Getting started with signal processing

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This document describes some of the basics of signal processing (measurements and data-analysis) using examples from a seismic isolation experiment. While it is written partly for our own reference, we hope it will also be useful for future students working on this project or on a related one.

I. KEY FORMULA

Consider simultaneous time series for two channels, $s_1(t)$ and $s_2(t)$ with FFTs denoted $\tilde{s}_1 f$ and $\tilde{s}_2(f)$. The transfer function from channel 1 to channel 2 is:

$$T_{21}(f) \equiv \frac{\tilde{s}_2(f)}{\tilde{s}_1(f)} = a_{21}(f)\, e^{i\,\phi_{21}(f)}. \tag{1}$$

The transfer function is complex valued and it is typical to plot the amplitude $a_{21}(f)$ and phase $\phi_{21}(f)$ separately. The transfer function describes how an excitation in one channel couples into the second channel as a fuction of frequency.

In order to determine if a transfer function measurement is trustworthy, one must also calculate coherence:

$$\text{coh}(f) \equiv \frac{\left|\sum_{I=1}^{N} \tilde{s}_1^\star(f,I)\tilde{s}_2(f,I)\right|^2}{\left(\sum_{I=1}^{N} |\tilde{s}_1^\star(f,I)|^2\right)\left(\sum_{I=1}^{N} |\tilde{s}_2^\star(f,I)|^2\right)}. \tag{2}$$

Here we are summing over $I = 1...N$ data segments. The coherence can also be written in terms of average power:

$$\text{coh}(f) = \frac{\left|\left(\overline{P_{12}(f)}\right)\right|^2}{\overline{P_{11}(f)}\,\overline{P_{22}(f)}}, \tag{3}$$

where $\overline{P_{xy}(f)}$ is the average cross- or auto- power in channels x and y. When we perform a transfer function measurement, we want the coherence to be near to one. If it is not, then the two channels do not contain the same signal, and so the transfer function measurement is not telling us useful information. Examples of transfer function and coherence measurements are given in Fig. 1.

II. CALIBRATION AND UNITS

In this section we describe the absolute calibration of a photodiode shadow sensor using a pre-calibrated accelerometer (see Fig 2). In the process, we illustrate how to think about units in the time and frequency domains. We consider two channels: $s_a(t)$ represents the time series measurement of an accelerometer and $s_p(t)$ represents the time series measurement of a photo-diode shadow sensor. Both sensors are (simultaneously) measuring the motion of a test mass, which we drive at angular frequency $\omega_0 = 2\pi f_0$. Both channels are measured in mV, but the manufacturer of the accelerometer provide the following calibration information

$$\frac{1000\,\text{mV}_\text{a}}{g} = \frac{1000\,\text{mV}_\text{a}}{9.8\,\text{m}\,\text{s}^{-2}} \approx \frac{100\,\text{mV}_\text{a}}{\text{m}\,\text{s}^{-2}}. \tag{4}$$

The units mV_a denote accelerometer voltage, which is in contrast to other voltages we can measure.

Ultimately, however, we are interested in measuring length, not acceleration. In the Fourier domain, we have

$$\tilde{a}(f) = -\omega^2 \tilde{x}(f). \tag{5}$$

Thus, we can construct an accelerometer length measurement like so:

$$\tilde{x}_a(f) = \frac{\tilde{s_a}(f)}{\omega^2}\left(\frac{1\,\text{m}\,\text{s}^{-2}}{100\,\text{mV}_\text{a}}\right)\left(\frac{1000\,\text{mm}}{1\,\text{m}}\right). \tag{6}$$

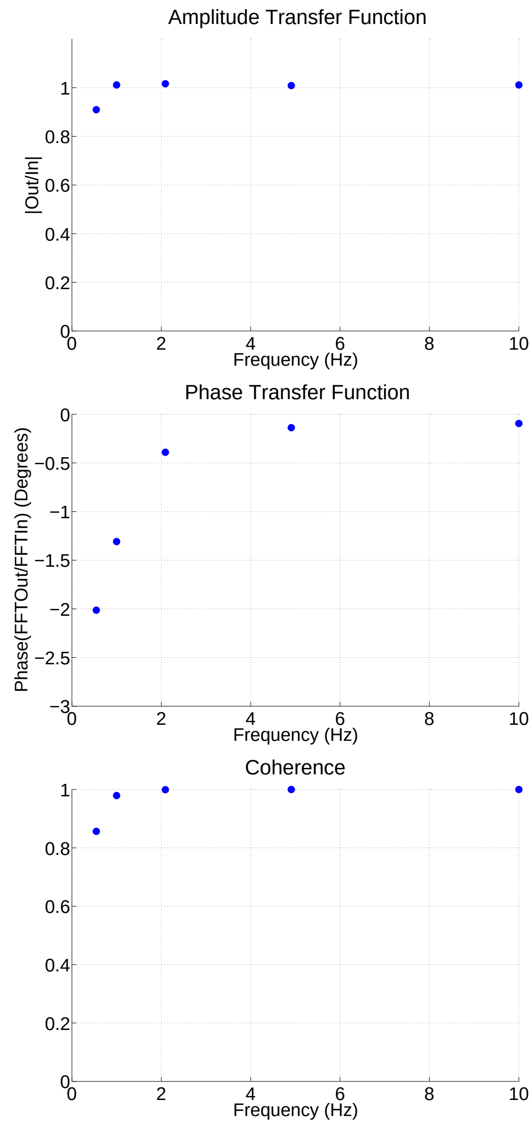

FIG. 1: Example transfer function and coherence measurements. These data are taken from two nearly parallel accelerometers. Top: amplitude of transfer function. Middle: phase of transfer function. Bottom: coherence.

The factor of ω^2 in the denominator will offset the s^{-2} in the numerator and the units work out to $[\tilde{x}_a(f)] = \mathrm{mm}$. Here we we have implicitly assumed a specific normalization for our FFT:

$$\tilde{s}_a(f) = \frac{2}{\mathtt{L}}\mathtt{fft}\left(s_a(t)\right), \tag{7}$$

where `fft` is the `matlab` fast Fourier transform and `L` is the length of the $s_a(t)$ array. Note that other different FFT normalizations are also commonly used.

Now we measure the same motion with the shadow sensor and the photodiode. From this measurement we can construct a transfer function

$$T_{ap}(f) = \left|\frac{\tilde{x_a}(f)}{\tilde{s_p}(f)}\right|, \tag{8}$$

with units $[T_{ap}] = \mathrm{mm\,mV_p^{-1}}$, which characterizes the mm length change measured by the accelerometer per change in photodiode voltage. We have labeled the photodiode voltage in units of $\mathrm{mV_p}$ to distinguish it from accelerometer voltage. Once calibrated, both devices will measure length in mm, but $3\,\mathrm{mV_p}$ is not equivalent to $3\,\mathrm{mV_a}$ of accelerometer voltage. If T_{ap} is nearly constant across the frequency range of interest, the absolute-calibrated photodiode length sensing measurement is given by

$$\tilde{x}_p = T_{ap}\,\tilde{s}_p, \tag{9}$$

where $[\tilde{x}_p] = \mathrm{mm}$. ■ [COMMENT by Eric: include a plot of $T_{ap}(f)$ vs f.]

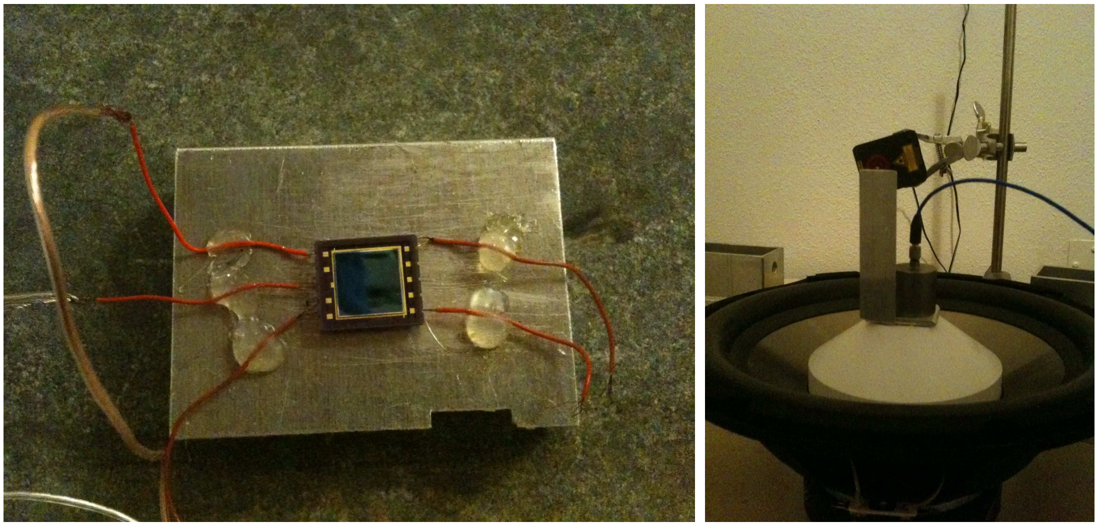

FIG. 2: Left: Hamamatsu (S8650) photodiode used in the shadow sensor. Right: the actuation speaker mounted with an accelerometer (PCB 393B04) and a shadow-sensor flag. In the background is a HeNe laser for the shadow sensor.

III. NOISE FLOOR

Now we attempt to answer the question: (approximately) how sensitive is our shadow sensor? We drive our actuator at a fixed frequency f_a and observe the resultant peak measured at f_a in the the amplitdude spectral density $|\tilde{s}_p(f)|$ of the photodiode data. We gradually reduce the actuation amplitude until the peak is no longer sticking up clearly above the surrounding noise. ■ [TODO: Include two figures of $|\tilde{s}_p(f)|$. Mark f_0 with red and all other frequencies with blue. One plot is for a high actuation amplitude and the other plot is for a marginal actuation amplitude...right before the peak is no longer clearly visible. Include the actuation voltage in the title of the plots. The y-axes should be in units of mm.] We find that the photodiode sensitivity at $f_a = 10\,\mathrm{Hz}$ is $\delta x = 100\,\mathrm{nm}$. ■ [COMMENT by Eric: Show how the sensitivity scales with f_a.]

IV. IMPLICATIONS FOR SEISMIC ISOLATION

How does our photodiode sensitivity affect our ability to measure the attenuation of seismic motion with a pendulum? The transfer fucntion from the hanging point to the (ideal) pendulum bob is given by

$$T(f) = \frac{f_0^2}{f_0^2 - f^2}, \tag{10}$$

where $\omega_0 = 2\pi f_0 = \sqrt{g/L}$ is the resonant frequency of the pendulum.