 \times \mathrm{PSL}_3^\varepsilon(q).3.$$

Now suppose $E = 3_+^{1+2} \leq G$ with $z_{\mathrm{B}} \in E$. In this case, $E = \langle z_{\mathrm{B}}, y\rangle$ for some $y \in N_G(3\overline{\mathrm{B}}) \setminus C_G(3\overline{\mathrm{B}})$, and, up to conjugacy in $\mathrm{Spin}_8^+(q)$, the element y induces the same action on $\mathrm{Spin}_8^+(q)$ as γ_i for some $i \in \{1, 2\}$. We may therefore suppose that $y^{-1}y_i \in C_{N_G(3\overline{\mathrm{B}})}(\mathrm{Spin}_8^+(q)) = (q-\varepsilon)^2$, and so $y = ty_i$ for some $t \in (q-\varepsilon)^2$. Note that for every $t \in (q-\varepsilon)^2$ the element $y = ty_i$ has order 3 and satisfies $E = \langle z_{\mathrm{B}}, y\rangle \cong 3_+^{1+2}$. If $s = (u, v) \in (q-\varepsilon)^2$, then

$$y^s = s^{-1}s^{\mu^2}y = (u^{-2}v^{-1}, v^{-1}u)y.$$

This shows that, up to conjugacy in $(q-\varepsilon)^2$, we can assume that $t = (t_1u^3, 1)$ with $u \in (q-\varepsilon)$: first conjugate with $s = (1, t_2)$, and then with $(1, u^{-1})$. In particular, if t_1 is a $3'$-element, then we can assume that $t = (1, 1)$, and so $E =_{C_G(3\overline{\mathrm{B}})} Y_i$. In conclusion, up to conjugacy in $C_G(3\overline{\mathrm{B}})$, we can suppose that $E = \langle z_{\mathrm{B}}, y\rangle$ with $y = ty_i$ for some $t \in \{(1, 1), (\alpha, 1), (\alpha^2, 1)\}$, where $|\alpha| = 3^a$.

If $a = 1$, then $t = (\alpha, 1) \in O_3(C_G(3\overline{\mathrm{B}})) \leq E$, and hence $E =_G Y_i$; in this case, up to conjugacy, there are exactly two groups $E = 3_+^{1+2}$ containing $3\overline{\mathrm{B}}$ and satisfying $Z(E) = Z$, namely, Y_1 and Y_2 with

$$C_G(Y_1) = 3 \times G_2(q) \quad \text{and} \quad C_G(Y_2) = 3 \times \mathrm{PSL}_3^\varepsilon(q).3,$$

where the outside 3 of $C_G(Y_2)$ acts as an outer-diagonal automorphism; define $Y_i' = Y_i'' = Y_i$ for $i = 1, 2$.

Now suppose $a \geq 2$. Every $E = 3_+^{1+2} \leq G$ with $z_{\mathrm{B}} \in E$ and $Z(E) = Z$ is G-conjugate to one of

$$Y_i = \langle z_{\mathrm{B}}, y_i\rangle, \quad Y_i' = \langle z_{\mathrm{B}}, (\alpha, 1)y_i\rangle, \quad Y_i'' = \langle z_{\mathrm{B}}, (\alpha^2, 1)y_i\rangle \quad \text{for } i = 1, 2$$

with

$$C_G(Y) = \begin{cases} 3 \times G_2(q) & \text{if } Y \in \{Y_1, Y_1', Y_1''\} \\ 3 \times \mathrm{PSL}_3^\varepsilon(q).3 & \text{if } Y \in \{Y_2, Y_2', Y_2''\}. \end{cases}$$

All the subgroups of type 3^2 of Y_1, Y_1' and Y_1'' containing Z have projective type $3\overline{\mathrm{B}}$ since $G_2(q) \not\leq C_G(3\overline{\mathrm{C}})$.

(2) We show that if $a \geq 2$, then Y_1, Y_1', Y_1'' are non-conjugate in G; recall that $Y_1 = Y_1' = Y_1''$ if $a = 1$. It follows from [6, (5.7)(6)] and [17, Table 1] that $\overline{G}$ has a maximal subgroup $\overline{M} = \mathrm{SL}_3 \times G_2$. In [17, Table 3] it is shown that the fixed-point set of $\overline{M}/Z$ under the Frobenius map σ satisfies $O^{r'}((\overline{M}/Z)^\sigma) = \mathrm{PSL}_3^\varepsilon(q) \times G_2(q)$; since $a \geq 2$, the group $G_2(q)$ is simple, and we conclude that $M = A \times G_2(q)$ with $A = \mathrm{SL}_3^\varepsilon(q)$ is a maximal subgroup of G. Since $|Z| = 3$, we must have $Z \leq Z(M)$, hence $Z(A) = Z$. Since $a \geq 2$, Lemma 5.3 shows that A contains exactly three A-classes of subgroups 3_+^{1+2}, with representatives E_1, E_1, E_3, such that $\mathrm{Out}_A(E_i) = \mathrm{SL}_2(3)$. Hence $C_M(E_i) = Z \times G_2(q)$. Note that $G_2(q) \not\leq C_G(3\overline{\mathrm{C}})$, hence $\langle z, g\rangle =_G 3\overline{\mathrm{B}}$ for every $g \in E_i \setminus Z$. Part (1) shows that we can assume $\{E_1, E_2, E_3\} \subseteq \{Y_1, Y_1', Y_1''\}$, hence $C_M(E_i) = C_G(E_i)$. Suppose $E_i^h = E_j$ for some $h \in G$ and $i, j \in \{1, 2, 3\}$, so that $O^{r'}(C_G(E_i))^h = O^{r'}(C_G(E_j)) = G_2(q)$, which forces $h \in M$; the latter follows from $M = N_G(G_2(q))$ since $M < G$ is maximal. Thus we can decompose $h = h_1h_2$ for some $h_1 \in A$ and $h_2 \in G_2(q)$; now $E_i^{h_1} = E_j$ forces $i = j$. This proves the claim of Part (2).

(3) This is a preliminary step. Note that each $H \in \{G_2(q), \mathrm{PSL}_3^\varepsilon(q).3\}$ has an element $y \in H$ such that $C_H(y) = \mathrm{GL}_2^\varepsilon(q)$: If $H = G_2(q)$, then this follows from [16, Table 4.7.3(A)]. If $H = \mathrm{PSL}_3^\varepsilon(q).3$ and $a = 1$, then we can choose $y \in \mathrm{PSL}_3^\varepsilon(q).3 \setminus \mathrm{PSL}_3^\varepsilon(q)$ of order 3, induced by $\mathrm{diag}(1, 1, \tau)$; if $a \geq 2$, then there exists a suitable element $y \in \mathrm{PSL}_3^\varepsilon(q)$ of order 3, induced by $\mathrm{diag}(\tau^{3^{a-2}}, \tau^{3^{a-2}}, ((\tau^{3^{a-2}})^{-2})$. Now let $E \in \{Y_1, Y_1', Y_1'', Y_2, Y_2', Y_2''\}$ and define

$$Q = E \times \langle y\rangle$$

such that $C_G(Q) = Z \times \mathrm{GL}_2^{\varepsilon}(q) = (3 \times (q-\varepsilon)) \circ_{2^*} (\mathrm{SL}_2(q).2^*)$. The aim of Part (3) is to prove that

$$X = \Omega_1(Z(Q)) = 3^2 =_G 3\overline{\mathrm{A}}.$$

If $X = 3\overline{\mathrm{B}}$, then $Q \leq C_G(3\overline{\mathrm{B}}) = (q-\varepsilon)^2 \circ_{(2^*)^2} (\mathrm{Spin}_8^+(q).(2^*)^2)$, so $X = \Omega_1(O_3((q-\varepsilon)^2)))$. If S is a Sylow 3-subgroup of $C_G(3\overline{\mathrm{B}})$ containing Q, then $Z(G) = [Q,Q] \leq [S,S] \leq \mathrm{Spin}_8^+(q)$, which is impossible, hence $X \neq_G 3\overline{\mathrm{B}}$.

Now suppose, for a contradiction, that $X = 3\overline{\mathrm{C}}$, so $Q \leq C_G(X) = L.x_{\mathrm{C}}$ and $X = Z(L)$. Writing $x_{\mathrm{C}} = x_1{:}x_2{:}x_3$ and $J_i = \langle L_i, x_i\rangle$, we have $Q \leq L.x_{\mathrm{C}} \leq J/D = (J_1 \times J_2 \times J_3)/D$. Recall that each x_i acts as $o_i = \mathrm{diag}(1,1,\tau)$ on L_i. For $i \in \{1,2,3\}$ denote by Q_i the projection of Q to J_iD/D; note that $J_iD/D \cong J_i$, so we consider Q_i as a subgroup of J_i. First suppose that Q_i is nonabelian and consider $u \in (Q_i \cap L_i) \setminus Z(L_i)$, so that $|u| = 3$ and we may suppose $u = \mathrm{diag}(1,\omega,\omega^{-1})$. Since Q_i is nonabelian, there is $v \in Q_i \setminus \langle Z(L_i), u\rangle$ such that v permutes the eigenspaces of u cyclically; this yields $C_{L_i}(Q_i) = Z(L_i)$. If Q_i is abelian, then

$$C_{L_i}(Q_i) \in \{L_i, H_i = \mathrm{GL}_2^{\varepsilon}(q), T_i = (q-\varepsilon)^2, V_i = (q^2+\varepsilon q+1)\}.$$

Since $O^{r'}(C_G(Q)) = \mathrm{SL}_2(q)$, it follows that there is a unique i such that $C_{L_i}(Q_i) = \mathrm{GL}_2^{\varepsilon}(q)$; we can assume that $i = 1$. Since the exponent of Q is 3, the exponent of Q_1 is 3, and hence $Q_1 \leq J_1$ is not a subgroup of L_1. Note that $z \in Q$ and $\Omega_1(O_3(\mathrm{GL}_2^{\varepsilon}(q))) = Z(L_1) < Q_1$, and so $Q_1 = \langle Z(L_1), y_1\rangle$ with $y_1 \in \langle Z(L_1), x_1\rangle$; hence we may suppose that $x_1 \in Q_1$. In conclusion, we can assume that $xD \in Q$ for some $x = x_1{:}t_2x_2{:}t_3x_3$ with $t_2 \in L_2$ and $t_3 \in L_3$. Note that $Z(Q) \leq Z(L.x_{\mathrm{C}})$, so $C_{L_1D/D}(Q_1) \leq C_G(Q)$. Since $C_G(Q) = 3 \times \mathrm{GL}_2^{\varepsilon}(q) = C_{C_G(3\overline{\mathrm{C}})}(Q)$ and each $C_{L_i}(Q_i) \leq C_{C_G(3\overline{\mathrm{C}})}(Q)$, it follows from the list of possible centralisers $C_{L_i}(Q_i)$ above that both Q_2 and Q_3 are nonabelian. For $i \in \{1,2,3\}$ let E_i be the projection of E into $J_iD/D \cong J_i$. In the following let $j \in \{2,3\}$. Note that $t_jx_j \in Q_j$ and Q_j is nonabelian, hence $E \cong E_j < Q_j$. Moreover, $E_j \leq L_j$ by Lemma 5.3. In conclusion, $Q \cong Q_j$ and $Z(Q_j) = 3^2$. This is impossible since $Z(Q_j) \leq C_{L_j}(Q_j) = Z(L_j) = 3$, as shown above. Thus, $X \neq_G 3\overline{\mathrm{C}}$, and so $X =_G 3\overline{\mathrm{A}}$.

(4) We show that if $a = 1$, then $Y_1 =_G K_1$, and if $a \geq 2$, then $\{Y_1, Y_1', Y_1''\} =_G \{K_1, K_2, K_3\}$ as defined in Proposition 5.1. We continue with the notation of Part (3); let $E \in \{Y_1, Y_1', Y_1''\}$ and $Q = \langle E, y\rangle$ with $X = \Omega_1(Z(Q)) = 3\overline{\mathrm{A}}$. Thus, we have

$$Q \leq C_G(X) = (q-\varepsilon) \circ_{2^*} (\mathrm{SL}_6^{\varepsilon}(q).2^*).$$

Define $K = O^{r'}(C_G(X)) = \mathrm{SL}_6^{\varepsilon}(q)$ and $P = Q \cap K$, so that $P \cong 3_+^{1+2}$ and $Q = \langle P, z_{\mathrm{A}}\rangle$. In particular, $C_G(Q) = C_{C_G(X)}(P)$, and $O^{r'}(C_G(Q)) = \mathrm{SL}_2(q)$ yields $O^{r'}(C_K(P)) = \mathrm{SL}_2(q)$. By Corollary 5.2, the group P is radical in K, and we can apply Lemma 5.4b). Let $E \in \{Y_1, Y_1', Y_1''\}$ such that $Q = \langle E, y\rangle = \langle P, z_{\mathrm{A}}\rangle$ with $P = Q \cap K = 3_+^{1+2}$. Now Proposition 5.1 shows that if $a = 1$, then $P =_K K_1$; if $a \geq 2$, then $P \in_K \{K_1, K_2, K_3\}$, and we can assume that $Q = \langle K_i, z_{\mathrm{A}}\rangle$ for some i. Recall from Part (1) that every $3^2 \leq E$ containing Z has type $3\overline{\mathrm{B}}$, that is, $E \cap 3\mathrm{C} = \emptyset$; now Lemma 5.4b) applied to $E < Q$ yields that $E = P = K_i$. In particular, we can assume that $E \leq K$, and hence $E \in_K \{K_1, K_2, K_3\}$. By Part (2), the groups in $\{Y_1, Y_1', Y_1''\}$ are pairwise non-conjugate in G, thus $\{Y_1, Y_1', Y_1''\} =_G \{K_1, K_2, K_3\}$, as claimed.

(5) We show that $E \cap 3\mathrm{C} \neq \emptyset$ for each $E \in \{Y_2, Y_2', Y_2''\}$. We continue with the notation of Part (3); let $E \in \{Y_2, Y_2', Y_2''\}$ and $Q = \langle E, y\rangle$ with $X = \Omega_1(Z(Q)) = 3\overline{\mathrm{A}}$. We can assume that $Q = \langle P, z_{\mathrm{A}}\rangle$ for $P = Q \cap K$ with $K = O^{r'}(C_G(X)) = \mathrm{SL}_6^{\varepsilon}(q)$. As in Part (4), we have $O^{r'}(C_K(P)) = \mathrm{SL}_2(q)$ and Corollary 5.2 shows that P is radical in K. Now it follows from Part (4) that $P \in_G \{Y_1, Y_1', Y_1''\}$; in particular, $P \cap 3\mathrm{B} \neq \emptyset$, and $\mathrm{Out}_K(P) = Q_8$ if $a = 1$, and $Q_8 \leq \mathrm{Out}_K(P) = \mathrm{SL}_2(3)$ if $a \geq 2$. Since $P \cap 3\mathrm{B} \neq \emptyset$ and $Q_8 \leq \mathrm{SL}_2(3)$ acts transitively on the nontrivial elements of $P/Z(P)$, we have $P \cap 3\mathrm{C} = \emptyset$. If $E \cap 3\mathrm{C} = \emptyset$, then Lemma 5.4 shows that $E = P$, so $E = 3_+^{1+2} \leq K$, and Part (4) yields $E \in_G \{Y_1, Y_1', Y_1''\}$; the latter is a contradiction to the local structure determined in Part (1). Thus, $E \cap 3\mathrm{C} \neq \emptyset$.

(6) We show that Y_2, Y_2', Y_2'' are pairwise non-conjugate in G if $a \geq 2$; recall that $Y_2 = Y_2' = Y_2''$ if $a = 1$. Let $E \in \{Y_2, Y_2', Y_2''\}$ and define $Q = \langle E, y\rangle$ as Part (5), so that $Z(Q) =_G 3\overline{\mathrm{A}}$ and we may suppose $Y_2, Y_2', Y_2'' \leq C_G(3\overline{\mathrm{A}})$. Recall that $C_G(3\overline{\mathrm{A}}) = (q-\varepsilon) \circ_{2^*} (K.2^*)$ with $K = \mathrm{SL}_6^{\varepsilon}(q)$, and $K_1, K_2, K_3 \leq K$ as defined in Proposition 5.1; by Part (4) we can assume that $\{K_1, K_2, K_3\} = \{Y_1, Y_1', Y_1''\}$. We define $U_1 = \langle z_{\mathrm{A}}, Y_2\rangle, U_2 = \langle z_{\mathrm{A}}, Y_2'\rangle, U_3 = \langle z_{\mathrm{A}}, Y_2''\rangle$, and $V_i = \langle z_{\mathrm{A}}, K_i\rangle$ for $i \in \{1,2,3\}$; note that each of these groups has center $\langle z, z_{\mathrm{A}}\rangle = 3\overline{\mathrm{A}}$. If we write $K_i = \langle g, h\rangle$, then $\langle z_{\mathrm{A}} g, z_{\mathrm{A}} h\rangle \cong 3_+^{1+2}$ and $\langle z, z_{\mathrm{A}} g\rangle =_G 3\overline{\mathrm{C}}$; the latter follows from Lemma 3.2d) and the fact that every subgroup 3^2 of K_i containing Z is of type $3\overline{\mathrm{B}}$, see Part (1). Thus we can assume that $\langle z_{\mathrm{A}} g, z_{\mathrm{A}} h\rangle \in_G \{Y_2, Y_2', Y_2''\}$, and so $\{V_1, V_2, V_3\} \subseteq \{U_1, U_2, U_3\}$. If $V_i^w = V_j$, then $w \in N_G(3\overline{\mathrm{A}})$, and we can assume that $w \in K$. Since K_u is the only extraspecial subgroup of V_u with all non-central elements being of type 3B, it follows that $K_i^w = K_j$, and hence $i = j$. This proves that $V_i \neq_G V_j$ when $i \neq j$, and so $\{U_1, U_2, U_3\} = \{V_1, V_2, V_3\}$ are three pairwise non-conjugate subgroups. Now suppose two distinct subgroups in $\{Y_2, Y_2', Y_2''\}$ are G-conjugate, say $Y_2^w = Y_2'$ for some $w \in G$. Since $C_G(Y_2) = C_G(Y_2')$, it follows that w normalises $O^{r'}(C_G(Y_2)) = \mathrm{PSL}_3^{\varepsilon}(q)$. Define $Q = \langle Y_2, y\rangle$ and $Q' = \langle Y_2', y'\rangle$ with $y, y' \in \mathrm{PSL}_3^{\varepsilon}(q).3$ such that $O^{r'}(C_G(Q)) = \mathrm{SL}_2(q)$ and $O^{r'}(C_G(Q')) = \mathrm{SL}_2(q)$. It follows that $C_{\mathrm{PSL}_3^{\varepsilon}(q)}(Q)$ and $C_{\mathrm{PSL}_3^{\varepsilon}(q)}(Q')$ are Levi subgroups of $\mathrm{PSL}_3^{\varepsilon}(q)$, so there exists $t \in \mathrm{PSL}_3^{\varepsilon}(q) \leq C_G(Y_2) = C_G(Y_2')$ such that $C_{\mathrm{PSL}_3^{\varepsilon}(q)}(Q)^{wt} = C_{\mathrm{PSL}_3^{\varepsilon}(q)}(Q)$; in particular, wt normalises $\mathrm{SL}_2(q) = O^{r'}(C_{\mathrm{PSL}_3^{\varepsilon}(q)}(Q))$. Note that

$$Z(Q) = \Omega_1(O_3(C_{C_G(Y_2)}(\mathrm{SL}_2(q)))),$$

and hence $Z(Q') = Z(Q)^{wt} = Z(Q)$; this implies $y^{wt} \in Q'$, and hence $Q^{wt} = Q'$. But this is impossible since Q and Q' are conjugate to two distinct elements in $\{U_1, U_2, U_3\}$, as shown above. Using a analogous argument, we establish that any two distinct elements of $\{Y_2, Y_2', Y_2''\}$ are non-conjugate in G.

(7) We now prove part a) of the proposition and classify $E = 3_+^{1+2} \leq G$ with $z, z_{\mathrm{C}} \in E$. Let $C_G(3\overline{\mathrm{C}}) = L.x_{\mathrm{C}}$ and $N_G(3\overline{\mathrm{C}}) = C_G(3\overline{\mathrm{C}}).\langle \gamma_{\mathrm{C}}, \xi\rangle$ as before. Note that ξ centralises the generator z of Z, thus we may suppose $z = (d, d^{-1}, 1)D \in Z(L)$ and $z_{\mathrm{C}} = (d, 1, 1)D$, and so $Y = \langle z_{\mathrm{C}}, \xi\rangle \cong 3_+^{1+2}$ with $Z(Y) = Z$. In the following let σ be the Frobenius morphism with $\overline{G}^{\sigma} = G$. We have seen in Table II that $C_{\overline{G}}(3\overline{\mathrm{C}}) = (\mathrm{SL}_3)^3/D$ and $L.x_{\mathrm{C}} = (C_{\overline{G}}(3\overline{\mathrm{C}}))^{\sigma}$. Note that $\xi \in G = \overline{G}^{\sigma}$, so $C_{L.x_{\mathrm{C}}}(\xi) = (C_{(\mathrm{SL}_3)^3/D}(\xi))^{\sigma}$. It is shown in [16, Table 4.7.1] that ξ permutes the three factors of $(\mathrm{SL}_3)^3/D$ cyclically, so $C_{(\mathrm{SL}_3)^3/D}(\xi) = 3 \times \mathrm{SL}_3/3$; note that $D = \langle (d,d,d)\rangle$ and so $(d, d^2, 1)D \in C_{(\mathrm{SL}_3)^3/D}(\xi)$. Since $(C_{\overline{G}}(3\overline{\mathrm{C}}))^{\sigma} = L.3$, we have $(\mathrm{SL}_3/3)^{\sigma} = \mathrm{PSL}_3^{\varepsilon}(q).3$, and so we deduce that

$$C_G(Y) = C_{C_G(3\overline{\mathrm{C}})}(\xi) = 3 \times \mathrm{PSL}_3^{\varepsilon}(q).3.$$

By [16, Table 4.7.1] again, γ_{C} acts as inverse-transpose on the first factor of $(\mathrm{SL}_3)^3/D$ and interchanges the last two factors. Note also γ_{C} acts on $C_{(\mathrm{SL}_3)^3/D}(\xi)$, since $\xi^{\gamma_{\mathrm{C}}} = f\xi^{-1}$ for some $f \in Z(C_{\overline{G}}(3\overline{\mathrm{C}}))$, and $C_{(\mathrm{SL}_3)^3/3}(f\xi^{-1}) = C_{(\mathrm{SL}_3)^3/3}(\xi)$. We deduce that γ_{C} acts as inverse-transpose on $C_{(\mathrm{SL}_3)^3/D}(\xi) = \mathrm{SL}_3/3$. Note that $3 \times \Delta(L^{\varepsilon})/D = 3 \times \mathrm{PSL}_3^{\varepsilon}(q) = L \cap C_{C_G(3\overline{\mathrm{C}})}(\xi)$, hence there exists $u \in L.x_{\mathrm{C}} \setminus L$ which induces the outer 3 in $C_{C_G(3\overline{\mathrm{C}})}(\xi) = \mathrm{PSL}_3^{\varepsilon}(q).3$; in particular, $[u, \xi] = 1$ by construction, and u acts as an outer-diagonal automorphism of order 3 on $\mathrm{PSL}_3^{\varepsilon}(q)$.

Now suppose $E = 3_+^{1+2} \leq N_G(3\overline{\mathrm{C}})$ with $z_{\mathrm{C}}, z \in E$, so $E = \langle z_{\mathrm{C}}, w\rangle$ for some $w \in N_G(3\overline{\mathrm{C}}) \setminus C_G(3\overline{\mathrm{C}})$. We can assume $w = t\xi$ for some $t \in L.3$, say $t = (t_1, t_2, t_3)Ds$ for some $s = (s_1{:}s_2{:}s_3) = u^{\ell}$ with $\ell \in \{0,1,2\}$ and each $t_i \in L_i$. We consider conjugates of w. If $v = (v_1, v_2, v_2)D \in L$, then

$$w^v = (v_1^{-1} t_1 (v_3)^{s_1}, v_2^{-1} t_2 (v_1)^{s_2}, v_3^{-1} t_3 (v_2)^{s_2}) D s \xi.$$

Taking $v_2 = t_2(v_1)^{s_2}$ and $v_3 = t_3(v_2)^{s_2}$, we can suppose that $t = (t_1, 1, 1)sD$. Since $|w| = 3$ and $[\xi, s] = 1$, we further know that $(t_1, t_1^{s_2}, t_1^{s_3}) \in D$. Thus $t_1 \in Z(L^{\varepsilon})$, and so $w = z_{\mathrm{C}}^k s\xi$ for some $k \in \{0,1,2\}$.

Replacing w by $z_{\mathrm{C}}^{-k}w$, we may assume that $w \in \{\xi, u\xi, u^2\xi\}$, hence, up to conjugacy, there are at most three groups $3_+^{1+2} \leq N_G(3\overline{\mathrm{C}})$ containing z_{C} with center Z, namely

$$Y_3 = Y = \langle 3\overline{\mathrm{C}}, \xi\rangle, \quad Y_4 = \langle 3\overline{\mathrm{C}}, u\xi\rangle, \quad Y_5 = \langle 3\overline{\mathrm{C}}, u^2\xi\rangle.$$

In particular, if these groups exist (which will be shown below), then $|\xi| = |u\xi| = |u^2\xi| = 3$ follows. In conclusion, if $E \cong 3_+^{1+2}$ with $Z(E) = Z$ contains z_{C}, then we may suppose

$$E \in_G \{Y_3, Y_4, Y_5\}.$$

Let $a = 1$. Recall that $C_{C_G(3\overline{\mathrm{C}})}(\xi) = \mathrm{PSL}_3^\varepsilon(q).u$ and u acts like the outer-diagonal automorphism induced by $\mathrm{diag}(1, 1, \alpha)$ for some $\alpha \in \mathrm{GF}(q)^\times$ of order 3; since $\mathrm{PSL}_3^\varepsilon(q)$ has trivial center, this implies that $C_{C_G(3\overline{\mathrm{C}})}(\xi) = \mathrm{PSL}_3^\varepsilon(q) \rtimes u$. Recall that $\Delta(L^\varepsilon)/D = \mathrm{PSL}_3^\varepsilon(q)$, which is the diagonal subgroup of L/D; since u induces an order 3 outerdiagonal automorphism on $\Delta(L^\varepsilon)/D$, it also induces an order 3 outerdiagonal automorphism on each factor of L/D, as x_{C} does. Thus, we assume that $x_{\mathrm{C}} = u$. Now let $a \geq 2$. We have seen in Part (5) that $U \cap 3\overline{\mathrm{C}} \neq \emptyset$ for each $U \in \{Y_2, Y_2', Y_2''\}$, so we can assume that $\{Y_2, Y_2', Y_2''\} \subseteq_G \{Y_3, Y_4, Y_5\}$. By Part (6), the groups $\{Y_2, Y_2', Y_2''\}$ are pairwise non-conjugate in G; this proves that $\{Y_2, Y_2', Y_2''\} = \{Y_3, Y_4, Y_5\}$; in particular, $|\xi| = |u\xi| = |u^2\xi| = 3$. We may suppose that $Y_2' = Y_4 = \langle z_{\mathrm{C}}, u\xi\rangle$. As in the case $a = 1$, we can replace x_{C} by u, that is, we can assume that $x_{\mathrm{C}} = u$ and hence $[\xi, x_{\mathrm{C}}] = 1$. In both cases, $a = 1$ and $a \geq 2$, the element γ_{C} acts as inverse-transpose on $\mathrm{SL}_3/3$, and so also on $\Delta(L^\varepsilon)/D = \mathrm{PSL}_3^\varepsilon(q)$. In particular, $x_{\mathrm{C}}^{\gamma_{\mathrm{C}}}$ acts as x_{C}^{-1} on $\Delta(L^\varepsilon)/D$, hence $x_{\mathrm{C}}^{\gamma_C} = x_{\mathrm{C}}$ as claimed.

(8) We now show that $E = 3_+^{1+2\gamma} \leq G$ with $Z(E) = Z$ forces $\gamma = 1$. Suppose, for a contradiction, that $\gamma \geq 2$. Then $E = U_1 \circ_3 U_2$ with $U_1 = 3_+^{1+2}$ and $U_2 = 3_+^{1+2(\gamma-1)}$. Parts (1) and (7) show that $U_2 \leq C_G(U_1) = Z \times H$ where $H = G_2(q)$ or $H = \mathrm{PSL}_3^\varepsilon(q).3$. This implies $U_2 = Z \times V$ for some $V \leq H$, hence $Z(U_2) = 3 \times Z(V) \geq 3^2$, which is impossible. This contradiction proves $\gamma = 1$, as claimed.

(9) We continue with the notation of Part (4) and determine $\mathrm{Out}_G(E)$ for $E \in \{Y_1, Y_1', Y_1''\} = \{K_1, K_2, K_3\}$. For $a = 1$ let $O_a = Q_8$, and $O_a = \mathrm{SL}_2(3)$ for $a \geq 2$. Since each $\mathrm{Out}_K(K_i) = O_a$, it follows that $O_a \leq \mathrm{Out}_G(K_i)$. Define

$$\mathrm{Out}^0(3_+^{1+3}) = C_{\mathrm{Out}(3_+^{1+2})}(Z(3_+^{1+2})) = \mathrm{SL}_2(3);$$

since $\mathrm{Out}_G(K_i) \leq \mathrm{Out}^0(3_+^{1+3})$ and $O_a = \mathrm{SL}_2(3)$ for $a \geq 2$, it follows that $\mathrm{Out}_G(K_i) = \mathrm{SL}_2(3)$ when $a \geq 2$. Now let $a = 1$, so $Y_1 =_G Y_1' =_G Y_1''$, and we can suppose that $E = \langle z_{\mathrm{B}}, u\rangle$ with $u = y_1 = \mu{:}\gamma_1$ as defined in Part (1). Note that $Q_8 \leq \mathrm{Out}_G(E)$. Suppose, for a contradiction, that $Q_8 < \mathrm{Out}_G(E)$. Since Q_8 is a maximal subgroup of $\mathrm{Out}(E) = \mathrm{SL}_2(3)$, this implies that $\mathrm{Out}_G(E) = \mathrm{SL}_2(3)$. Thus, there exists $w \in N_G(E)$ such that w induces an order 3-element in $\mathrm{SL}_2(3)$ and w fixes $z_{\mathrm{B}}Z$, that is, it satisfies $(z_{\mathrm{B}}Z)^w = z_{\mathrm{B}}Z$ and $(uZ)^w = z_{\mathrm{B}}uZ$. In particular, w normalizes $3\overline{\mathrm{B}}$, and hence $w \in N_G(3\overline{\mathrm{B}}) = C_G(3\overline{\mathrm{B}}).\langle \gamma_{\mathrm{B}}, u\rangle$. We may suppose that w is a 3-element, hence $w \in C_G(3\overline{\mathrm{B}}).u$. Replacing w by wu^k if necessary, we may suppose $w \in C_G(3\overline{\mathrm{B}})$. (Note that $wu^k = 1$ is not possible since $(uZ)^w = z_{\mathrm{B}}uZ$.) Thus, $(uZ)^w = z_{\mathrm{B}}^\ell u^k Z$ for some $\ell, k \in \{1, 2\}$. Since $C_G(3\overline{\mathrm{B}}) = (q-\varepsilon)^2 \circ_{(2^*)^2} (\mathrm{Spin}_8^+(q).(2^*)^2)$ and $O_3((q-\varepsilon)^2) = 3\overline{\mathrm{B}} \leq E$, we may suppose that $w \in \mathrm{Spin}_8^+(q)$, and so $w^u = w^{\gamma_1} \in \mathrm{Spin}_8^+(q)$, say $[w, u] = [w, \gamma_1] = v \in \mathrm{Spin}_8^+(q)$. This yields $u^w Z = uv^{-1}Z = z_{\mathrm{B}}^\ell u^k Z$; but $uv^{-1}Z = z_{\mathrm{B}}^\ell u^k$ is not possible as $u \notin C_G(3\overline{\mathrm{B}})$ and $z_{\mathrm{B}} \notin Z \times \mathrm{Spin}_8^+(q)$. This contradiction proves that $\mathrm{Out}_G(E) = Q_8$ for $a = 1$.

(10) We determine $\mathrm{Out}_G(E)$ for $E \in \{Y_2, Y_2', Y_2''\}$. Parts (1), (5), and (6) show that $E \cap 3\mathrm{C} \neq \emptyset$ and $E \cap 3\mathrm{B} \neq \emptyset$; in particular, $\mathrm{Out}_G(E)$ acts reducibly on E/Z. Recall that $Y_2 = \langle z_{\mathrm{B}}, u\rangle$ with $u = \mu{:}\gamma_2$, and so $\gamma_{\mathrm{B}} \in N_G(3\overline{\mathrm{B}})$ normalises E, as it normalises $\langle u\rangle$ and interchanges the two factors of $(q-\varepsilon)^2 = Z(C_G(3\overline{\mathrm{B}}))$; similarly for Y_2' and Y_2'', which shows that $\mathrm{Out}_G(E) \geq 2$. First consider the case $a \geq 2$; we claim that $\mathrm{Out}_G(E) = 6$. Recall that $E = \langle z_{\mathrm{B}}, u\rangle$ with $u \in \{\mu{:}\gamma_2, (\alpha, 1)\mu{:}\gamma_2, (\alpha^2, 1)\mu{:}\gamma_2\}$; the action of μ implies that if $X = \langle (q-\varepsilon)^2, E\rangle$ with $(q-\varepsilon)^2 = Z(C_G(3\overline{\mathrm{B}}))$, then $C_X(E) = Z(E) = 3$. Since E is not a Sylow 3-subgroup of X, we deduce that $|\mathrm{Out}_X(E)|_3 \geq 3$, and so $\mathrm{Out}_{N_G(3\overline{\mathrm{B}})}(E) \geq 3$. Since $6 \leq \mathrm{SL}_2(3) = \mathrm{Out}(Y_2)$ is maximal, the previous results imply that $\mathrm{Out}_G(Y_2) = 6$. Now consider $a = 1$, so that $E = Y_2$; we claim that $\mathrm{Out}_G(E) = 2$. Suppose, for a contradiction, that $\mathrm{Out}_G(E) > 2$. Since $\mathrm{Out}_G(E)$ acts reducibly on E/Z, a direct computation shows that $\mathrm{Out}_G(E) = 6$. This implies that there is an element $w \in N_G(E)$ which has order 3 modulo $C_G(E)E$. We show that this is impossible. Since $\mathrm{Out}(E) = \mathrm{GL}_2(3)$, the action of w on E stabilises a generator of $E = 3_+^{1+2}$ modulo Z, that is, $w \in N_G(3\overline{\mathrm{B}})$ or $N_G(3\overline{\mathrm{C}})$. If $w \in N_G(3\overline{\mathrm{B}})$, then we may suppose $(z_{\mathrm{B}}Z)^w = z_{\mathrm{B}}Z$ and $(uZ)^w = z_{\mathrm{B}}uZ$; however, the same argument as in Part (9) for $E = Y_1$ and $a = 1$ shows that this is impossible. Thus $w \in N_G(3\overline{\mathrm{C}})$ and we may suppose $E = Y_3 = \langle z, z_{\mathrm{C}}, \xi\rangle$ with $(\xi Z)^w = z_{\mathrm{C}}\xi Z$. Since w is a 3-element in $N_G(3\overline{\mathrm{C}}) = (L.x_{\mathrm{C}}).\langle \xi, \gamma_{\mathrm{C}}\rangle$, we may suppose $w \in L.x_{\mathrm{C}}.\xi$, that is, $w \in C_G(E)E$. Since $x_{\mathrm{C}} \in C_G(E)$, replacing w by wt for some $t \in \langle x_{\mathrm{C}}, \xi\rangle$ if necessary, we can assume that $w \in L$ and $(\xi Z)^w = z_{\mathrm{C}}^\ell \xi Z$ for some $\ell \in \{1,2\}$; the latter follows together with our assumption that $\mathrm{Out}_G(E) = 6$. Thus $[w, \xi^{-1}] = v$ for some $v \in L$, and so $\xi^w Z = v\xi Z = z_{\mathrm{C}}^\ell \xi Z$ and $vZ = z_{\mathrm{C}}^\ell Z$. Since $w^{\xi^{-1}} = wv$ and $v \in X = Z(C_G(3\overline{\mathrm{C}})) = 3\overline{\mathrm{C}}$, it follows that $(wX)^\xi = (wX)$. But $L/X = L_1/Z(L_1) \times L_2/Z(L_2) \times L_3/Z(L_3)$ and ξ permutes these direct factors cyclically, so $w = (w_1, w_1, w_1)hD$ for some $h \in X$ and $w_1 \in \mathrm{SL}_3^\varepsilon(q)$. Note that if $h = z^\ell z_{\mathrm{C}}^t$ for some ℓ, t, then $\xi^w = \xi^h = \xi^{(z_{\mathrm{C}})^t}$. Together with $E = \langle z_{\mathrm{C}}, \xi\rangle$ and $[\xi, z_{\mathrm{C}}] = z$, this yields $\xi^w \in \langle z, \xi\rangle$, and so $\xi^w Z = \xi^k Z$ for some $k \in \{1,2\}$. As shown above, we also have $(\xi Z)^w = z_{\mathrm{C}}^\ell \xi Z$, which implies that $z_{\mathrm{C}}^\ell \in \xi^{k-1}Z$, which is impossible as $[\xi, z_{\mathrm{C}}] = z$. This contradiction shows that $\mathrm{Out}_G(Y_2) = 2$.

(11) Again, let $E = 3_+^{1+2} \leq G$ with $z \in E$. We prove Part d) of the theorem and determine when $N_G(E)$ is maximal-proper 3-local. If $E \cap 3\mathrm{C} = \emptyset$, then every $X < E$ with $Z < X$ satisfies $X =_G 3\overline{\mathrm{B}}$. Suppose, for a contradiction, that $N_G(E) \leq N_G(3\overline{\mathrm{B}})$, so that $\mathrm{Out}_G(E) \leq \mathrm{Out}_{N_G(3\overline{\mathrm{B}})}(E)$. Each element in $\mathrm{Out}_{N_G(3\overline{\mathrm{B}})}(E)$ stabilises the line $3\overline{\mathrm{B}}/Z \leq E/Z(E)$, so $\mathrm{Out}_{N_G(3\overline{\mathrm{B}})}(E) \leq 2 \times 6$ is a parabolic subgroup of $\mathrm{GL}_2(3) = \mathrm{Out}(E)$; but this is impossible by Part (9) where we have shown that $Q_8 \leq \mathrm{Out}_G(E)$. Thus, $N_G(E) \not\leq N_G(3\overline{\mathrm{B}})$ and Lemma 2.6b) implies that $N_G(E)$ is maximal-proper 3-local. As shown in Part (10), if $E \cap 3\mathrm{C} \neq \emptyset$, then $E \cap 3\mathrm{B} \neq \emptyset$ and $\mathrm{Out}_G(E)$ acts reducibly on E/Z. In particular, we have shown that if $a \geq 2$, then $N_G(E) \leq N_G(3\overline{\mathrm{B}})$ (as $\mathrm{Out}_G(E) = 6 = \langle \mathrm{Out}_{(q-\epsilon)^2}(E), \gamma_B\rangle$), and if $a = 1$, then $N_G(E) \leq_G N_G(3X)$ for any $X \in \{\overline{\mathrm{B}}, \overline{\mathrm{C}}\}$ (as $\mathrm{Out}_G(E) = 2$). Lemma 2.6b) shows that $N_G(E)$ is not maximal-proper 3-local. □

6. Maximal 3-local subgroups

Using the results of the previous sections, it is straightforward to classify the maximal-proper 3-local subgroups of $G = E_6^\varepsilon(q)$ with $3 \mid q - \varepsilon$. Recall that $n_\varepsilon = \gcd(n, q - \varepsilon)$

Theorem 6.1. *Up to conjugacy, the maximal-proper 3-local subgroups of $G = E_6^\varepsilon(q)$ with $3 \mid q - \varepsilon$ are $M_1, \ldots, M_{15}$ as given in Table III, where M_i is only defined if the conditions on q and a listed in the right column of the table is met.*

PROOF. Let E_i be defined as in Table II and Proposition 4.1. By Lemma 2.6a), every maximal-proper 3-local $M \leq G$ is conjugate to some $N_G(E_i)$ or to $N_G(E)$ for some extraspecial E with $Z(G) < E$. Now the result follows from Corollary 3.5, Proposition 4.1, and Proposition 5.5. □

References

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	$N_G(E)$	E	$C_G(E)$	condition
M_1	$((q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{SL}_6^\varepsilon(q).2_\varepsilon)).2$	$3\overline{\mathrm{A}}$	$(q-\varepsilon)\circ_{2_\varepsilon}(\mathrm{SL}_6^\varepsilon(q).2_\varepsilon)$	–
M_2	$((q-\varepsilon)^2\circ_{(2_\varepsilon)^2}(\mathrm{Spin}_8^+(q).(2_\varepsilon)^2)).\mathrm{S}_3$	$3\overline{\mathrm{B}}$	$(q-\varepsilon)^2\circ_{(2_\varepsilon)^2}(\mathrm{Spin}_8^+(q).(2_\varepsilon)^2)$	–
M_3	$(((\mathrm{SL}_3^\varepsilon(q))^3/D).3).\mathrm{S}_3$	$3\overline{\mathrm{C}}$	$(\mathrm{SL}_3^\varepsilon(q)^3/D).3$	–
M_4	$[(q-\varepsilon)^3\circ_{4_\varepsilon}(\mathrm{SL}_4^\varepsilon(q).4_\varepsilon)].D_8$	$3\overline{\mathrm{A}}_2\overline{\mathrm{B}}_2$	$(q-\varepsilon)^3\circ_{4_\varepsilon}(\mathrm{SL}_4^\varepsilon(q).4_\varepsilon)$	–
M_5	$((q^2+\varepsilon q+1)^3.3_+^{1+2}).\mathrm{SL}_2(3)$	$(3\overline{\mathrm{C}}^2)_2$	$(q^2+\varepsilon q+1)^3.3$	$q\geq 4$
M_6	$[(q-\varepsilon)^4\circ_{(2_\varepsilon)^2}((\mathrm{SL}_2(q))^2.(2_\varepsilon)^2)].(2\times\mathrm{S}_4)$	$3\overline{\mathrm{A}}_6\overline{\mathrm{B}}_3\overline{\mathrm{C}}_4$	$(q-\varepsilon)^4\circ_{(2_\varepsilon)^2}((\mathrm{SL}_2(q))^2.(2_\varepsilon)^2)$	$q\geq 4$
M_7	$3^4.3^3.\mathrm{SL}_3(3)$	$(3\overline{\mathrm{C}}^3)_1$	3^4	–
M_8	$[(q-\varepsilon)^5\circ_{2_\varepsilon}(\mathrm{SL}_2(q).2_\varepsilon)].\mathrm{S}_6$	$3\overline{\mathrm{A}}_{15}\overline{\mathrm{B}}_{15}\overline{\mathrm{C}}_{10}$	$(q-\varepsilon)^5\circ_{2_\varepsilon}(\mathrm{SL}_2(q).2_\varepsilon)$	$q\geq 4$
M_9	$(q-\varepsilon)^6.W$	$3\overline{\mathrm{A}}_{36}\overline{\mathrm{B}}_{45}\overline{\mathrm{C}}_{40}$	$(q-\varepsilon)^6$	–
M_{10}	$(q-\varepsilon)\circ_{4_\varepsilon}(\mathrm{Spin}_{10}^+(q).4_\varepsilon)$	9	$(q-\varepsilon)\circ_{4_\varepsilon}(\mathrm{Spin}_{10}^+(q).4_\varepsilon)$	$a\geq 2$
M_{11}	$(q-\varepsilon)\circ_{10_\varepsilon}((\mathrm{SL}_5^\varepsilon(q)\times\mathrm{SL}_2(q)).10_\varepsilon)$	9	$(q-\varepsilon)\circ_{10_\varepsilon}((\mathrm{SL}_5^\varepsilon(q)\times\mathrm{SL}_2(q)).10_\varepsilon$	$a\geq 2$
M_{12}	$(3_+^{1+2}.Q_8)\times G_2(q)$	3_+^{1+2}	$3\times G_2(q)$	$a=1$
M_{13}	$(3_+^{1+2}.\mathrm{SL}_2(3))\times G_2(q)$	3_+^{1+2}	$3\times G_2(q)$	$a\geq 2$
M_{14}	$(3_+^{1+2}.\mathrm{SL}_2(3))\times G_2(q)$	3_+^{1+2}	$3\times G_2(q)$	$a\geq 2$
M_{15}	$(3_+^{1+2}.\mathrm{SL}_2(3))\times G_2(q)$	3_+^{1+2}	$3\times G_2(q)$	$a\geq 2$

Table III: Maximal 3-local subgroups of $G=E_6^\varepsilon(q)$ with $3\mid q-\varepsilon$ as listed in Theorem 6.1.

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