Vertex Partitions of Chordal Graphs

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Abstract: A *k-tree* is a chordal graph with no $(k+2)$-clique. An *ℓ-tree-partition* of a graph G is a vertex partition of G into 'bags,' such that contracting each bag to a single vertex gives an ℓ-tree (after deleting loops and replacing parallel edges by a single edge). We prove that for all $k \geq \ell \geq 0$, every k-tree has an ℓ-tree-partition in which each bag induces a connected $\lfloor k/(\ell+1) \rfloor$-tree. An analogous result is proved for oriented k-trees. J Graph Theory 53: 167–172, 2006

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1. INTRODUCTION

Let G be an (undirected, simple, finite) graph with vertex set $V(G)$ and edge set $E(G)$. Let $\Delta(G)$ be the maximum degree of G. The neighborhood of a vertex v of G is denoted by $N(v) = \{w \in V(G) : vw \in E(G)\}$. A *chord* of a cycle C is an edge

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not in C whose endpoints are both in C. G is *chordal* if every cycle on at least four vertices has a chord. A k-clique ($k \geq 0$) is a set of k pairwise adjacent vertices. A *k-tree* is a chordal graph with no $(k+2)$-clique. The *tree-width* of G, denoted by $\mathsf{tw}(G)$, is the minimum k such that G is a subgraph of a k-tree. It is well known that this definition agrees with the standard definition of tree-width in terms of tree decompositions. It is also well known that G is a k-tree[1] if and only if $V(G) = \emptyset$, or G has a vertex v such that $G \setminus v$ is a k-tree, and $N(v)$ is a k'-clique for some $k' \leq k$.

Let G and H be graphs. The elements of $V(H)$ are called *nodes*. Let $\{H_x \subseteq V(G) : x \in V(H)\}$ be a set of subsets of $V(G)$ indexed by the nodes of H. Each set H_x is called a *bag*. The pair $(H, \{H_x \subseteq V(G) : x \in V(H)\})$ is an *H-partition* of G if:

- for every vertex v of G, there is a node x of H with $v \in H_x$, and
- for all distinct nodes x and y of H, $H_x \cap H_y = \emptyset$, and
- for every edge vw of G, either
  - — there is a node x of H with $v \in H_x$ and $w \in H_x$, or
  - — there is an edge xy of H with $v \in H_x$ and $w \in H_y$.

For brevity we say H is a partition of G. A *k-tree-partition* is an H-partition for some k-tree H. A *tree-partition* is a 1-tree-partition. Tree-partitions were independently introduced by Seese [16] and Halin [12], and have since been investigated by a number of authors [2,3,6,7,11,12,16]. The main property of tree-partitions that has been studied is the maximum cardinality of a bag, called the *width* of the tree-partition. The minimum width over all tree-partitions of a graph G is the *tree-partition-width*[2] of G, denoted by $\mathsf{tpw}(G)$.

A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width [7]. In particular, Seese [16] proved the lower bound,

$$2 \cdot \mathsf{tpw}(G) \geq \mathsf{tw}(G) + 1,$$

which is tight for even complete graphs. The best known upper bound is

$$\mathsf{tpw}(G) \leq 2\big(\mathsf{tw}(G) + 1\big)\big(9\,\Delta(G) - 1\big),$$

which was obtained by the author [18] using a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [6]. See [1,5,8,9,14] for other results related to tree-width and vertex partitions[3].

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[1] In the most common definition of k-tree, $N(v)$ is required to be a k-clique. Working in the slightly larger class of graphs enables cleaner results.

[2] Tree-partition-width has also been called *strong tree-width* [3,16].

[3] Vertex partitions also provide an approach for attacking Hadwiger's conjecture, which states that every graph G with no K_{t+1} minor has chromatic number $\chi(G) \leq t$. No $\chi(G) \leq \mathcal{O}(t)$ bound is currently known. Reed and Seymour [15] observed that "perhaps it is true" that G has an H-partition, such that H is chordal and each bag induces a connected bipartite subgraph of G. This would imply that $\chi(H) = \omega(H) \leq t$, and thus $\chi(G) \leq 2t$. Note that we only need H to be perfect for this conclusion to be reached.

Tree-partition-width is not bounded above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width, as observed by Bodlaender and Engelfriet [3]. Thus, it seems unavoidable that the maximum degree appears in an upper bound on the tree-partition-width. This fact, along with other applications, motivated Dujmović et al. [10] to study the structure of the bags in a tree-partition. In this article we continue this approach, and prove the following result (in Section 2).

Theorem 1. *Let k and ℓ be integers with $k \geq \ell \geq 0$. Let $t = \lfloor k/(\ell+1) \rfloor$. Every k-tree G has an ℓ-tree-partition in which each bag induces a connected t-tree.*

It is easily seen that Theorem 1 is tight for $G = K_{k+1}$ and for all ℓ. Note that Theorem 1 can be interpreted as a statement about chromomorphisms [13].

Dujmović et al. [10] proved that every k-tree has a tree-partition in which each bag induces a $(k-1)$-tree. Thus Theorem 1 with $\ell = 1$ improves this result. That said, the tree-partition of Dujmović et al. [10] has a number of additional properties that were important for the intended application. We generalize these additional properties in Section 3. The price paid is that each bag may now induce a $(k-\ell)$-tree, thus matching the result of Dujmović et al. [10] for $\ell = 1$. Note that the proof of Dujmović et al. [10] uses a different construction to the one given here.

2. PROOF OF THEOREM 1

We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of k and ℓ. Now suppose that $|V(G)| \geq 1$. Thus G has a vertex v such that $G \setminus v$ is a k-tree, and $N(v)$ is a k'-clique for some $k' \leq k$. By induction, $G \setminus v$ has an ℓ-tree-partition H in which each bag induces a connected t-tree. Let $C = \{x \in V(H) : N(v) \cap H_x \neq \emptyset\}$. Since $N(v)$ is a clique, C is a clique of H (by the definition of H-partition). Since H is an ℓ-tree, $|C| \leq \ell + 1$.

Case 1. $|C| \leq \ell$: Add one new node y to H adjacent to each node $x \in C$. Since C is a clique of H and $|C| \leq \ell$, H remains an ℓ-tree. Let $H_y = \{v\}$. The other bags remain unchanged. Since $t \geq 0$, H_y induces a connected t-tree ($= K_1$) in G. Thus H is now a partition of G in which each bag induces a connected t-tree in G.

Case 2. $|C| = \ell + 1$: There is a node $y \in C$ such that $|N(v) \cap H_y| \leq t$, as otherwise $|N(v)| \geq (t+1)|C| = (\lfloor k/(\ell+1) \rfloor + 1)(\ell+1) \geq k+1$. Add v to the bag H_y. Let $u \in N(v) \cap H_y$. Every neighbor of v not in H_y is adjacent to u (in $G \setminus v$). Thus H is a partition of G. H_y induces a connected t-tree in G, since $H_y \setminus \{v\}$ induces a connected t-tree in $G \setminus v$, and the neighborhood of v in H_y is a clique of at least one and at most t vertices. The other bags do not change. Thus each bag of H induces a connected t-tree in G. ■

3. ORIENTED PARTITIONS

Let G be an oriented graph with arc set $A(G)$. Let $\widehat{G}$ be the underlying undirected graph of G. The in- and out-neighborhoods of a vertex v of G are respectively denoted by $N^-(v) = \{u \in V(G) : uv \in A(G)\}$ and $N^+(v) = \{w \in V(G) : vw \in A(G)\}$. It is easily seen that an (undirected) graph G is a k-tree if and only if there is an acyclic orientation of G such that for every vertex v of G, $N^-(v)$ is a k'-clique for some $k' \leq k$. An oriented graph with this property is called an *oriented k-tree*. Let G and H be oriented graphs. An *oriented H-partition* of G is an $\widehat{H}$-partition of $\widehat{G}$ such that for every arc xy of H, and for every edge vw of $\widehat{G}$ with $v \in H_x$ and $w \in H_y$, vw is oriented from v to w. This concept is similar to an oriented homomorphism; see [4,17] for example.

Theorem 2. *Let k and ℓ be integers with $k \geq \ell \geq 0$. Let $t = k - \ell$. Every oriented k-tree G has an oriented ℓ-tree partition H in which each bag induces a weakly connected oriented t-tree in G. Moreover, for each node x of H, the set of vertices $Q(x) = \bigcup\{N^-(v) \setminus H_x : v \in H_x\}$ is a k'-clique of G for some $k' \leq k$.*

The construction in the proof of Theorem 2 differs from that of Theorem 1 in only the choice of the node y in Case 2.

Proof. We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of k and ℓ. Now suppose that $|V(G)| \geq 1$. Since G is acyclic, there is a vertex v of G such that $N^+(v) = \emptyset$, $N^-(v)$ is a k'-clique for some $k' \leq k$, and $G \setminus v$ is an oriented k-tree. By induction, there is an oriented ℓ-tree-partition H of $G \setminus v$ in which each bag induces a weakly connected oriented t-tree in $G \setminus v$. Moreover, for every node x of H, $Q(x)$ is a k'-clique for some $k' \leq k$. Let $C = \{x \in V(H) : N^-(v) \cap H_x \neq \emptyset\}$. Since $N^-(v)$ is a clique, C is a clique of H. Since H is an oriented ℓ-tree, $|C| \leq \ell + 1$.

Case 1. $|C| \leq \ell$: Add one new node y to H adjacent to each node $x \in C$. Orient each new edge from x to y. Obviously H remains acyclic. Since C is a clique of H and $|C| \leq \ell$, H remains an oriented ℓ-tree. Let $H_y = \{v\}$. The other bags are unchanged. Since $t \geq 0$, H_y induces a weakly connected oriented t-tree ($=K_1$) in G. All edges of G that are incident to a vertex in H_y are oriented into the vertex in H_y. Thus H is now an oriented partition of G in which each bag induces a weakly connected oriented t-tree in G. Now $Q(y) = N^-(v)$, which is a k'-clique for some $k' \leq k$. $Q(x)$ is unchanged for nodes $x \neq y$. Hence the theorem is satisfied.

Case 2. $|C| = \ell + 1$: The clique C induces an acyclic tournament in H. Let y be the sink of this tournament. Since $|N^-(v) \cap H_x| \geq 1$ for every node $x \in C \setminus \{y\}$, $|N^-(v) \cap H_y| \leq k' - (|C| - 1) \leq k - \ell = t$. Add v to the bag H_y.

Consider a neighbor u of v. Since $N^+(v) = \emptyset$, uv is oriented from u to v. Say $u \in H_z$ with $z \neq y$. Then z is in the clique C. Thus zy is an edge of H. Since y is a sink of C, zy is oriented from z to y. Thus H is now an oriented partition of G. H_y

induces a weakly connected oriented t-tree in G, since $H_y \setminus \{v\}$ induces an oriented t-tree in $G \setminus v$, and the in-neighborhood of v in H_y is a clique of at least one and at most t vertices. The other bags do not change. Thus each bag of H induces a weakly connected oriented t-tree in G.

$Q(y)$ is not changed by the addition of v to H_y, as there is at least one vertex $u \in N^-(v) \cap H_y$, and any vertex in $N^-(v) \setminus H_y$ is also in $N^-(u) \setminus H_y$. For nodes $x \neq y$, $Q(x)$ is unchanged by the addition of v to H_y, since v is not in the in-neighbourhood of any vertex. Hence the theorem is satisfied. ■

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