COLOURING THE TRIANGLES DETERMINED BY A POINT SET

Ruy Fabila-Monroy,[*] *David R. Wood*[†]

Abstract. Let P be a set of n points in general position in the plane. We study the chromatic number of the intersection graph of the open triangles determined by P. It is known that this chromatic number is at least $\frac{n^3}{27} + O(n^2)$, and if P is in convex position, the answer is $\frac{n^3}{24} + O(n^2)$. We prove that for arbitrary P, the chromatic number is at most $\frac{n^3}{19.259} + O(n^2)$.

1 Introduction

Let P be a set of n points in general position in the plane (that is, no three points are collinear). A triangle with vertices in P is said to be *determined by* P. Let G_P be the intersection graph of the set of all open triangles determined by P. That is, the vertices of G_P are the triangles determined by P, where two triangles are adjacent if and only if they have an interior point in common. This paper studies the chromatic number of G_P.

Consider a colour class X in a colouring of G_P. Then X is a set of triangles determined by P, no two of which have an interior point in common. If $P' \subseteq P$ is the union of the vertex sets of the triangles in X, then there is a triangulation of P' in which each triangle in X is an internal face. The converse also holds: the set of internal faces in a triangulation of a subset of P can all be assigned the same colour in a colouring of G_P. Thus $\chi(G_P)$ can be considered to be the minimum number of triangulations of subsets of P that cover all the triangles determined by P, where a triangulation T is said to *cover* each of its internal faces.

First consider $\chi(G_P)$ for small values of n. If $n = 3$ then $\chi(G_P) = 1$ trivially. If $n = 4$ then $\chi(G_P) = 2$, as illustrated in Figure 1. If $n = 5$ then $\chi(G_P) = 5$, as illustrated in Figure 2. Here $\omega(G_P)$ denotes the maximum order of a clique in G_P.

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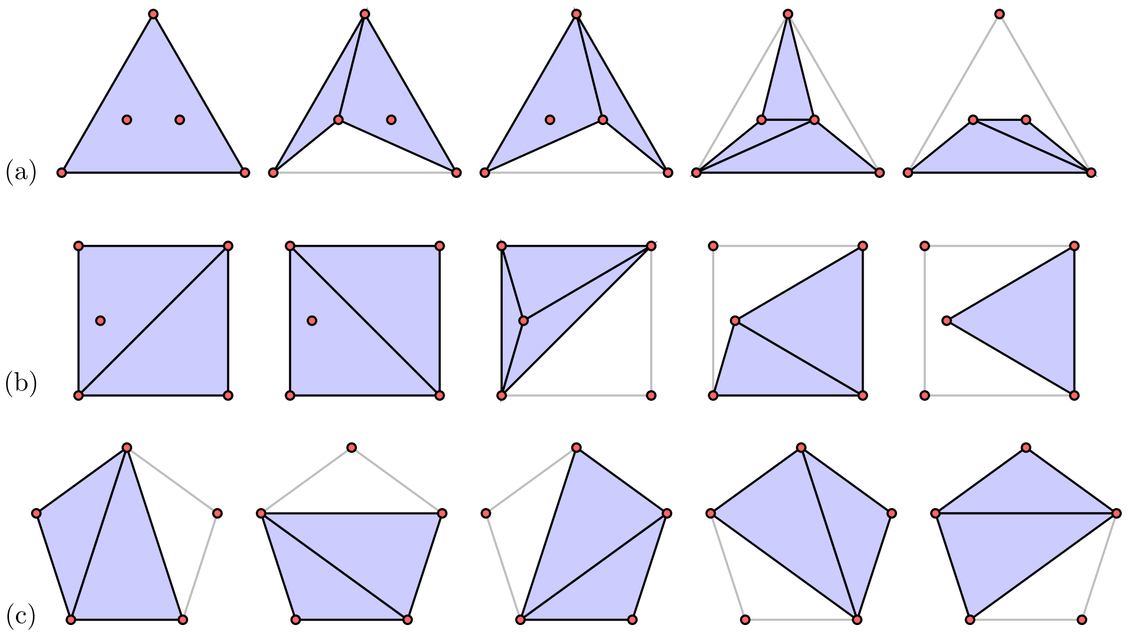

Figure 2: Colouring the triangles determined by five points: (a) three boundary points, (b) four boundary points, (c) five boundary points. In each case, $\chi(G_P) = \omega(G_P) = 5$.

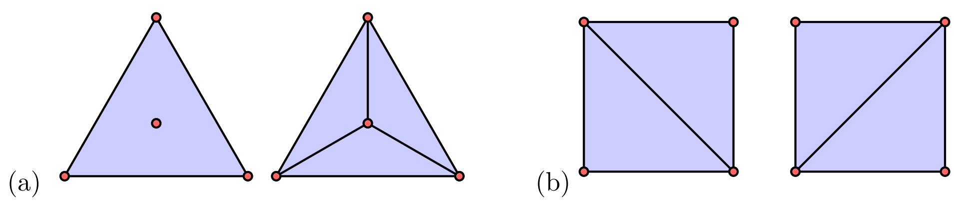

Figure 1: Colouring the triangles determined by four points: (a) non-convex position, (b) convex position. In both cases, $\chi(G_P) = \omega(G_P) = 2$.

For $n = 6$, we used the database of 16 distinct order types of 6 points in general position [1], and calculated $\chi(G_P)$ exactly for each such set using sage [14]. As shown in Appendix B, $\chi(G_P) = 8$ for each 6-point set P. This result will also be used in the proof of Theorem 1 below.

It is interesting that $\chi(G_P)$ is invariant for sets of n points, for each $n \leq 6$. However, this property does not hold for $n = 7$. If P consists of 7 points in convex position, then $\chi(G_P) = 14$, whereas we have found a set P of 7 points in general position for which $\chi(G_P) = 13$; see Appendix C.

Now consider $\chi(G_P)$ for arbitrarily large values of n. If P is in convex position then the problem is solved: results of Cano et al. [8] imply that

$$\chi(G_P) = \begin{cases} \frac{1}{24}(n-1)n(n+1) & \text{if } n \text{ is odd} \\ \frac{1}{24}(n-2)n(n+2) & \text{if } n \text{ is even} \end{cases} .$$

See Appendix A for a proof of this and other related results.

Our main contribution is to prove the following bound for arbitrary point sets.

Theorem 1. *For every set P of n points in general position in the plane,*

$$\tfrac{1}{27}n^3 \;\leq\; \chi(G_P) \;\leq\; \tfrac{27}{520}n^3 + O(n^2) \;=\; \tfrac{1}{19.259\ldots}n^3 + O(n^2)\ .$$

It is an interesting open problem whether the lower bound on $\chi(G_p)$ in Theorem 1 is tight. That is, are there infinitely many n-point-sets P for which $\chi(G_P) = \frac{n^3}{27} + O(n^2)$?

Note that all computer code used in this project is available from the first author upon request.

2 Proof of Theorem 1

The lower bound in Theorem 1 follows immediately from a theorem by Boros and Füredi [5], who proved that for every set P of n points in general position, there is a point q in the plane such that q is in the interior of at least $\frac{n^3}{27} + O(n^2)$ triangles determined by P. These triangles form a clique in G_P, implying $\chi(G_P) \geq \omega(G_p) \geq \frac{n^3}{27} + O(n^2)$. This result is called the 'first selection lemma' by Matoušek [12, Section 9.1]. See [6] for an alternative proof and see [3, 11] for generalisations.

Note that Boros and Füredi's theorem is stronger than simply saying that $\omega(G_p) \geq \frac{n^3}{27} + O(n^2)$. For example, for sets of n points in convex position, G_P is invariant. Moving the points around a circle does not change the graph, which is not true for the question of a point in many triangles. Indeed, Bukh et al. [7] proved that there is a set P of n points in convex position, such that every point in the plane is in the interior of at most $\frac{n^3}{27} + O(n^2)$ triangles determined by P (thus proving that the Boros-Füredi bound is best possible). However, in this case, $\omega(G_P) = \frac{n^3}{24} + O(n^2)$ by the result of Cano et al. [8] mentioned above.

The proof of the upper bound in Theorem 1 depends on the following lemmas. The first is a restatement of results from the literature on covering arrays; see [10] for the details.

Lemma 2 ([10])**.** *Let G be a complete k-partite graph with at most m vertices in each part. Then G contains $m^3 + O(m^2)$ copies of K_k such that each triangle of G appears in some copy.*

Lemma 3. *Let A and B be sets of n points in general position in the plane separated by a line. Let X be the set of open triangles that are determined by $A \cup B$ and have at least one vertex in each of A and B. Then the chromatic number of the intersection graph of X is at most $\frac{2}{5}n^3 + O(n^2)$*

Proof. We proceed by induction on n. It is easily seen that two colours suffice for $n \leq 2$.

If necessary, add a point to A and B so that $|A| = |B| = 2m$, where $m := \lceil \frac{n}{2} \rceil$. Adding points cannot decrease the chromatic number. By the Ham Sandwich Theorem there is a line ℓ such that in each open half-plane determined by ℓ, there are exactly m

points of A and m points of B. Without loss of generality, ℓ is horizontal. Let A_1 and A_2 respectively be the subsets of A consisting of points above and below ℓ. Define B_1 and B_2 analogously. Thus $|A_1| = |A_2| = |B_1| = |B_2| = m$. We call A_1, A_2, B_1 and B_2 *quadrants*.

Let G be the complete 4-partite graph with colour classes A_1, A_2, B_1, B_2. By Lemma 2, G contains $m^3 + O(m^2)$ copies of K_4 such that each triangle of G appears in some copy. Say $\{a_1, a_2, b_1, b_2\}$ induce such a copy of K_4, where $a_i \in A_i$ and $b_i \in B_i$. The intersection graph of the open triangles determined by any set of four points is 2-colourable, as illustrated in Figure 1. Thus $2m^3 + O(m^2)$ colours suffice for the triangles with vertices in distinct quadrants.

For each $i, j \in \{1, 2\}$, by induction, $\frac{2}{5}m^3 + O(m^2)$ colours suffice for the triangles in X determined by $A_i \cup B_j$. Moreover, the triangles determined by $A_1 \cup B_1$ can share the same set of colours as the triangles determined by $A_2 \cup B_2$. Thus $\frac{6}{5}m^3 + O(m^2)$ colours suffice for the triangles with vertices in two quadrants. This accounts for all triangles in X. The total number of colours is $(2 + \frac{6}{5})m^3 + O(m^2) = \frac{2}{5}n^3 + O(n^2)$. □

Proof of the Upper Bound in Theorem 1. We proceed by induction on n. As shown in Section 1, for $n = 3, 4, 5, 6$ every point set P with $|P| = n$ satisfies $\chi(G_P) = 1, 2, 5, 8$ respectively. Now assume that $n \geq 7$.

Ceder [9] proved that there are three concurrent lines that divide the plane into six parts each containing at least $\frac{n}{6} - 1$ points in its interior. So each part has at most $n - 5(\frac{n}{6} - 1) = \frac{n}{6} + 5$ points. Add points if necessary so that each part contains exactly $m := \lfloor \frac{n}{6} \rfloor + 5$ points. Adding points cannot decrease the chromatic number. Let $P_1, P_2, \dots, P_6$ be the partition of P determined by the six parts, in clockwise order about the point of concurrency. Each P_i is called a *sector*. Let G be the complete 6-partite graph, with colour classes $P_1, P_2, \dots, P_6$.

By Lemma 2, G contains $m^3 + O(m^2)$ copies of K_6 such that each triangle of G appears in some copy. Each copy of K_6 corresponds to a set of points $\{x_1, \dots, x_6\}$ such that each $x_i \in P_i$. The chromatic number of the intersection graph of open triangles determined by $\{x_1, \dots, x_6\}$ is 8; see Appendix B. Thus $8m^3 + O(m^2)$ colours suffice for the triangles determined by P with vertices in distinct sectors, as illustrated in Figure 3(a).

For $i, j \in \{1, \dots, 6\}$, let $X_{i,j}$ be the set of triangles determined by $P_i \cup P_j$ that have at least one endpoint in each of P_i and P_j.

By induction, $\frac{27}{520}(2m)^3 + O(m^2)$ colours suffice for the triangles determined by $P_1 \cup P_2$. The same set of colours can be used for the triangles determined by $P_3 \cup P_4$, and for the triangles determined by $P_5 \cup P_6$. This accounts for all triangles contained in a single sector, as well as $X_{1,2} \cup X_{3,4} \cup X_{5,6}$, as illustrated in Figure 3(b).

We now colour $X_{i,j}$ for other values of i, j. Note that P_i and P_j are separated by a line. Thus, by Lemma 3, $\frac{2}{5}m^3 + O(m^2)$ colours suffice for the triangles in $X_{i,j}$. Moreover, $X_{2,3} \cup X_{4,5} \cup X_{6,1}$ can use the same set of colours, as can $X_{1,5} \cup X_{2,4}$ and $X_{1,3} \cup X_{4,6}$ and $X_{3,5} \cup X_{2,6}$. These cases are illustrated in Figures 3(c)–(f). Each of $X_{1,4}$, $X_{2,5}$ and $X_{3,6}$ use their own set of colours, as illustrated in Figures 3(g)–(i). In total the number of colours is

$$8m^3 + O(m^2) + \frac{27}{520}(2m)^3 + O(m^2) + \frac{14}{5}m^3 + O(m^2) = \frac{27}{520}n^3 + O(n^2) .$$

□

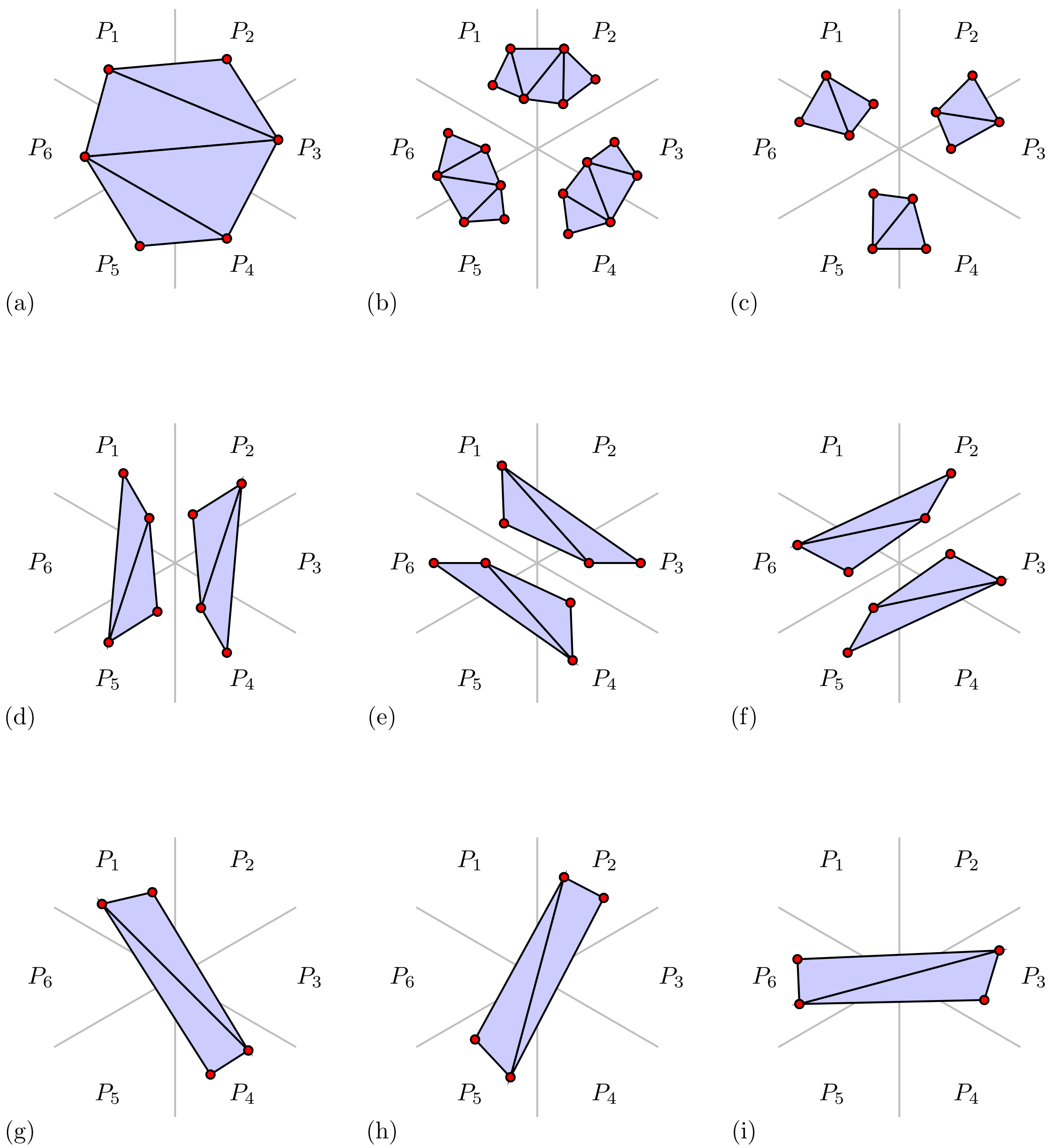

Figure 3: Partition of triangles in Theorem 1.

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A Related Results

The following theorem is obtained by combining results by Boros and Füredi [4, 5] and Cano et al. [8]. In particular, Boros and Füredi [4, 5] proved that (A) = (B) = (F) and Cano et al. [8] proved that (E) = (F). We include the proof for completeness. See [13] for other combinatorial objects counted by the same formula. A *tournament* is an orientation of a complete graph.

Theorem 4. *The following are equal:*

(A) the maximum number of directed 3-cycles in a tournament on n vertices,

(B) the maximum number of triangles determined by n points in general position with an interior point in common,

(C) the maximum number of triangles determined by n points in convex position with an interior point in common,

(D) the clique number of the intersection graph of the open triangles determined by n points in convex position,

(E) the chromatic number of the intersection graph of the open triangles determined by n points in convex position,

(F)

$$\begin{cases} \frac{1}{24}(n-1)n(n+1) & \text{if } n \text{ is odd} \\ \frac{1}{24}(n-2)n(n+2) & \text{if } n \text{ is even} \end{cases}.$$

Proof. **(A) $\leq$ (F):** (This is an exercise in [2, page 33].) Let G be a tournament on n vertices. Let $\deg^+(u)$ be the outdegree of each vertex u of G. Let X be the set of directed 3-cycles in G. For each triple $\{u, v, w\}$ of vertices not in X, exactly one of u, v, w, say u, has outdegree 2 in $G[\{u, v, w\}]$. Charge this triple to u. Exactly $\binom{\deg^+(u)}{2}$ such triples are charged to u. Thus the number of triples not in X equals $\sum_u \binom{\deg^+(u)}{2}$. Hence

$$|X| = \binom{n}{3} - \sum_u \binom{\deg^+(u)}{2}, \tag{1}$$

which is maximised when the outdegrees are as equal as possible (subject to $\sum_u \deg^+(u) = \binom{n}{2}$). Thus when n is odd, $|X|$ is maximised when every vertex has outdegree $\frac{n-1}{2}$. Hence $|X| \leq \binom{n}{3} - n\binom{(n-1)/2}{2} = \frac{1}{24}(n-1)n(n+1)$. When n is even, $|X|$ is maximised when half the vertices have outdegree $\frac{n-2}{2}$ and the other half have outdegree $\frac{n}{2}$. Hence $|X| \leq \binom{n}{3} - \frac{n}{2}\binom{(n-2)/2}{2} - \frac{n}{2}\binom{n/2}{2} = \frac{1}{24}(n-2)n(n+2)$.

(B) $\leq$ (A): Let P be a set of n points in general position. Let X be a set of triangles determined by P that contain a common interior point q. Let G be the n-vertex tournament with vertex set P, where the edge vw is directed from v to w whenever w is clockwise from v in the triangle vwq. If vwq are collinear then orient vw arbitrarily in G. A triangle in X is a directed 3-cycle in G. Thus $|X|$ is at most the maximum number of directed 3-cycles in an n-vertex tournament.

(C) $\leq$ (B): This follows immediately from the definitions.

(C) $\leq$ (D): If P is a set of points, and X is a set of triangles determined by P with an interior point in common, then X is a clique in G_P. Thus (D) $\geq$ (C).

(D) $\leq$ (E): The chromatic number of every graph is at least its clique number.

(E) $\leq$ (D): For sets P of n points in convex position, G_P does not depend on the particular choice of P. Thus we may assume that P consists of n equally spaced points around a circle. Below we define a specific point q at or near the centre of the circle. Say a triangle determined by P is *central* if it contains q in its interior. Thus the set of central triangles are a clique in G_P. For each central triangle uvw, we define an independent set of triangles (including uvw) that is said to *belong* to uvw. We prove that each triangle is in an independent set belonging to some central triangle. Thus these independent sets define a colouring of G_P, with one colour for each central triangle.

First suppose that n is even. For each point $v \in P$, let v' be the point on the circle antipodal to v. Since n is even, $v' \in P$. A triangle determined by P is *long* if it contains an antipodal pair of vertices. Let q be a point near the centre of the circle, such that for all consecutive points $v, w \in P$, exactly one of the long triangles $vv'w$ and $vv'w'$ contain q in their interior. If uvw is a non-long central triangle, then each of uvw', $uv'w$ and $u'vw$ is not central, and $\{uvw, uvw', uv'w, u'vw\}$ is the independent set that belongs to uvw. If $vv'w$ is a long central triangle, then $vv'w'$ is not central, and $\{vv'w, vv'w'\}$ is the independent set that belongs to $vv'w$. We claim that every triangle determined by P is in an independent set that belongs to a central triangle. Let uvw be a non-central triangle. Without loss of generality, vw separates u from q, implying $u'vw$ is a central triangle, and uvw is in the independent set that belongs to $u'vw$ (regardless of whether $u'vw$ is long), as claimed.

Now assume that n is odd. For each point $v \in P$, let v' be the point in P immediately clockwise from the point on the circle antipodal to v (which is not in P since n is odd). Let q be the centre of the circle. If uvw is a central triangle, and no two of u, v, w are consecutive around the circle, then each of uvw', $uv'w$ and $u'vw$ is not central, and $\{uvw, uvw', uv'w, u'vw\}$ is the independent set in G_P that belongs to uvw. If uvw is a central triangle, and u and v are consecutive, then $uv'w$ and $u'vw$ are not central, and $\{uvw, uv'w, u'vw\}$ is the independent set in G_P that belongs to uvw. We claim that every triangle determined by P is in an independent set that belongs to a central triangle. Let uvw be a non-central triangle. Without loss of generality, vw separates u from q. Let x be the vertex immediately anticlockwise from u'. Then xvw is a central triangle, and $x' = u$. Thus uvw is in the independent set that belongs to xvw, as claimed.

Since there is one colour for each central triangle in the above colouring, the set of central triangles are a maximum clique in G_P, and $\chi(G_P) = \omega(G_P)$. That is, (D) = (E).

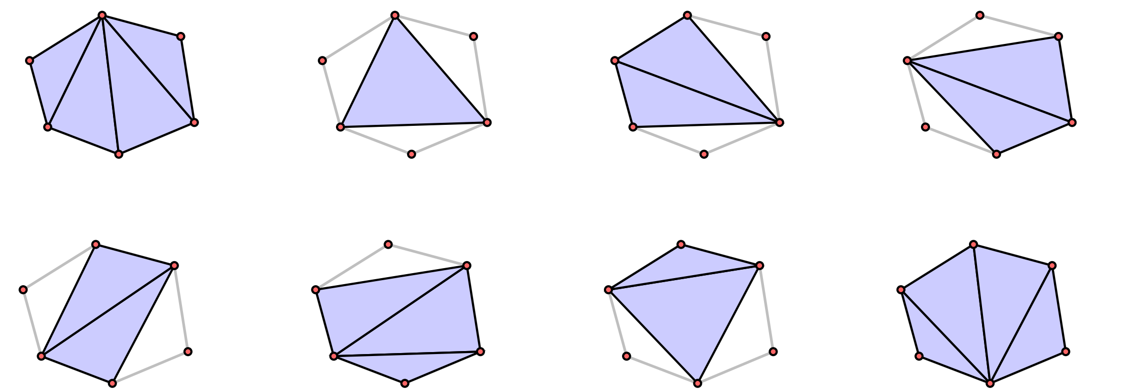
(F) $\leq$ (C): Let P be n evenly spaced points on a circle. Let q be the point near the centre of the circle defined in the proof that (E) $\leq$ (D). Let X be the set of triangles determined by P that contain q in their interior. Thus (C) $\geq |X|$. Let G be the n-vertex tournament with vertex set P, where the edge vw is directed from v to w whenever w is clockwise from v in the triangle vwq. Observe that if n is odd, then every vertex in G has outdegree $\frac{n-1}{2}$. And if n is even, then half the vertices in G have outdegree $\frac{n-2}{2}$ and the other half have outdegree $\frac{n}{2}$. The analysis in the proof that (A) $\leq$ (F) shows that $|X| =$ (F). Hence (C) $\geq$ (F).

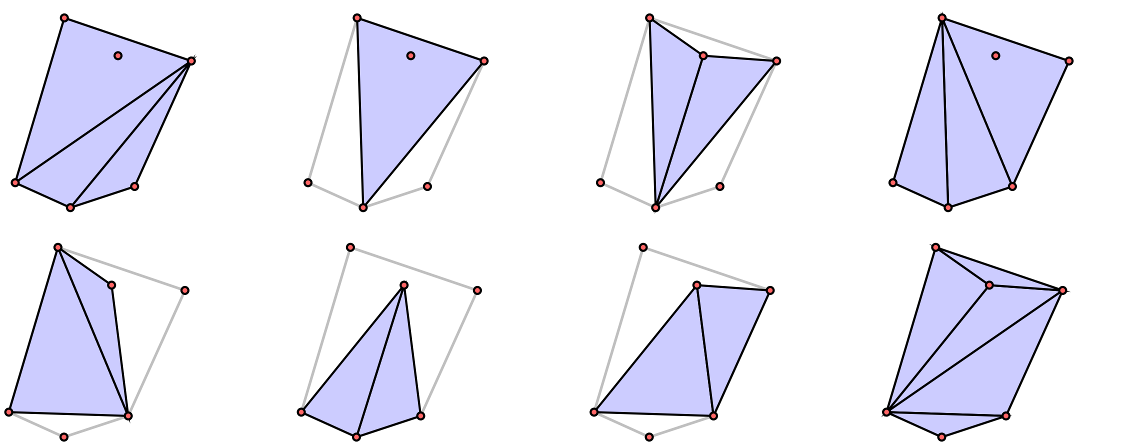
(E) $\leq$ (A): Let P be n evenly spaced points on a circle. Let q be the point near the centre of the circle defined in the proof that (E) $\leq$ (D). Let G be the n-vertex tournament with vertex set P, where the edge vw is directed from v to w whenever w is clockwise from v in the triangle vwq. Three vertices form a directed 3-cycle in G if and only if they form a central triangle. Thus (A) is at least the number of central triangles, which equals (E) by the proof that (E) $\leq$ (D).

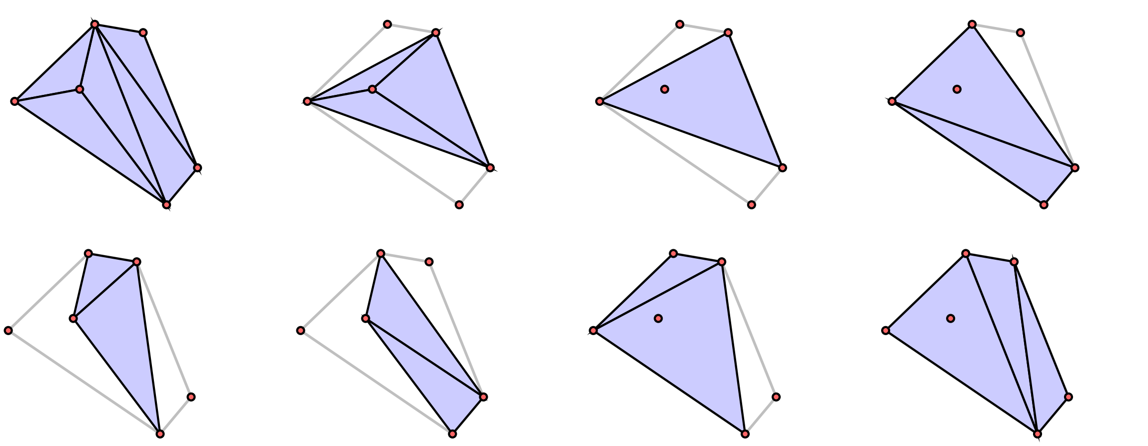
We have proved that (F) $\leq$ (C) $\leq$ (B) $\leq$ (A) $\leq$ (F) and (F) $\leq$ (C) $\leq$ (D) $\leq$ (E) $\leq$ (A) $\leq$ (F). Thus (A) = (B) = (C) = (D) = (E) = (F). ☐

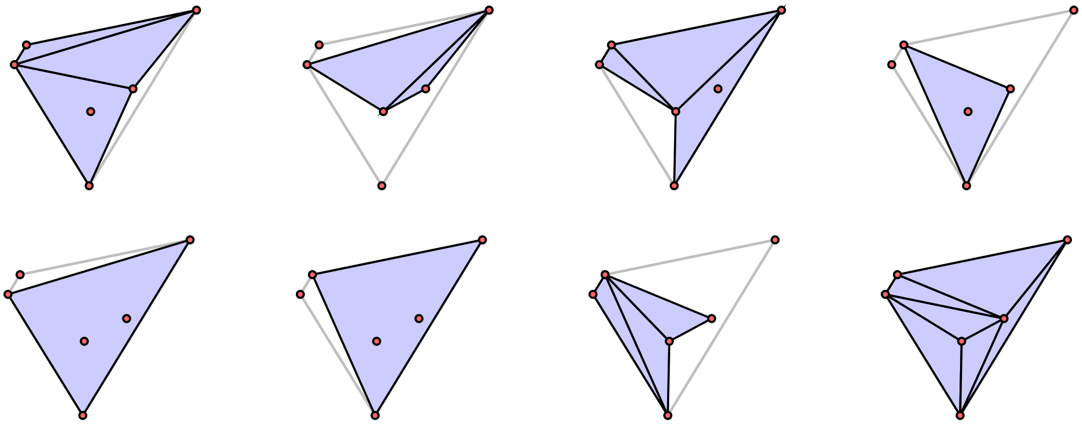
We conjecture that the maximum clique number of the intersection graph of the open triangles determined by n points in general position also equals the number in Theorem 4, as does the maximum chromatic number. It may even be true that $\chi(G_P) = \omega(G_P)$ for every set P of points in general position. We have verified by computer that $\chi(G_P) = \omega(G_P)$ for every set P of at most 7 points in general position.

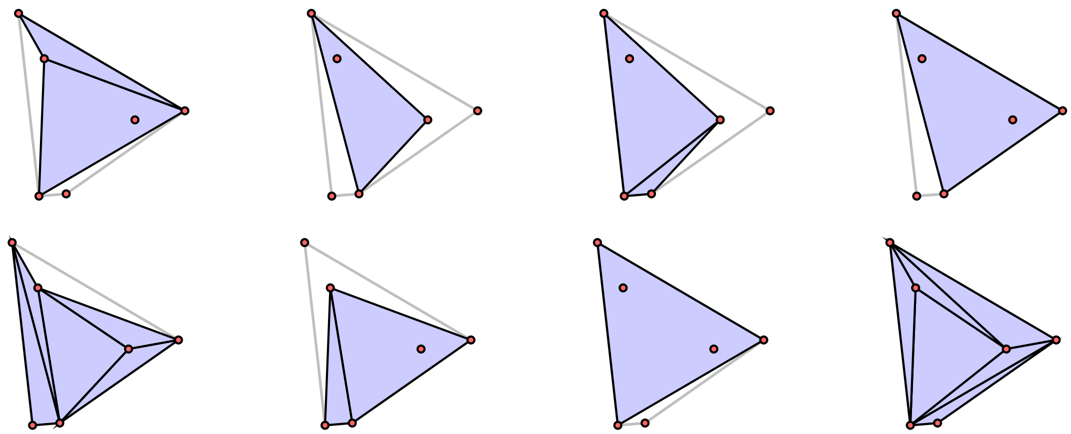
B 8-Colouring the Triangles Determined by 6 Points

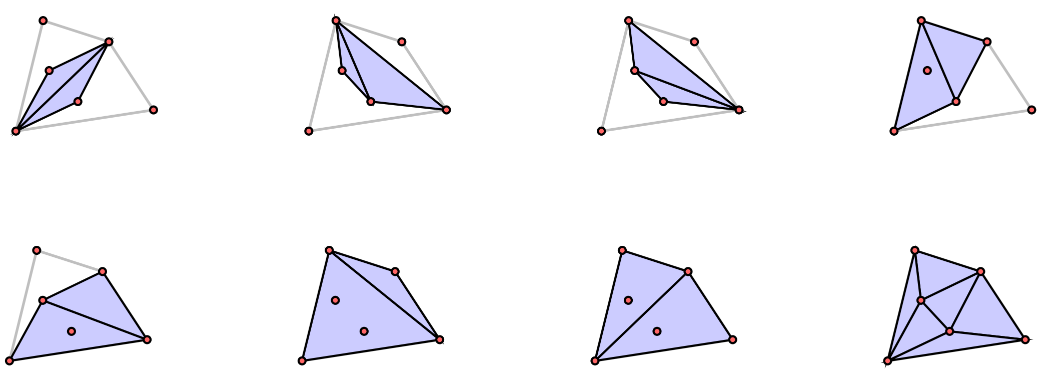


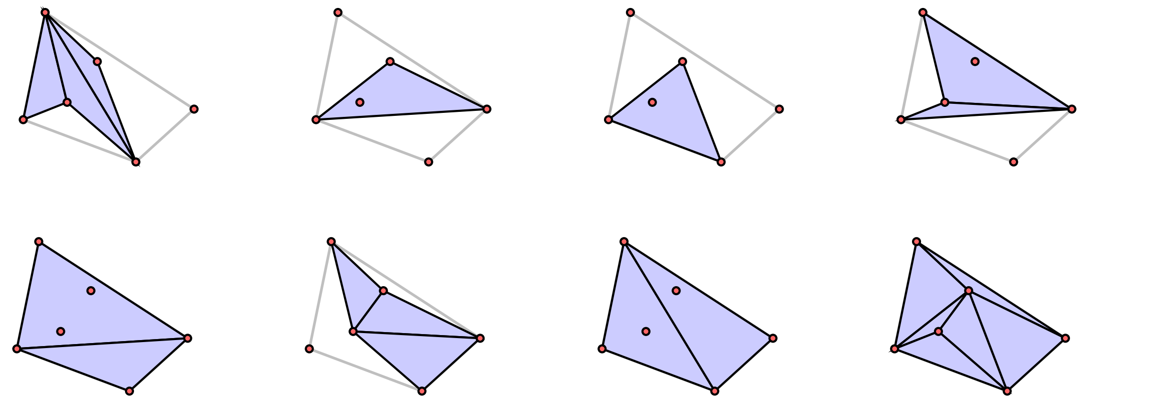


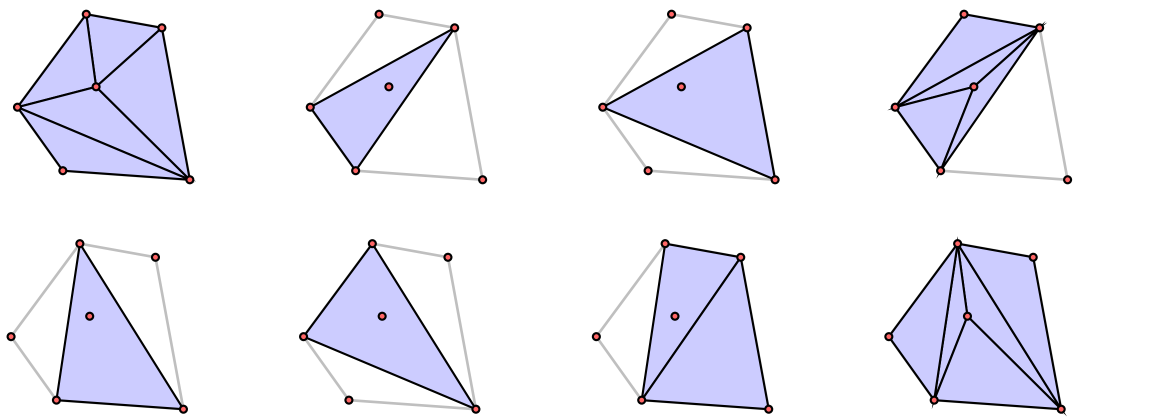


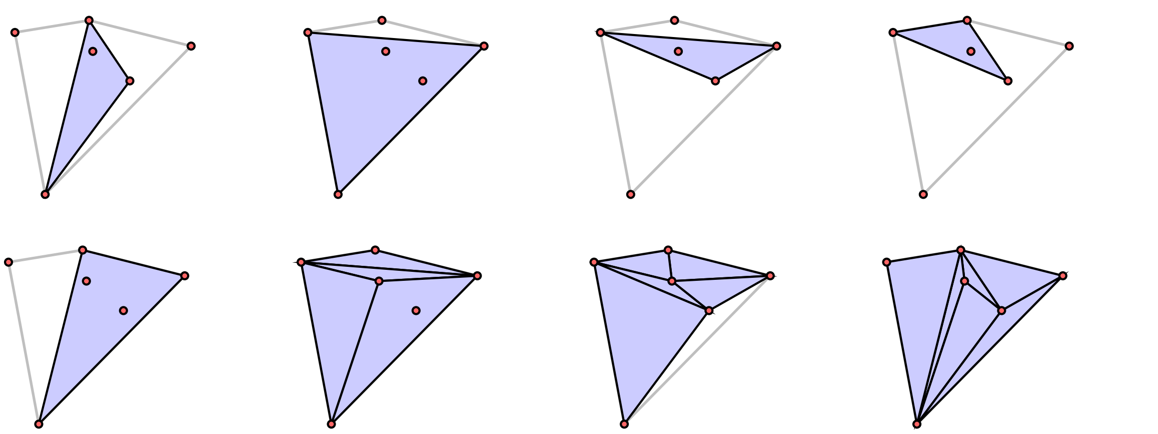


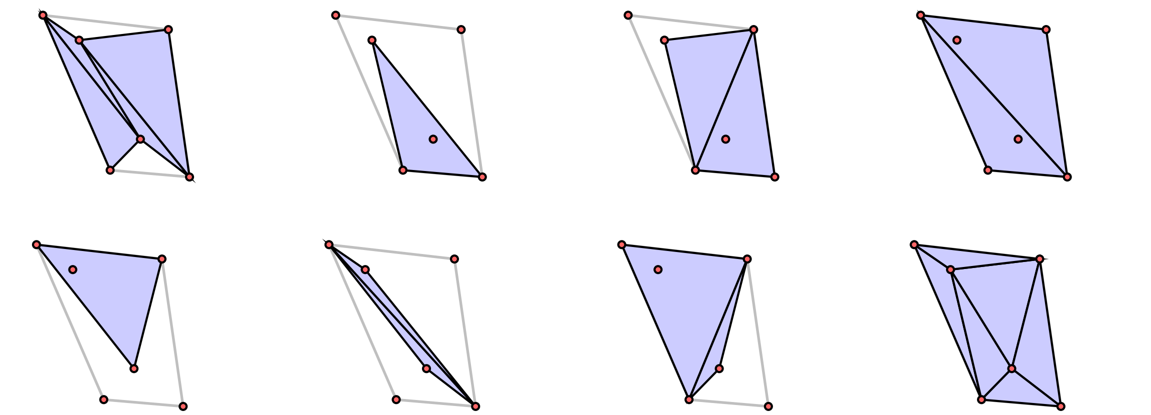


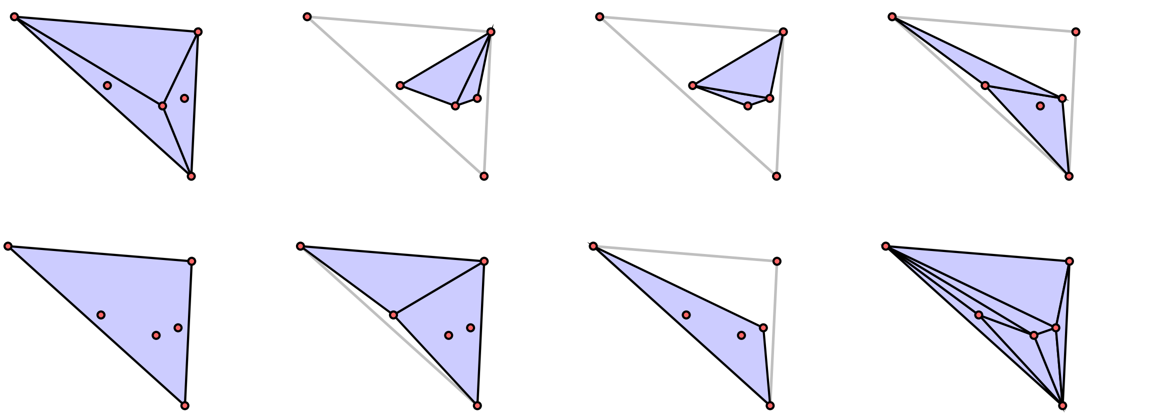


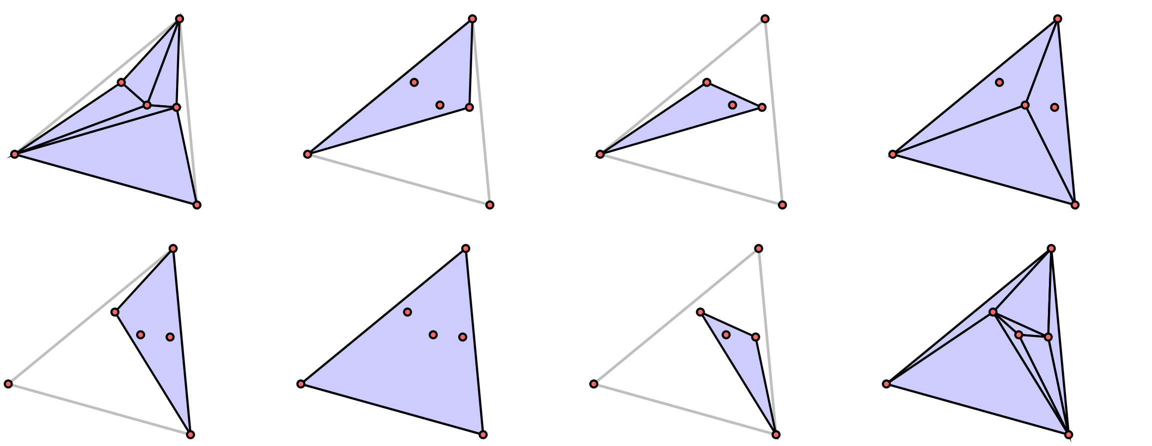


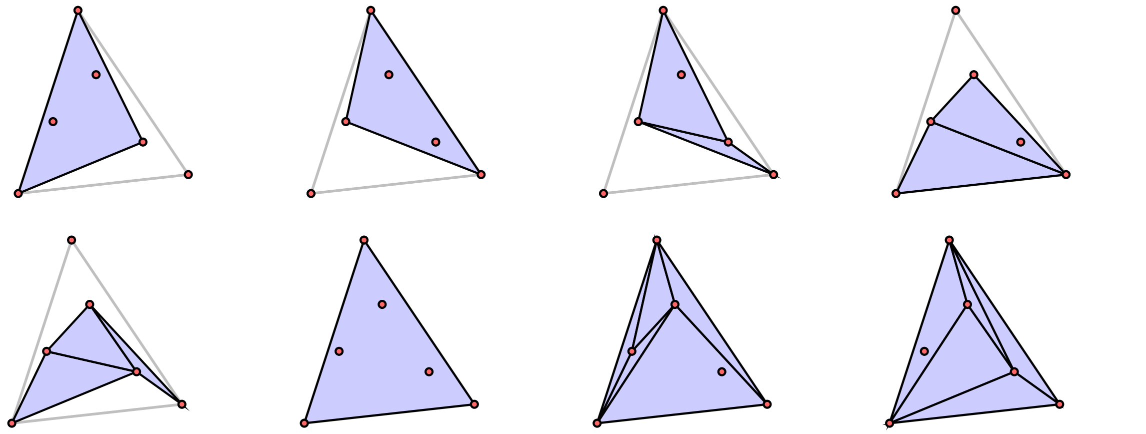


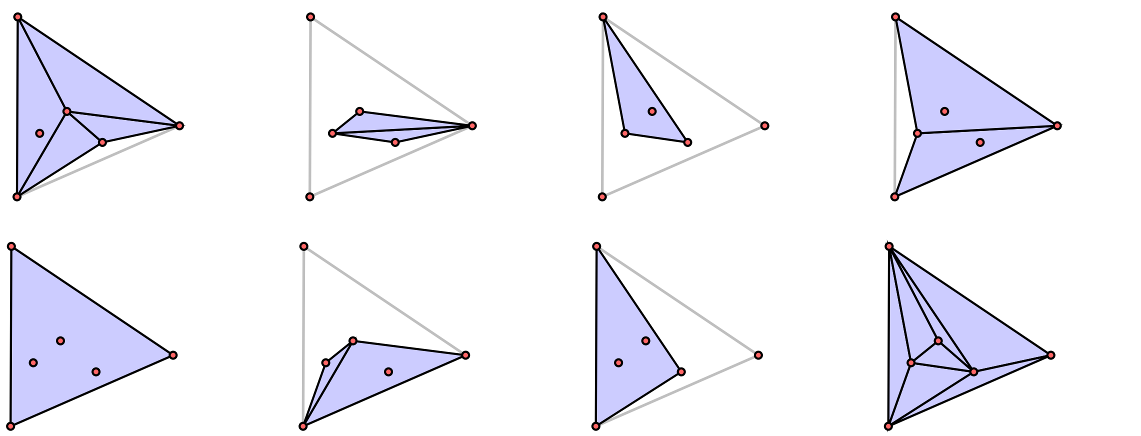


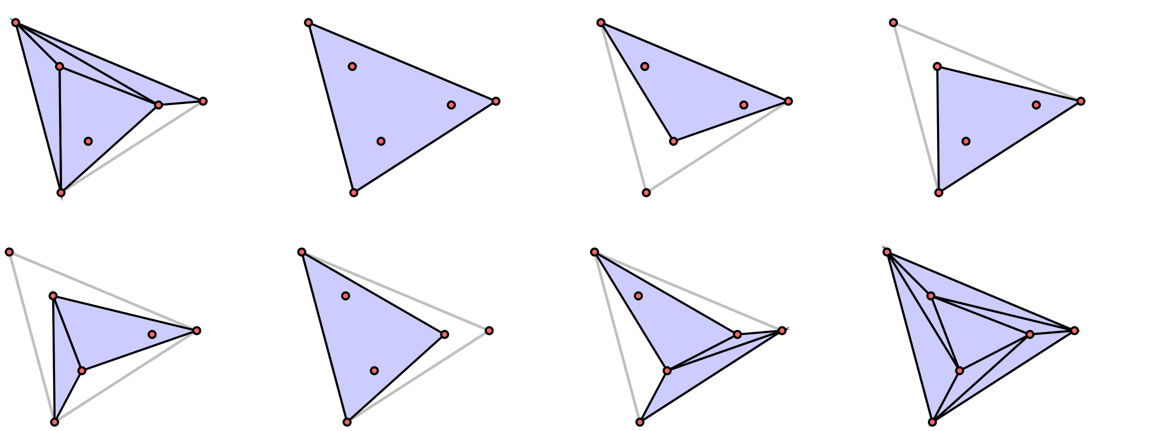


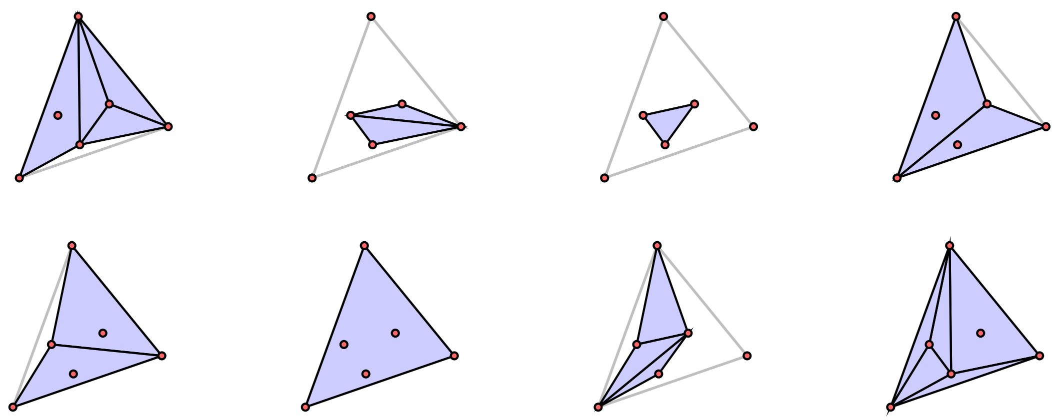












C 13-Colouring the Triangles Determined by a Particular Set of 7 Points