Math 4191 Stochastic Calculus
Summary

1 Theoretical

Suppose that $S(t)$ is the value of a stock at time t. If we hold H shares of stock, what is our profit at time T? Holding $H(t)$ shares at each time t leads to a profit of

$$\int_0^T H(t)S'(t)dt \tag{1}$$

if S is differentiable, but in many cases it is not. Under certain conditions, even when S is not differentiable the sum $\sum_{n=1}^N H(t_n)(S(t_{n+1})-S(t_n))$ converges independent of the choice of partitions. When this occurs, we label the limit as

$$\int_0^T H(t)dS(t) \tag{2}$$

and refer to it as a it Stieltjes integral. We can form a Stieltjes integral with any function which is the difference of two monotone functions. The variation of a function is

$$V_g([a,b]) = \sup_{\Delta} \sum_{n=1}^{N} |g(t_n) - g(t_{n-1})| \tag{3}$$

where $\Delta = \{a = t_0, t_1, \ldots, t_N = b\}$ is any partition of $[a,b]$. If $V_g < \infty$ we say that g is of *finite or bounded variation.* A function is the difference of two monotone functions if, and only if, it is of bounded variation. The quadratic variation of a function is:

$$[g]([a,b]) = \lim_{|\Delta| \longrightarrow 0} \sum_{n=1}^{N} (g(t_n) - g(t_{n-1}))^2 \tag{4}$$

where $\Delta = \{a = t_0, t_1, \ldots, t_N = b\}$ is any partition of $[a,b]$ and $|\Delta| = \max_n |t_{n+1} - t_n|$. Let X be a $\mathcal{F}$-measurable random variable on a space Ω, and let $\mathcal{G}$ be a σ-field on Ω with $\mathcal{G} \subseteq \mathcal{F}$, so that X is not necessarily $\mathcal{G}$-measurable. There is a $\mathcal{G}$-measurable random variable, denoted $E[X|\mathcal{G}]$ and referred to as the *conditional expectation of X with respect to $\mathcal{G}$*, such that $E[X1_A] = E[E[X|\mathcal{G}]1_A]$ for all $A \in \mathcal{G}$. There are a few rules for this:

- $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$.

- If $\mathcal{G} = \{\emptyset, \Omega\}$, then $E[X|\mathcal{G}] = E[X]$.
- If X is $\mathcal{G}$-measurable, then $E[X|\mathcal{G}] = X$, and more generally $E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$
- If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_1]$.
- If $\sigma(X)$ and $\mathcal{G}$ are independent, then $E[X|\mathcal{G}] = E[X]$.

Stochastic processes for which $E[X_t|\mathcal{F}_s] = X_s$ whenever $t > s$ are called martingales, and are very important. Note that $\mathcal{F}_s$ is usually $\sigma(X_r, 0 \leq r \leq s)$, the σ-field generated by all X_r's up till time s. A *stopping time* τ is a r.v. taking values in the nonnegative integers and ∞ such that

$$\{\tau \leq t\} \in \mathcal{F}_t \tag{5}$$

Stopping times are very important in the theory of martingales, in large part due to the following theorem.

Theorem 1 *If M_n is a martingale and τ is a bounded stopping time then $E[M_\tau] = E[M_0]$.*

Brownian motion is defined to be a stochastic process with the following properties.

- $B_t : [0, \infty) \longrightarrow R$ is continuous a.s.
- For $t_1 < \ldots < t_n$ the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.
- For $t' < t''$, $B_{t''} - B_{t'}$ is a normally distributed variable with mean 0 and variance $t'' - t'$.
- Generally we normalize with $B_0 = 0$ a.s.

Some further properties of Brownian motion:

- Brownian motion is a martingale.
- With probability 1, B_t is not differentiable at any t.
- Brownian motion is not of finite variation, and the quadratic variation of Brownian motion over $[0, t]$ is t.
- $B_t^2 - t$ is a martingale.
- For any a, $e^{aB_t - \frac{1}{2}a^2 t}$ is a martingale.
- (Strong Markov property) If τ is a bounded stopping time, and $\hat{B}_t = B_{\tau+t} - B_\tau$, then $\hat{B}_t$ is a Brownian motion.

- (Reflection principle) $P(\max_{0\leq t\leq T} B_t > a) = 2P(B_T > a)$.
- $P(\max_{0\leq t\leq 1} B_t > 0) = P(\min_{0\leq t\leq 1} B_t < 0) = 1$
- $P(B_t = a \text{ for some t}) = 1$

Brownian motion is not of finite variation, so as a Stieltjes integral $\int_0^T H(t)dB_t$ does not make sense. However, we can make sense of the integral by approximation by simple functions and taking a limit. The result is referred to as a stochastic, or Itō, integral. It has the following properties.

- $\int_0^T (aX_t + bY_t)dB_t = a\int_0^T X_t dB_t + b\int_0^T Y_t dB_t$.
- $\int_0^T 1_{[a,b]}(t)dB_t = B_b - B_a$.
- $E[\int_0^T X_t dB_t] = 0$
- (Itō isometry) $E[\Big(\int_0^T X_t dB_t\Big)^2] = \int_0^T E[X_t^2]dt$
- $\int_0^T X_t dB_t$ is a continuous martingale if $\int_0^T E[X_t^2]dt < \infty$(otherwise, local martingale).
- If $f(t)$ is a deterministic function, then $\int_0^T f(t)dB_t$ is $N(0,\sigma^2)$, where the value of σ^2 is given by the Itō isometry.
- If $Y_t = \int_0^t X_s dB_s$, then the quadratic variation $[Y](t)$ is given by $[Y](t) = \int_0^t X_s^2 ds$.

We then have the all-important Itō's rule. If f is twice continuously differentiable, then

$$f(B_T) - f(B_0) = \int_0^T f'(B_t)dB_t + \frac{1}{2}\int_0^T f''(B_t)dt \tag{6}$$

An Itō process is a process Y_t of the form

$$Y_t = Y_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB_s \tag{7}$$

for $0 \leq t \leq T$, where $\mu(s)$ and $\sigma(s)$ are stochastic processes adapted to the filtration of B, and $\int_0^T |\mu(s)|ds < \infty$ and $\int_0^T \sigma^2(s)ds < \infty$ almost surely. The quadratic variation of Y_t is given by

$$[Y](t) = Y([0,t]) = \int_0^t \sigma(s)^2 ds \tag{8}$$

We may write (7) in the shorthand

(9) $$dY_t = \mu(t)dt + \sigma(t)dB_t$$

We then may integrate:

(10) $$\int_0^t X_s dY_s = \int_0^t X_s \mu(s)ds + \int_0^t X_s \sigma(s)dB_s$$

Itō's formula takes the form

(11) $$d(f(Y_t)) = f'(Y_t)dY_t + \frac{1}{2}f''(Y_t)d[Y](t) = \Big(f'(Y_t)\mu(t) + \frac{1}{2}f''(Y_t)\sigma^2(t)\Big)dt + f'(Y_t)\sigma(t)dB_t$$

We have the following integration by parts formula.

(12) $$d(X_tY_t) = X_t dY_t + Y_t dX_t + d[X,Y](t)$$

To calculate $d[X,Y](t)$, we write $d[X,Y](t) = dX_t dY_t$ and use the following rules.

(13) $$(dt)^2 = 0, dB_t dt = 0, (dB_t)^2 = dt$$

A *Stochastic Differential Equation(SDE)* is an equation of the form

(14) $$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$$

The SDE

(15) $$dS_t = S_t(rdt + \sigma dB_t) = rS_t dt + \sigma S_t dB_t$$

has the solution

(16) $$S_t = S_0 e^{(r-\sigma^2/2)t + \sigma B_t}$$

which can be verified by the multi-dimensional Itō's formula. Let X_t be any Itō process, and consider the SDE

(17) $$dU_t = U_t dX_t$$

with $U_0 = 1$. This has the solution

(18) $$U_t = e^{X_t - X_0 - \frac{1}{2}[X](t)}$$

which is the *stochastic exponential* of X, which we write as $\mathcal{E}(X_t)$. If $U_t = \mathcal{E}(X_t) = e^{X_t - X_0 - \frac{1}{2}[X](t)}$, then we define the *stochastic logarithm* of U_t to be X_t, and write $\mathcal{L}(U_t) = X_t$. We can show that $\mathcal{L}(U_t)$ satisfies the SDE

(19) $$d(\mathcal{L}(U_t)) = \frac{dU_t}{U_t}$$

and find the explicit solution

(20) $$\mathcal{L}(U_t) = \ln\Big(\frac{U_t}{U_0}\Big) + \int_0^t \frac{d[U](t)}{2U_t^2}$$

The exponential and logarithms are inverses of each other, so under appropriate conditions $\mathcal{E}(\mathcal{L}(X_t)) = \mathcal{L}(\mathcal{E}(X_t)) = X_t$. *Linear SDE's*, that is, SDE's of the following form:

(21) $$dX_t = (\alpha(t) + \beta(t)X_t)dt + (\gamma(t) + \delta(t)X_t)dB_t$$

where $\alpha(t), \beta(t), \gamma(t), \delta(t)$ are continuous, adapted processes can be solved explicitly:

(22) $$X_t = U_t\Big(X_0 + \int_0^t \frac{\alpha(s) - \delta(s)\gamma(s)}{U_s}ds + \int_0^t \frac{\gamma(s)}{U_s}dB_s\Big)$$

where

(23) $$U_t = \exp\Big(\int_0^t (\beta(s) - \frac{1}{2}\delta^2(s))ds + \int_0^t \delta(s)dB_s\Big)$$

The following SDE

(24) $$dS_t = S_t(rdt + \sigma dB_t) = rS_tdt + \sigma S_tdB_t$$

gives *geometric* or *exponential Brownian motion*:

(25) $$S_t = S_0e^{(r-\sigma^2/2)t+\sigma B_t}$$

The solution of

(26) $$dX_t = \alpha(\theta - X_t)dt + \sigma dB_t$$

is the *Ornstein-Uhlenbeck process*:

(27) $$X_t = e^{-\alpha t}\Big(X_0 + \theta(e^{\alpha t} - 1) + \int_0^t \sigma e^{\alpha s}dB_s\Big)$$

We have the following existence theorem.

Theorem 2 *The SDE $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$ has a unique solution if the following conditions are met.*
1) μ and σ are locally Lipshitz in x uniformly in t. That is, for any T, N there is a $K > 0$ such that for $|x|, |y| < N$, $0 \le t \le T$ we have

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| < K|x - y| \tag{28}$$

2) There is a $K > 0$ such that

$$|\mu(x,t)| + |\sigma(x,t)| < K(1 + |x|) \tag{29}$$

3) X_0 (usually constant) is independent of B and $E[X_0^2] < \infty$.

The Lipschitz condition on σ can be weakened to a Hölder condition. That is, the theorem still holds if $|\sigma(x,t) - \sigma(y,t)| < K|x - y|^\alpha$ for some K and $\alpha \ge 1/2$. If X_t satisfies $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$, then the *generator* of X is the differential operator

$$L_t f(x,t) = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}(x,t) + \mu(x,t)\frac{\partial f}{\partial x}(x,t) \tag{30}$$

That is, L_t is an operation which you can apply to a differentiable function. If f is a smooth function, then

$$M_f(t) = f(X_t, t) - \int_0^t \Big(L_s f + \frac{\partial f}{\partial t}\Big)(X_s, s)ds \tag{31}$$

is a martingale. Since martingales have constant expectation, we find Dynkin's formula:

$$E[f(X_t, t)] = f(X_0, 0) + E\int_0^t \Big(L_s f + \frac{\partial f}{\partial t}\Big)(X_s, s)ds \tag{32}$$

We also have the *Feynman-Kac* formula:

Theorem 3 *Assume r and g are bounded functions, and let*

$$C(x,t) = E\Big(e^{-\int_t^T r(X_s,s)ds} g(X_T)|X_t = x\Big) \tag{33}$$

If there is a solution to

$$\frac{\partial f}{\partial t}(x,t) + L_t f(x,t) = r(x,t)f(x,t), \text{ with } f(x,T) = g(x) \tag{34}$$

Then the solution is unique and equal to $C(x,t)$.

This formula is commonly used in financial applications. A process X_t is called a sub(super)-martingale if $E[X_t|\mathcal{F}_s] \geq X_s(E[X_t|\mathcal{F}_s] \leq X_s)$. A uniformly integrable process is one for which

$$\lim_{n\longrightarrow\infty} \sup_t E\Big(|X_t| 1_{\{|x_t|>n\}}\Big) = 0 \tag{35}$$

They are important because uniformly integrable martingales converge a.s. as $t \longrightarrow \infty$. There are a number of other conditions that imply this(see p. 187). A local martingale is a stochastic process X_t with a sequence of stopping times $\tau_1, \tau_2, \ldots$ increasing to ∞ such that $X_{t\wedge\tau_n}$ is a martingale for each n. *Davis' inequality* and the *Burkholder-Gundy Inequality* relate the supremum of a local martingale with its quadratic variation. Levy's Theorem, which is quite important, states

Theorem 4 *A local martingale M with $M_0 = 0$ a.s. and $[M](t) = t$ is a Brownian motion.*

This leads to the idea that any martingale can be time-changed into a Brownian motion.

Theorem 5 *Let M_t be a continuous martingale with $M_0 = 0$ with quadratic variation process $[M](t)$. Then there is a Brownian motion B such that $M_t = B([M]_t)$. Conversely, if τ_t is defined to be the inverse of $[M]_t$ in the correct manner, we have $M_{\tau_t} = B_t$.*

Let P and Q be two probability measures. Then Q is *absolutely continuous* with respect to P, written $Q \ll P$, if $Q(A) = 0$ whenever $P(A) = 0$.

Theorem 6 *If $Q \ll P$, then there is a random variable Λ such that $Q(A) = E_P[\Lambda 1_A]$, and in general $E_Q[X] = E_P[X\Lambda]$.*

Λ is referred to as the *Radon-Nikodym derivative*, and we write $\Lambda = \frac{dQ}{dP}$. Changing measure is an important technique.

Theorem 7 *Let $X \sim N(0,1)$ under P, and let $\frac{dQ}{dP} = e^{\mu X - \mu^2/2}$. Then Q and P are equivalent(i.e. $Q \ll P$, $P \ll Q$), and $X \sim N(\mu, 1)$ under Q.*

Girsanov's Theorem extends this ideas to processes.

Theorem 8 *a) Let B_t be a Brownian motion under P, with $0 \leq t \leq T$. Let $\frac{dQ}{dP} = e^{-\mu B_T - \frac{1}{2}\mu^2 T}$. Then P and Q are equivalent, and $W_t = B_t + \mu t$ is a Brownian motion under Q. Furthermore $\frac{dP}{dQ} = e^{\mu W_T - \frac{1}{2}\mu^2 T}$.*

b) Let B_t be a Brownian motion under P, with $0 \leq t \leq T$. Suppose H_t is a process such that $X_t = \int_0^t H_s dB_s$ is defined and $\mathcal{E}(X_t)$ is a martingale. Let

$$\frac{dQ}{dP} = \mathcal{E}(X_T) = e^{-\int_0^T H_s dB_s - \frac{1}{2}\int_0^T H_s^2 ds}$$

Then $W_t = B_t + \int_0^t H_s ds$ is a Brownian motion under Q.

The behavior of conditional expectation under changes in measure is given by Bayes' formula:

Theorem 9 *a) If* $Q \ll P$ *and* $\frac{dQ}{dP} = \Lambda$*, then*

$$E_Q[X|\mathcal{G}] = \frac{E_P[X\Lambda|\mathcal{G}]}{E_P[\Lambda|\mathcal{G}]} \tag{36}$$

b) Let $\Lambda(t)$ *be a positive* P*-martingale such that* $E_P[\Lambda(T)] = 1$*. Suppose* $\frac{dQ}{dP} = \Lambda(T)$*. Then, if* X *is* $\mathcal{F}_t$ *measurable,*

$$E_Q[X|\mathcal{F}_s] = E_P[\frac{\Lambda(t)}{\Lambda(s)}X|\mathcal{F}_s] \tag{37}$$

2 Finance

We consider a model in which there are two ways in which a person can invest their money. One is in a stock, S_t, which possesses risk, or randomness, and the second is in a bond or savings account, β_t, which is risk free, i.e. deterministic. We will create a *portfolio*, which is a trading strategy of buying $a(S_t, t) = a(t)$ units of stocks and $b(S_t, t) = b(t)$ units of bonds. The value of the portfolio at any time t is

$$V_t = a(t)S_t + b(t)\beta_t \tag{38}$$

We require this process to be nonnegative, although $a(t)$ and $b(t)$ are each allowed to be negative. We also require that the process be *self-financing*, that is, any change in the amount of money invested can only be funded by money earned or lost by the portfolio. We express this mathematically as $dV_t = a(t)dS_t + b(t)d\beta_t$. A portfolio is *admissible* if it is self-financing and satisfies an integrability condition. A *claim* is simply a non-negative random variable representing some sort of payoff at time T, but we will mainly be interested in *attainable claims*. These are claims for which there is an admissible portfolio such that $V(T) = X$. Claims are priced under the principle of *no-arbitrage*. Arbitrage is essentially risk-free profit. That is, an arbitrage is an admissible trading strategy such that $V(0) = 0$ but $E[V(T)] > 0$(remember V_t must be nonnegative as well). In order to avoid arbitrage, the price of a claim must be equal to the price of any replicating portfolio. The first step is generally to find an *equivalent martingale measure*(EMM) Q such that Q and P are equivalent and under which the discounted stock price $\frac{S_t}{\beta_t}$ is a martingale. Then we have

Theorem 10 *Suppose we can find an EMM* Q *under which* $Z_t = \frac{S_t}{\beta_t}$ *is a martingale. Then for any admissible* V_t*, the process* $\frac{V_t}{\mathcal{B}_t}$ *is a* Q*-martingale as well.*

The following is the First Fundamental Theorem of asset pricing.

Theorem 11 *A market does not admit arbitrage if and only if there is an EMM.*

In this case, we can price a claim by any replicating portfolio.

Theorem 12 *If a market model does not admit arbitrage, then any attainable claim X with admissible strategy can be priced by*

$$C(t) = V_t = E_Q[\frac{\beta_t}{\beta_T} X | \mathcal{F}_t] \tag{39}$$

A market model in which any claim is attainable is called *complete*, and

Theorem 13 *A market model is complete if and only if the EMM Q is unique.*

A good way to determine whether a portfolio is self-financing is the following result.

Theorem 14 *A portfolio $V(t)$ is self-financing if and only if the discounted value process $\frac{V(t)}{\beta_t}$ is a stochastic integral with respect to the discounted price process $Z_t = \frac{S_t}{\beta_t}$:*

$$\frac{V(t)}{\beta_t} = V(0) + \int_0^t a(u) dZ_u \tag{40}$$

This leads to the following result on attainable claims.

Theorem 15 *Let X be an integrable claim and $M_t = E_Q[\frac{X}{\mathcal{B}_T} | \mathcal{F}_t]$. Then X is attainable if and only if*

$$M_t = M_0 + \int_0^t H_u dZ_u \tag{41}$$

for some measurable process H_u. Furthermore $\frac{V_t}{\beta_t}$ is the same for any replicating portfolio V.

The *numeraire* is the factor which we discount by. So, in the examples above, the value of the risk-free account, β_t, was the numeraire. However we can discount by other factors, such as the stock price. The following gives the correct formula for change of numeraire.

Theorem 16 *Suppose $\frac{S_t}{\beta_t}$ is a positive Q-martingale. Let $\frac{dQ_1}{dQ} = \frac{S(T)/S(0)}{\beta(T)/\beta(0)}$. Then $\frac{\beta_t}{S_t}$ is a Q_1-martingale. Furthermore if $C(t)$ is the price of an attainable claim X, then*

$$C(t) = E_Q[\frac{\beta_t}{\beta_T} X | \mathcal{F}_t] = E_{Q_1}[\frac{S_t}{S_T} X | \mathcal{F}_t] \tag{42}$$

The concept of change of numeraire leads to a general formula for the value of a call option.

Theorem 17 *Let a call option have payoff* $C(T) = (S(T) - K)^+$. *Then if* $\frac{dQ_1}{dQ} = \frac{S(T)/S(0)}{\beta(T)/\beta(0)}$ *we have*

$$C = \frac{\beta_0}{S_0} Q_1(S(T) > K) - \frac{K}{\beta(T)} Q(S(T) > K) \tag{43}$$

The Black-Scholes model is currently the leading model for pricing claims, although it is not without its critics. We begin by assuming that a stock price S_t satisfies the SDE $dS_t = \mu S_t dt + \sigma S_t dB_t$. Changing measure transforms this SDE into $dS_t = rS_t dt + \sigma S_t dB_t$, where r is the interest rate, so $d\beta_t = r\beta_t dt$. If a claim is given by $g(S(T))$, for some function g, then the value of the claim at time t is given by $C_t = e^{-r(T-t)} E_Q[g(S(T))|\mathcal{F}_t]$. Set $C(x,t) = e^{-r(T-t)} E_Q[g(S(T))|S_t = x]$. Then, either by the Feynman-Kac formula or by replicating portfolio arguments we find that $C(x,t)$ satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C(x,t)}{\partial x^2} + rx \frac{\partial C(x,t)}{\partial x} + \frac{\partial C(x,t)}{\partial t} - rC = 0 \tag{44}$$

This is known as the *Black-scholes PDE*. We use $g(S(T))$ as the boundary conditions, and this can in many cases be solved explicitly. We can price call and put options, with formula given in the book and elsewhere(wikipedia, for example). This formula can also be obtained through other arguments(including Theorem 17).