Journal of Graph Algorithms and Applications

http://www.cs.brown.edu/publications/jgaa/

vol. 7, no. 1, pp. 33–77 (2003)

Lower Bounds for the Number of Bends in Three-Dimensional Orthogonal Graph Drawings

David R. Wood

School of Computer Science
Carleton University
Ottawa, Canada

davidw@scs.carleton.ca
http://www.scs.carleton.ca/~davidw

Abstract

This paper presents the first non-trivial lower bounds for the total number of bends in 3-D orthogonal graph drawings with vertices represented by points. In particular, we prove lower bounds for the number of bends in 3-D orthogonal drawings of complete simple graphs and multigraphs, which are tight in most cases. These result are used as the basis for the construction of infinite classes of c-connected simple graphs, multigraphs, and pseudographs ($2 \leq c \leq 6$) of maximum degree Δ ($3 \leq \Delta \leq 6$), with lower bounds on the total number of bends for all members of the class. We also present lower bounds for the number of bends in general position 3-D orthogonal graph drawings. These results have significant ramifications for the '2-bends problem', which is one of the most important open problems in the field.

Communicated by Dorothea Wagner: submitted January 2001; revised December 2002.

A preliminary version of this paper appeared in *Proc. 8th International Symp. on Graph Drawing* (GD 2000). Volume 1984 of Lecture Notes in Comput. Sci., pages 259-271, Springer, 2001.

Supported by NSERC. Partially completed while a Ph.D. student in the School of Computer Science and Software Engineering, Monash University (Melbourne, Australia), and at the School of Information Technologies, The University of Sydney (Sydney, Australia), where supported by the ARC.

1 Introduction

The *3-D orthogonal grid* consists of *grid-points* in 3-space with integer coordinates, together with the axis-parallel *grid-lines* determined by these points. Two grid-points are said to be *collinear* if they are contained in a single grid-line, and are *coplanar* if they are contained in a single grid-plane. A *3-D orthogonal drawing* of a graph positions each vertex at a distinct grid-point, and routes each edge as a polygonal chain composed of contiguous sequences of axis-parallel segments contained in grid-lines, such that (a) the end-points of an edge route are the grid-points representing the end-vertices of the edge, and (b) distinct edge routes only intersect at a common end-vertex.

For brevity we say a 3-D orthogonal graph drawing is a *drawing.* A drawing with no more than b bends per edge is called a *b-bend drawing.* The graph-theoretic terms 'vertex' and 'edge' also refer to their representation in a drawing. The *ports* at a vertex v are the six directions, denoted by X_v^+, X_v^-, Y_v^+, Y_v^-, Z_v^+ and Z_v^-, which the edges incident with v can use. For each dimension $I \in \{X, Y, Z\}$, the I_v^+ (respectively, I_v^-) port at a vertex v is said to be *extremal* if v has maximum (minimum) I-coordinate taken over all vertices.

Clearly, 3-D orthogonal drawings can only exist for graphs with maximum degree at most six. 3-D orthogonal drawings of maximum degree six graphs have been studied in [3, 4, 6, 10–13, 17, 19, 21, 22, 33, 35–37]. By representing a vertex by a grid-box, 3-D orthogonal drawings of arbitrary degree graphs have also been considered; see for example [5, 8, 21]. 3-D graph drawing has applications in VLSI circuit design [1, 2, 18, 23, 26] and software engineering [15, 16, 24, 25] for example. Note that there is some experimental evidence suggesting that displaying a graph in three dimensions is better than in two [28, 29].

Drawings with many bends appear cluttered and are difficult to visualise. In VLSI layouts, bends in the wires increase the cost of production and the chance of circuit failure. Therefore minimising the number of bends, along with minimising the bounding box volume, have been the most commonly proposed aesthetic criteria for measuring the quality of a drawing. Using straightforward extensions of the corresponding 2-D NP-hardness results, optimising each of these criteria is NP-hard [12]. Kolmogorov and Barzdin [17] established a lower bound of $\Omega(n^{3/2})$ on the bounding box volume of drawings of n-vertex graphs. In this paper we establish the first non-trivial lower bounds for the number of bends in 3-D orthogonal drawings. Lower bounds for the number of bends in 2-D orthogonal drawings have been established by Tamassia *et al.* [27] and Biedl [7].

A graph with no parallel edges and no loops is *simple*; a *multigraph* may have parallel edges but no loops; and a *pseudograph* may have parallel edges and loops. We consider n-vertex m-edge graphs G with maximum degree at most six, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively.

A j-edge matching is denoted by M_j; that is, M_j consists of j edges with no end-vertex in common. $K_p \setminus M_j$ is the graph obtained from the complete graph K_p by deleting a j-edge matching M_j (where $2j \leq p$). The 2-vertex multigraph with j edges is denoted by $j \cdot K_2$, and L_j is the 1-vertex pseudograph with j loops. An *m-path* is a path with m edges. By C_m we denote the cycle with m edges, which is also called an m-cycle. A *chord* of a cycle C is an edge not in C whose end-vertices are both in C. We say two cycles are *chord-disjoint* if they do not have a chord in common. Note that chord-disjoint cycles may share a vertex or edge. A *chordal path* of a cycle C is a path P whose end-vertices are in C, but the internal vertices of P and the edges of P are not in C.

Lower bounds for the maximum number of bends per edge:

Obviously every drawing of K_3 has at least one bend. It follows from results in multi-dimensional orthogonal graph drawing by Wood [32, 35] that every drawing of K_5 has an edge with at least two bends. It is well known that every drawing of $6 \cdot K_2$ has an edge with at least three bends, and it is easily seen that $2 \cdot K_2$ and $3 \cdot K_2$ have at least one edge with at least one and two bends, respectively.

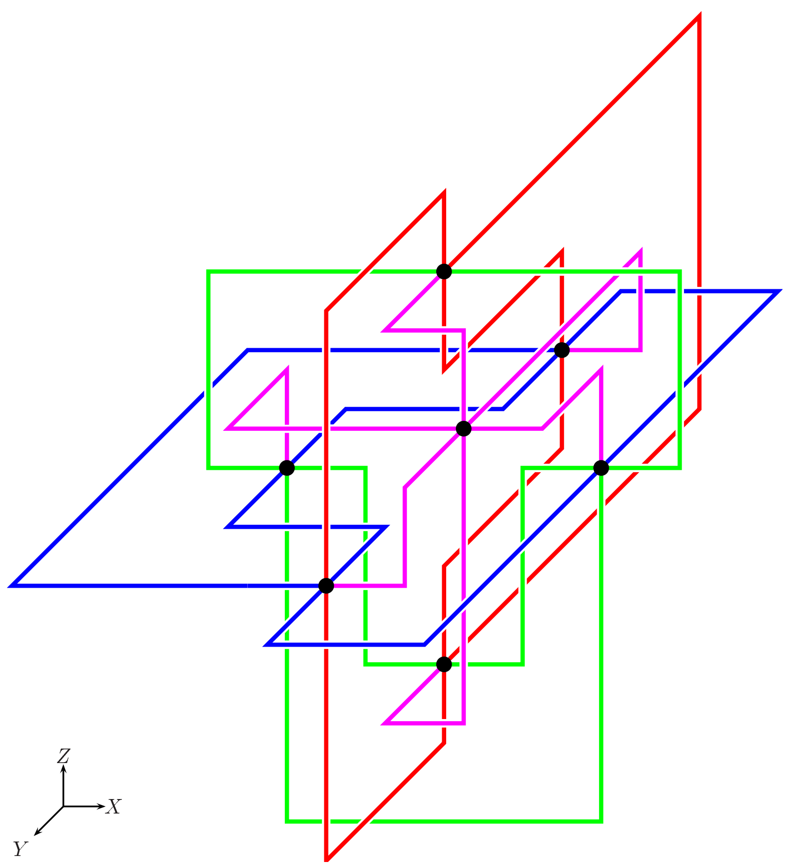


Figure 1: A 2-bend drawing of K_7.

Eades *et al.* [13] originally conjectured that every drawing of K_7 has an edge with at least three bends. A counterexample to this conjecture, namely a drawing of K_7 with at most two bends per edge, was first exhibited by Wood [32]. A more symmetric drawing of K_7 with at most two bends per edge is illustrated in Figures 1 and 2. This drawing[1] has the interesting feature of rotational symmetry about the line $X = Y = Z$.

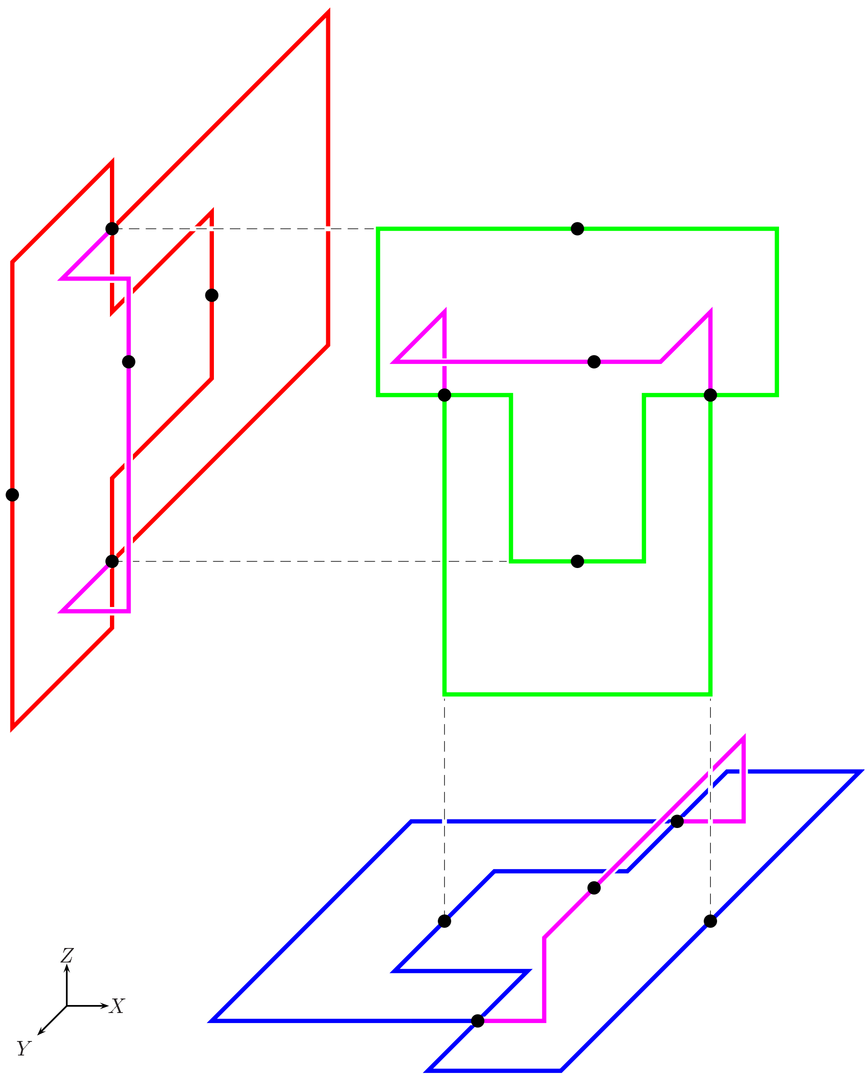


Figure 2: Components of a 2-bend drawing of K_7.

[1] A physical model of this drawing is on display at the School of Computer Science and Software Engineering, Monash University, Clayton, Melbourne, Australia.

One may consider the other 6-regular complete multi-partite graphs $K_{6,6}$, $K_{3,3,3}$ and $K_{2,2,2,2}$ to be potential examples of simple graphs requiring an edge with at least three bends. However, 2-bend drawings of these graphs were discovered by Wood [35].

Lower bounds for the total number of bends:

The main result in this paper is the construction of infinite families of graphs of given connectivity and maximum degree, with a lower bound on the average number of bends in a drawing of each graph in the class. As a first step toward this goal we establish lower bounds on the minimum number of bends in drawings of small complete graphs, and the graphs obtained from small complete graphs by deleting a matching; see Table 1. For many of these graphs the obtained lower bound is tight; that is, there is a drawing with this many bends. The main exception being K_7 and the graphs derived from K_7 by deleting a matching. In particular, we prove a lower bound of $20 - 3j$ for the number of bends in drawings of $K_7 \setminus M_j$, whereas the best known drawings have $24 - 4j$ edges; see Figure 19. We conjecture that there is no drawing of $K_7 \setminus M_j$ with fewer than $24 - 4j$ edges for each $j \in \{0, 1, 2, 3\}$. There is also a gap in our bounds in the case of $K_6 \setminus M_3$. Here we have a lower bound of seven bends, whereas the best known drawing of $K_6 \setminus M_3$ has eight bends, which we conjecture is bend-minimum.

Table 1: Bounds for the minimum number of bends in drawings of complete graphs minus a matching $K_p \setminus M_j$.

	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$
$j = 0$	1	3	7	12	$20 \dots 24$
$j = 1$	0	2	5	10	$17 \dots 20$
$j = 2$	-	1	4	8	$14 \dots 16$
$j = 3$	-	-	-	$7 \dots 8$	$11 \dots 12$

Furthermore we show that drawings of the multigraphs $j \cdot K_2$ with $2 \leq j \leq 6$ have at least 2, 4, 6, 8 and 12 bends, respectively. Since a loop has at least three bends in every drawing, the pseudograph L_j with $1 \leq j \leq 3$ has at least $3j$ bends.

We use the above lower bounds as the basis for the construction of infinite families of c-connected graphs of maximum degree Δ with lower bounds on the number of bends for each member of the class, as summarised in Table 2.

Upper bounds:

A number of algorithms have been proposed for 3-D orthogonal graph drawing [3, 6, 9–13, 17, 21, 22, 33, 35–37]. We now summarise the best known upper

Table 2: Lower bounds on the average number of bends in drawings of an infinite family of c-connected graphs with maximum degree at most Δ.

type	simple graphs					multigraphs					pseudographs				
Δ	6	5	4	3	2	6	5	4	3	2	6	5	4	3	2
$c=0$	$\frac{20}{21}$	$\frac{4}{5}$	$\frac{7}{10}$	$\frac{1}{2}$	$\frac{1}{3}$	2	$\frac{8}{5}$	$\frac{3}{2}$	$\frac{4}{3}$	1	3	3	3	3	3
$c=2$	$\frac{17}{21}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	-	$\frac{4}{3}$	$\frac{6}{5}$	1	$\frac{2}{3}$	-	2	$\frac{3}{2}$	$\frac{3}{2}$	-	-
$c=3$	$\frac{8}{11}$	$\frac{14}{23}$	$\frac{8}{19}$	$\frac{2}{9}$	-	1	$\frac{4}{5}$	$\frac{1}{2}$	-	-	$\frac{4}{3}$	$\frac{6}{5}$	-	-	-
$c=4$	$\frac{12}{17}$	$\frac{7}{12}$	$\frac{2}{5}$	-	-	$\frac{2}{3}$	$\frac{2}{5}$	-	-	-	1	-	-	-	-
$c=5$	$\frac{24}{35}$	$\frac{14}{25}$	-	-	-	$\frac{1}{3}$	-	-	-	-	-	-	-	-	-
$c=6$	$\frac{2}{3}$	-	-	-	-	-	-	-	-	-	-	-	-	-	-

bounds for the number of bends in 3-D orthogonal drawings. The 3-Bends algorithm of Eades *et al.* [13] and the Incremental algorithm of Papakostas and Tollis [21] both produce 3-bend drawings[2] of multigraphs[3] with maximum degree six. As discussed above there exist simple graphs with at least one edge having at least two bends in every drawing. The following open problem is therefore of interest:

2-*Bends Problem:* Does every (simple) graph with maximum degree at most six admit a 2-bend drawing? [13]

The Diagonal Layout & Movement algorithm of Wood [37] (also see [33]) solves the 2-bends problem in the affirmative for simple graphs with maximum degree five. For maximum degree six simple graphs, the same algorithm uses a total of at most $\frac{16}{7}m$ bends, which is the best known upper bound for the total number of bends in 3-D orthogonal drawings.

In this paper we provide a negative result related to the 2-bends problem. A 3-D orthogonal graph drawing is said to be in *general position* if no two vertices lie in a common grid-plane. The general position model is used in the 3-Bends and Diagonal Layout & Movement algorithms. In this paper we show that the general position model, and the natural variation of this model where pairs of vertices share a common plane, cannot be used to solve the 2-bends problem, at least for 2-connected graphs.

The remainder of this paper is organised as follows. In Section 2 we establish

[2]The 3-Bends algorithm [13] produces drawings with $27n^3$ volume. By deleting grid-planes not containing a vertex or a bend the volume is reduced to $8n^3$. The Incremental algorithm [21] produces drawings with $4.63n^3$ volume. A modification of the 3-Bends algorithm by Wood [36] produces drawings with $n^3 + o(n^3)$ volume.

[3]The 3-Bends algorithm [13] explicitly works for multigraphs. The Incremental algorithm, as stated in [21], only works for simple graphs, however with a suitable modification it also works for multigraphs [A. Papakostas, private communication, 1998].

a number of introductory results concerning 0-bend drawings of cycles. These results are used to prove our lower bounds on the total number of bends in drawings of complete graphs and graphs obtained from complete graphs by deleting a matching, which are established in Section 3. In Section 4 we use these lower bounds as the basis for lower bounds on the number of bends in infinite families of graphs. In Section 5 we present lower bounds for the number of bends in general position drawings. These results have important implications for the nature of any solution to the 2-bends problem, which are discussed in Section 6.

Some technical aspects of our proofs are presented in the appendices. In particular, in Appendix A we prove a number of results concerning the existence of cycles and other small subgraphs in graphs of a certain size; in Appendix B we establish the connectivity of the graphs used in our lower bounds; and in Appendix C we prove a result, which is not directly used in other parts of the paper, but may be of independent interest.

2 Drawings of Cycles

In this section we characterise the 0-bend drawings of the cycles C_k $(k \leq 7)$. We then show that if a drawing of a complete graph contains such a 0-bend drawing of a cycle then there are many edges with at least three bends in the drawing of the complete graph. These results are used in Section 3 in the proofs of our lower bounds on the total number of bends in drawings of complete graphs.

A straight-line path in a 0-bend drawing of a cycle is called a *side*. A side parallel to the I-axis for some $I \in \{X, Y, Z\}$ is called an I-side, and I is called the *dimension* of the side. Clearly the dimension of adjacent sides is different. Thus in a 2-dimensional drawing the dimension of the sides alternate around the cycle. We therefore have the following observation.

Observation 1. *There is no 2-dimensional* 0*-bend drawing of a cycle with an odd number of sides.*

If there is an I-side in a drawing of a cycle for some $I \in \{X, Y, Z\}$ then clearly there is at least two I-sides. Therefore a drawing of a cycle with X-, Y- and Z-sides, which we call *truly 3-dimensional*, has at least six sides. Hence there is no truly 3-dimensional 3-, 4- or 5-sided 0-bend drawing of a cycle. By Observation 1 there is also no two-dimensional 3- or 5-sided 0-bend drawing of a cycle. We therefore have the following observations.

Observation 2. *There is no 3- or 5-sided* 0*-bend drawing of a cycle.*

Observation 3. *There is no* 0*-bend drawing of* C_3*.*

Observation 4. *All* 0*-bend drawings of* C_4 *and* C_5 *have four sides.*

Lemma 1. *If a drawing of a complete graph contains a* 0*-bend 4-cycle* (*respectively, 5-cycle*) *then at least two* (*four*) *chords of the cycle each have at least three bends.*

Proof. By Observation 4 all 0-bend drawings of C_4 and of C_5 have four sides. As illustrated in Figure 3(a), the chord connecting diagonally opposite vertices in a 4-sided drawing of a cycle has at least three bends. Hence, if a drawing of a complete graph contains a 0-bend C_4, then the two chords each have at least three bends. Also, in the case of C_5, the edges from the vertex not at the intersection of two sides to the diagonally opposite vertices both have at least three bends, as illustrated in Figure 3(b). Hence, if a drawing of a complete graph contains a 0-bend C_5, then the four chords each have at least three bends. □

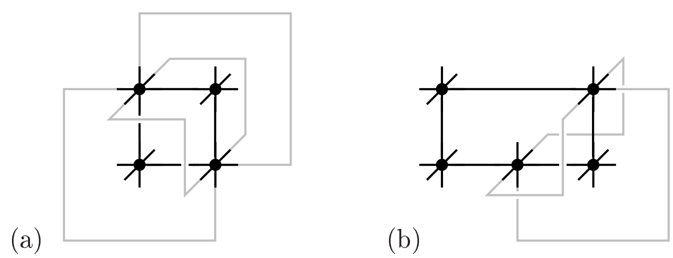


Figure 3: 3-bend edge 'across' the 4- and 5-cycle.

Observation 5. *$K_{2,3}$ does not have a 0-bend drawing.*

Proof. $K_{2,3}$ contains C_4. By Observation 4, all 0-bend drawing of C_4 have four sides. As in Lemma 1, an edge between the diagonally opposite vertices of a 4-sided cycle has at least three bends. Hence the 2-path in $K_{2,3}$ between the non-adjacent vertices of the 4-cycle has at least one bend, as illustrated in Figure 4(b). Hence $K_{2,3}$ does not have a 0-bend drawing. □

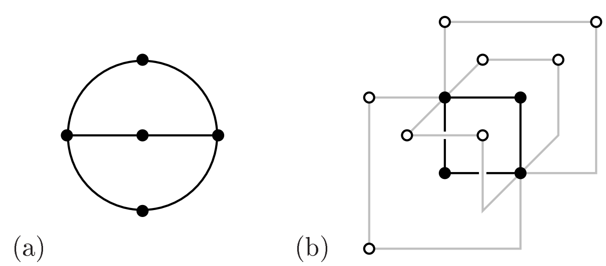


Figure 4: (a) The graph $K_{2,3}$. (b) $K_{2,3}$ does not have a 0-bend drawing.

The proof of the following lemma is almost identical to that of Observation 5 and is omitted.

Observation 6. *If a drawing of a graph contains a 0-bend 4-cycle (a, b, c, d) with a chordal 2-path $P \in \{(a, x, c), (b, x, d)\}$, then P has at least two bends.* □

We now classify the 0-bend drawings of C_6.

Lemma 2. *The only 6-sided* 0*-bend drawings of* C_6 *are those in Figure 5* (*up to symmetry and the deletion of grid-planes not containing a vertex*).

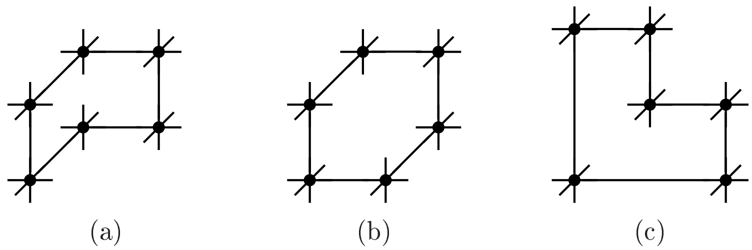


Figure 5: 6-sided 0-bend drawings of C_6.

Proof. Let S be the cyclic sequence of dimensions of the sides around an arbitrary, but fixed, 6-sided 0-bend drawing of C_6.

First suppose the drawing is 2-dimensional. Since adjacent sides are perpendicular, without loss of generality three sides are X-sides and three sides are Y-sides. Therefore S is (X, Y, X, Y, X, Y). The length of one of the X-sides equals the sum of the lengths of the other two X-sides, and similarly for the Y-sides. Label these long sides X^* and Y^*. If the long sides are adjacent then S is (X^*, Y^*, X, Y, X, Y), which corresponds to the drawing in Figure 5(c). If the long sides are not adjacent then S is (X^*, Y, X, Y^*, X, Y), which corresponds to the 'drawing' in Figure 6, which contains an edge crossing.

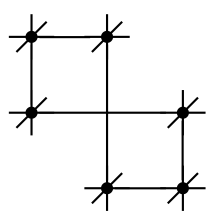


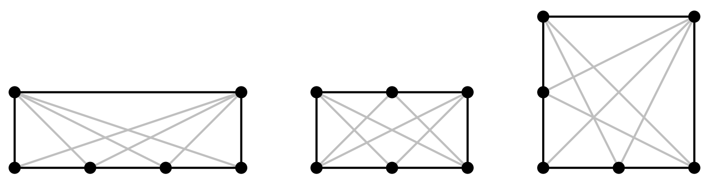
Figure 6: 6-sided 0-bend 'drawing' of C_6 with an edge crossing.

Now suppose the drawing is truly 3-dimensional. Clearly there are two X-sides, two Y-sides and two Z-sides. Let x be the number of sides between the two X-sides in S. Clearly x is one or two. Define y and z similarly for the Y- and Z-sides. We can assume without loss of generality that $x \leq y \leq z$.

If $x = 1$ and $y = 1$ then S is (X, Z, X, Y, Z, Y), and $z = 2$. This sequence corresponds to the drawing in Figure 5(a). If $x = 1$ and $y = 2$ then S is (X, Y, X, Z, Y, Z), and $z = 1$ which is a contradiction. Otherwise $x = y = z = 2$ and S is (X, Y, Z, X, Y, Z) without loss of generality, which corresponds to the drawing in Figure 5(b). □

Lemma 3. *If a drawing of a complete graph contains a* 0*-bend 6-cycle then there are at least six chords of the cycle each with at least three bends.*

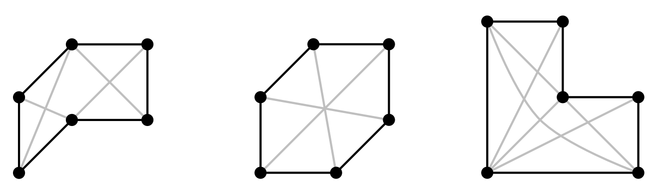
Proof. We can assume without loss of generality that the complete graph in question is K_6. By Observation 2, all 0-bend drawings of C_6 are 4- or 6-sided. In a 4-sided 0-bend drawing of C_6 the two vertices not at the intersection of adjacent sides can be (a) on the same side, (b) on opposite sides, or (c) on adjacent sides, as illustrated in Figure 7. In each case there are at least six chords each with at least three bends if the 0-bend drawing of C_6 is contained in a drawing of K_6.



(a) (b) (c)

Figure 7: Edges with at least three bends in a drawing of K_6 containing a 4-sided 0-bend drawing of C_6.

By Lemma 2, the only 6-sided 0-bend drawings of C_6 (up to symmetry) are those in Figure 5. For each such drawing of C_6, if this is a sub-drawing of a drawing of K_6, then those chords of C_6 illustrated in Figure 8 each require at least three bends (compare with Figure 3). In the case of the drawing in Figure 8(c) there are at least six chords each requiring at least three bends.



(a) (b) (c)

Figure 8: Edges with at least three bends in a drawing of K_6 containing a 6-sided 0-bend C_6.

Consider the drawing in Figure 8(a) which forces at least four chords to have at least three bends if a sub-drawing of a drawing of K_6. As illustrated in Figure 9(a), any drawing of the edges vu and vw with at most two bends per edge passes through the same point. Hence one of these edges has at least three bends. We can make the same argument for the edges xw and xu. Hence if K_6 contains the sub-drawing of C_6 illustrated in Figure 8(a) then there are at least six chords each with at least three bends.

Now consider the drawing in Figure 8(b) which forces at least three chords to have at least three bends if a sub-drawing of a drawing of K_6. As illustrated in Figure 9(b), any drawing of the edges vu, uw and vw with at most two bends per edge passes through the same point. Hence two of these edges have at least three bends. We can make the same argument for three edges connecting the other three vertices. Hence if K_6 contains the sub-drawing of C_6 illustrated in Figure 8(b) then there are at least seven chords each with at least three bends. The result follows. □

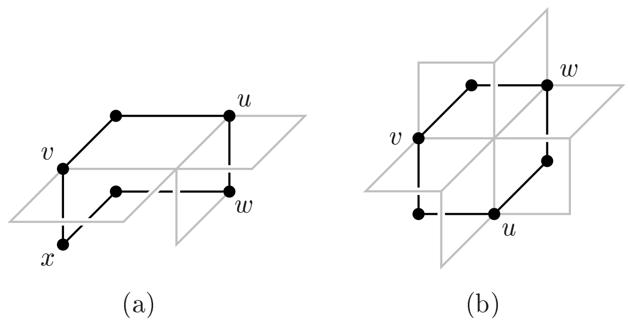


Figure 9: Intersecting 1- and 2-bend edges.

Lemma 4. *The only 7-sided* 0*-bend drawings of* C_7 *are those in Figure 10* (*up to symmetry and the deletion of grid-planes not containing a vertex*).

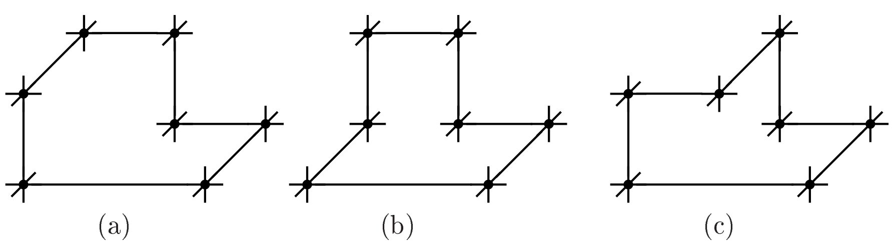


Figure 10: 7-sided 0-bend drawings of C_7.

Proof. Consider an arbitrary, but fixed, 7-sided 0-bend drawing of C_7. By Observation 2, there is no 2-dimensional 0-bend drawing of an odd cycle, and if there is an I-side in a drawing of a cycle for some $I \in \{X, Y, Z\}$, then there are at least two I-sides. Therefore in a 7-sided cycle, without loss of generality three of the sides are X-sides, two are Y-sides and two are Z-sides. Clearly the length of one of the X-sides equals the sum of the lengths of the other two X-sides. Label this long side X^*.

Let S be the cyclic sequence of the dimensions of the sides around C_7, which without loss of generality begins with the X^*-side. Therefore S is (i) $(X^*, 1, X, 2, X, 3, 4)$, (ii) $(X^*, 1, X, 2, 3, X, 4)$, or (iii) $(X^*, 1, 2, X, 3, X, 4)$, where the numbered locations refer to a Y- or Z-side.

In case (i), the dimensions of the '3' and '4' sides are different, hence the dimensions of the '1' and '2' sides are also different. Without loss of generality '1' is a Y-side and '2' is a Z-side. Therefore S is either (X^*, Y, X, Z, X, Y, Z) or (X^*, Y, X, Z, X, Z, Y), which correspond to the drawings in Figure 10(a) and Figure 10(b), respectively.

In case (ii), the dimensions of the '2' and '3' sides are different, hence the dimensions of the '1' and '4' sides are also different. Without loss of generality '1' is a Y-side and '4' is a Z-side. Therefore S is either (X^*, Y, X, Z, Y, X, Z), which corresponds to the drawing in Figure 10(c), or (X^*, Y, X, Y, Z, X, Z) which corresponds to the 'drawing' in Figure 11 with an edge crossing.

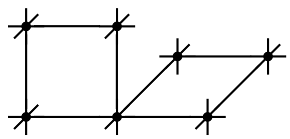


Figure 11: 7-sided 0-bend 'drawing' of C_7 with an edge intersection.

In case (iii), S is simply the reverse sequence of S in case (i). We therefore have classified all 7-sided 0-bend drawings of C_7 up to symmetry and after removing grid-planes not containing a vertex. □

Lemma 5. *If a drawing of K_7 contains a 0-bend 7-cycle then there are at least six chords of the cycle each with at least three bends.*

Proof. By Observation 2, a 0-bend drawing of C_7 has four, six or seven sides.

In a 4-sided 0-bend drawing of C_7, as illustrated in Figure 12, the three vertices not at the intersection of two adjacent sides can be (a) all on the same side, (b) two on one side and one on an adjacent side, (c) two on one side and one on the opposite side, or (d) all on different sides. For each drawing, if the 7-cycle is a sub-drawing of a drawing of K_7, then eight chords of the cycle have at least three bends (compare with Figure 3).

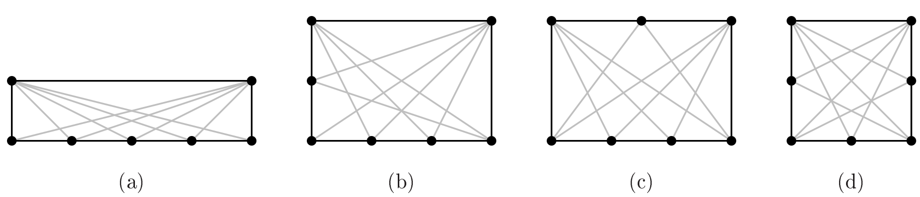


Figure 12: Edges with at least three bends in a 4-sided 0-bend drawing of C_7.

Any 6-sided 0-bend drawing of C_7 can be obtained by placing a new vertex on one side of a 6-sided 0-bend drawing of C_6. Thus, by Lemma 3 if a drawing of K_7 contains a 6-sided 0-bend drawing of C_7 then at least six of the chords have at least three bends.

By Lemma 4, the only 7-sided drawings of C_7 are those illustrated in Figure 10. For each such drawing, if the 7-cycle is a sub-drawing of a drawing of K_7, Figure 13 shows chords of the cycle which need at least three bends. The drawings in Figure 13(a), (b) and (c) have four, six, and four chords, respectively, which need at least three bends.

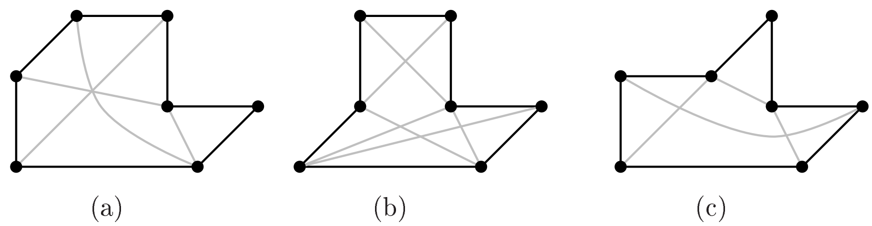


Figure 13: Edges with at least three bends in a 7-sided 0-bend drawing of C_7.

Consider the drawing in Figure 13(a). As illustrated in Figure 14(a), any drawing of the edges vu, vw and vx with at most two bends per edge passes through the same grid-point. Hence two of these edges have at least three bends. Therefore if K_7 contains the sub-drawing of C_7 illustrated in Figure 13(a) then there are at least six edges each with at least three bends.

Now consider the drawing in Figure 13(c). As illustrated in Figure 14(b), there is one route for the edge vu with at most two bends, one route for the edge vw with at most two bends, and three routes for the edge vx with at most two bends. Any two of these edge routes for distinct edges pass through the same point. Hence two of these edges have at least three bends. Therefore if K_7 contains the sub-drawing of C_7 illustrated in Figure 13(c), then there are at least six edges each with at least three bends.

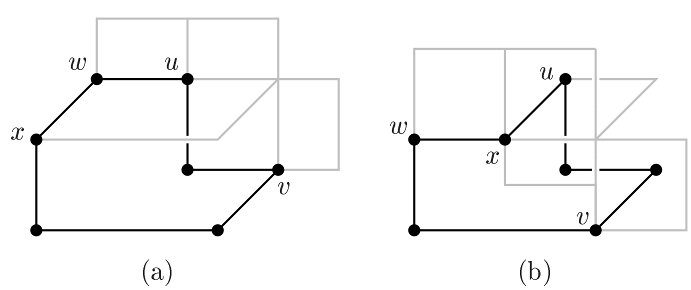


Figure 14: Intersecting 1- and 2-bend edges.

Therefore if a drawing of a complete graph contains a 0-bend 7-cycle, then there are at least six chords of the cycle each with at least three bends. □

The results in this section are summarised by the following immediate corollary of Lemmata 1, 3 and 5.

Theorem 1. *If a drawing of $K_p \setminus M_j$ contains a 0-bend 4-cycle (respectively, 5-cycle, 6-cycle, or 7-cycle), then there are at least $2-j$ $(4-j,\ 6-j,\ 6-j)$ chords of the cycle each with at least three bends.* □

3 Drawings of Complete Graphs

In this section we establish lower bounds for the total number of bends in drawings of the complete graphs K_4, K_5, K_6 and K_7, and the graphs obtained from these complete graphs by deleting a matching. We complete the section by establishing lower bounds for the number of bends in drawings of the multigraphs $j \cdot K_2$ for $2 \le j \le 6$.

The 0-*bend subgraph* of a given drawing consists of those edges drawn with no bends. The following proofs typically proceed by case analysis on the size of the 0-bend subgraph.

Figure 15 shows a drawing of K_4 with three bends. Deleting one of the 1-bend edges produces a drawing of $K_4 \setminus M_1$ with two bends. We now prove that both of these drawings are bend-minimum. This elementary result is indicative of the method of proof for the corresponding results for larger complete graphs which follow.

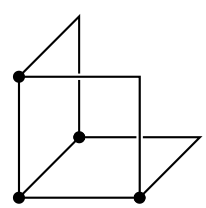


Figure 15: A drawing of K_4 with three bends.

Theorem 2. *Let $j \in \{0,1\}$. Every drawing of $K_4 \setminus M_j$ has at least $3-j$ bends.*

Proof. Let k_0 be the number of 0-bend edges in a drawing of $K_4 \setminus M_j$. If $k_0 \le 3$ then there are at least $3-j$ edges each with at least one bend, and we are done. Otherwise $k_0 \ge 4$. The 0-bend subgraph has no 3-cycle by Observation 3. The only 4-vertex graph with at least four edges and no 3-cycle is a 4-cycle. Thus the 0-bend subgraph is a 4-cycle. By Theorem 1, if $K_4 \setminus M_j$ (with $j \le 1$) contains a 0-bend 4-cycle, then there is at least one chord with at least three bends. □

We now establish tight lower bounds for the total number of bends in drawings of K_5 and the graphs obtained from K_5 by deleting a matching. To prove that the drawing of K_5 illustrated in Figure 16 is bend-minimum we use the following result, which may be of independent interest.

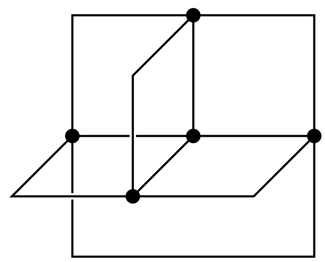


Figure 16: A 2-bend drawing of K_5 with seven bends.

Lemma 6. *For every set S of grid-points in the 3-D orthogonal grid, either*
(1) two points in S are non-coplanar, or
(2) at least $\lceil 2|S|/3 \rceil$ grid-points are in a single grid-plane.

Proof. Suppose (1) does not hold; that is, every pair of points in S share at least one coordinate. Without loss of generality assume that there is a point $v \in S$ positioned at $(0,0,0)$. Thus every point in S has at least one coordinate equal to 0. If some point $w \in S$ has exactly one coordinate equal to 0 then, supposing this is the X-coordinate, all points in S have an X-coordinate of 0 (to be coplanar with v and w); that is, all points are in a single grid-plane, and the result follows. Otherwise every point in S, except v, has exactly two coordinates equal to 0; that is, every point lies on an axis. Let x, y and z be the number of points in S, not counting v, on the X, Y and Z axes, respectively. Then $x+y+z = |S|-1$. Without loss of generality $x \geq y \geq z$. Clearly $x \geq \lceil (|S|-1)/3 \rceil$ and $y \geq \lfloor (|S|-1)/3 \rfloor$, which implies there are $x+y+1 \geq \lceil 2|S|/3 \rceil$ points in S on the X- or Y-axes, and thus in a single grid-plane. □

Wood [32] shows that a 1-bend drawing of K_n in a multi-dimensional orthogonal grid requires at least $n-1$ dimensions. We now provide a simple proof of an equivalent formulation of this result in the case of $n = 5$.

Theorem 3. *Every drawing of K_5 has an edge with at least two bends.*

Proof. In a layout of the vertices of K_5, if two vertices are non-coplanar, then the edge between them has at least two bends. By Lemma 6 with $S = V(K_5)$, if all the vertices are pairwise coplanar then four of the vertices are coplanar. Consider the K_4 subgraph H induced by these four coplanar vertices. If any edge of H leaves the plane containing the vertices then it has at least two bends. Otherwise we have a plane orthogonal drawing of K_4, which has an edge with at least two bends [14]. □

We now prove that the drawing of K_5 in Figure 16 is bend-minimum.

Theorem 4. *Every drawing of K_5 has at least seven bends.*

Proof. Let k_0 be the number of 0-bend edges in a drawing of K_5. Since every subgraph of K_5 with at least eight edges contains a 3-cycle, and by Observation 3, the 0-bend subgraph has no 3-cycle, $k_0 \leq 7$. If $k_0 \leq 4$ then there are at least six edges each with at least one bend, and by Theorem 3, one of these edges has at least two bends, implying there are at least seven bends in total, and we are done. Now assume $5 \leq k_0 \leq 7$. Thus the 0-bend subgraph contains a cycle, which by Observation 3, is a 4-cycle or a 5-cycle C. By Lemma 1, C has at least two chords each with at least three bends. Since $k_0 \leq 7$ there is at least one additional edge with at least one bend, implying a total of at least seven bends. □

By deleting the 2-bend edge in the drawing of K_5 illustrated in Figure 16, we obtain a drawing of $K_5 \setminus M_1$ with five bends in total. By deleting the appropriate 1-bend edge from this drawing, we obtain a drawing of $K_5 \setminus M_2$ with four bends in total. We now prove that both of these drawings are bend-minimum.

Theorem 5. *For each $j \in \{1, 2\}$, every drawing of $K_5 \setminus M_j$ has at least $6 - j$ bends.*

Proof. Let k_0 be the number of 0-bend edges in a drawing of $K_5 \setminus M_j$ for some $j \in \{1, 2\}$. If $k_0 \leq 4$ then there are at least $6 - j$ edges each with at least one bend, and we are done. Now assume $k_0 \geq 5$. By Lemma 13 in Appendix A, every subgraph of $K_5 \setminus M_j$ with at least six edges contains C_3 or $K_{2,3}$, and by Observations 3 and 5, the 0-bend subgraph does not contain C_3 or $K_{2,3}$. Thus $k_0 = 5$. Since every 5-edge subgraph of $K_5 \setminus M_j$ contains a cycle, and the 0-bend subgraph does not contain a 3-cycle (by Observation 3), there is a 0-bend 4-cycle or 5-cycle C.

Suppose C has a chord, which is guaranteed by Theorem 1 in the case of C being a 5-cycle. Then by Lemma 1 the chord has at least three bends. There are a further $4 - j$ edges each with at least one bend, giving a total of at least $7 - j$ bends. If C is chordless then $j = 2$ and C is a 4-cycle. Thus C is spanned by two edge-disjoint chordal 2-paths, each of which has at least two bends by Observation 6. Thus the drawing has at least four bends, and we are done. □

We now establish tight lower bounds for the total number of bends in drawings of K_6, and the graphs obtained from K_6 by deleting a matching, except in the case of $K_6 \setminus M_3$ for which there is a difference of one bend between our lower bound and the best known drawing. Figure 17 shows the well-known drawing of K_6 with two 2-bend edges and a total of twelve bends.

Theorem 6. *Every drawing of K_6 has at least two edges each with at least two bends.*

Proof. By Theorem 3 every drawing of K_5, and thus K_6, has an edge with at least two bends. Suppose that there is a drawing of K_6 with exactly one edge vw with at least two bends. By removing v and all the edges incident to v we obtain a 1-bend drawing of K_5, which contradicts Theorem 3. Thus every drawing of K_6 has at least two edges each with at least two bends. □

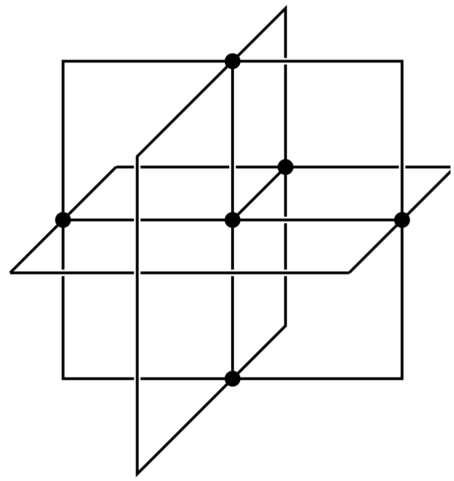


Figure 17: A 2-bend drawing of K_6 with 12 bends.

Note that the above result can be strengthened to say that in every drawing of K_6 there are two non-adjacent edges each with at least two bends. We now prove that the drawing of K_6 illustrated in Figure 17 is bend-minimum, as is the drawing obtained by deleting one or both of the 2-bend edges.

Theorem 7. *For each $j \in \{0,1,2\}$, every drawing of $K_6 \setminus M_j$ has at least $12-2j$ bends.*

Proof. Let k_i ($i \geq 0$) be the number of i-bend edges in a drawing of $K_6 \setminus M_j$ for some $j \in \{0,1,2\}$. Observe that $K_6 \setminus M_j$ has $15-j$ edges. By Lemma 14 in Appendix A, every subgraph of $K_6 \setminus M_j$ with at least eight edges contains C_3 or $K_{2,3}$. By Observations 3 and 5, the 0-bend subgraph does not contain C_3 or $K_{2,3}$. Thus $k_0 \leq 7$, and hence there are at least $8-j$ edges each with at least one bend.

If $k_0 \leq 5$ then at least $10-j$ edges have at least one bend, and since there are at least $2-j$ edges each with at least two bends (by Theorem 6), there are at least $12-2j$ bends, and we are done. Now assume $k_0 \geq 6$. Thus the 0-bend subgraph contains a cycle C, which by Observation 3, is not a 3-cycle. If C has at least two chords then by Theorem 1, each chord has at least three bends, giving a total of at least $12-j \geq 12-2j$ bends, and we are done. Also by Theorem 1, if $j=0$ or C is a 5- or 6-cycle, then C has at least two chords. Thus, we now assume that $C=(a,b,c,d)$ is a 4-cycle and $j \in \{1,2\}$.

If $j=1$ then by Theorem 1, C has exactly one chord, say ac. Let x and y be the other two vertices in $K_6 \setminus M_1$. The edge ac has at least three bends, and each of the chordal 2-paths axc, ayc, bxd and byd have at least two bends, by Observation 6. Thus there is a total of at least $11 > 12-2j$ bends, and we are done. Now assume $j=2$. If C has one chord then this chord has at least three bends, giving a total of at least $8 = 12-2j$ bends, and we are done. Otherwise

C has no chords. Let x and y be the other two vertices in $K_6 \setminus M_2$. Each of the chordal 2-paths axc, ayc, bxd and byd have at least two bends by Observation 6, giving a total of at least $8 = 12 - 2j$ bends. This completes the proof. □

We now prove a lower bound of seven on the number of bends in drawings of $K_6 \setminus M_3$, the octahedron graph. The drawing of $K_6 \setminus M_3$ obtained from the drawing of K_6 illustrated in Figure 17 by deleting the two 2-bend edges and the 0-bend edge has eight bends. We conjecture that every drawing of $K_6 \setminus M_3$ has at least eight bends.

Theorem 8. *Every drawing of $K_6 \setminus M_3$ has at least seven bends.*

Proof. Let k_i $(i \geq 0)$ be the number of i-bend edges in a drawing of $K_6 \setminus M_3$. If $k_0 \leq 5$ then at least seven edges each have at least one bend, and we are done. Now assume $k_0 \geq 6$. Thus the 0-bend subgraph contains a cycle C which is not a 3-cycle by Observation 3. Hence C is a 4-, 5- or 6-cycle. Let $V(K_6 \setminus M_3) = \{a, b, c, 1, 2, 3\}$ with $a1, b2, c3 \notin E(K_6 \setminus M_3)$.

Case 1. *C* is a 4-cycle: Since $K_4 \not\subseteq K_6 \setminus M_3$, C has at most one chord. Initially suppose C has no chords. Without loss of generality $C = (a, b, 1, 2)$. Then $ac1$, $bc2$, $a31$, and $b32$ are edge-disjoint chordal 2-paths between diagonally opposite vertices on C. By Observation 6, each of these chordal 2-paths have at least two bends, giving a total of eight bends, and we are done. Now suppose C has one chord. Without loss of generality $C = (a, b, 3, 2)$, with the chord $a3$ having at least three bends (by Theorem 1). Thus $bc2$ and $b12$ are chordal 2-paths and $ac13$ is a chordal 3-path between diagonally opposite vertices on C. By Observation 6, these chordal 2-paths each have at least two bends and, similarly, the chordal 3-path has at least one bend. Therefore there is a total of at least eight bends.

Case 2. *C* is a 5-cycle: Since $K_6 \setminus M_3$ is vertex-transitive, the vertices of C induce the graph illustrated in Figure 18(a).

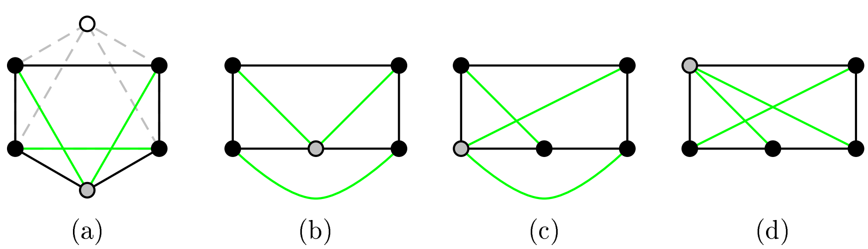


Figure 18: 0-bend 5-cycle in drawings of $K_6 \setminus M_3$.

Since there is only one drawing of a 0-bend 5-cycle, by symmetry there are three different ways to draw C, as illustrated in Figure 18(b), (c) and (d). In each case there are two chords of C each with at least three bends, and a further chord with at least two bends, giving a total of at least eight bends.

Case 3. C is a 6-cycle: By Theorem 1, C has at least three chords each with at least three bends. Thus the drawing has at least nine bends, and we are done. □

We now establish lower bounds for the number of bends in drawings of $K_7 \setminus M_j$ for each $j \in \{0, 1, 2, 3\}$. Figure 19 shows a 4-bend drawing of K_7 with a total of 24 bends. (Compare this with the total of 42 bends in the 2-bend drawing of K_7 in Figure 1.) Deleting j of the three 4-bend edges from this drawing produces a drawing of $K_7 \setminus M_j$ with $24 - 4j$ bends. The following lower bound is thus within $4 - j$ bends of being tight.

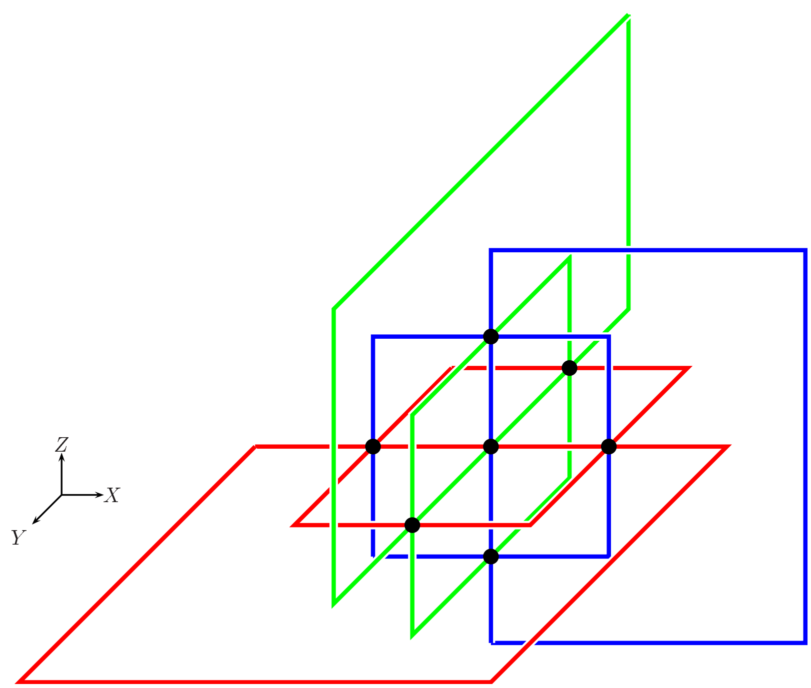


Figure 19: A 4-bend drawing of K_7 with 24 bends.

Theorem 9. *For each $j \in \{0, 1, 2, 3\}$, every drawing of $K_7 \setminus M_j$ has at least $20 - 3j$ bends.*

Proof. Suppose to the contrary, that for some $j \in \{0, 1, 2, 3\}$, there is a drawing of $K_7 \setminus M_j$ with at most $19 - 3j$ bends. Let k_i ($i \geq 0$) be the number of i-bend edges. Then

$$\sum_{i \geq 1} i k_i \leq 19 - 3j \ ,$$

and

$$\sum_{i \geq 1} k_i \leq (19 - 3j) - \sum_{i \geq 2} (i - 1) k_i \ .$$

Since $K_7 \setminus M_j$ has $21 - j$ edges,

$$21 - j \;=\; \sum_{i \geq 0} k_i \;\leq\; k_0 + (19 - 3j) - \sum_{i \geq 2} (i-1) k_i \;\;.$$

Hence,

$$\sum_{i \geq 2} (i-1) k_i \;\leq\; k_0 + (19 - 3j) - (21 - j) \;=\; k_0 - 2j - 2 \;\;. \qquad (1)$$

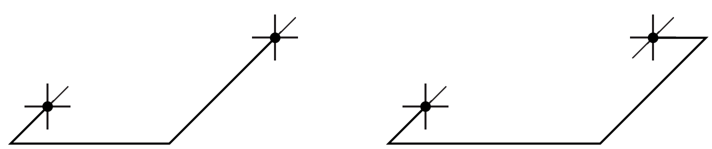
By Lemma 15 in Appendix A, every subgraph of $K_7 \setminus M_j$ with at least ten edges contains C_3 or $K_{2,3}$. By Observations 3 and 5, the 0-bend subgraph does not contain C_3 or $K_{2,3}$; thus $k_0 \leq 9$.

Case 1. $k_0 = 8$ or $k_0 = 9$: By Lemma 16 in Appendix A, every subgraph of $K_7 \setminus M_j$ with at least eight edges contains a cycle C_k ($k \neq 4$), two chord-disjoint cycles, or $K_{2,3}$. Therefore the 0-bend subgraph contains a cycle C_k ($k \geq 5$) or two chord-disjoint 4-cycles (since C_3 and $K_{2,3}$ do not have 0-bend drawings by Observations 3 and 5, respectively). In either case, by Theorem 1 there are at least $4 - j$ chords of these cycles each with at least three bends. Thus $\sum_{i \geq 3} k_i \geq 4 - j$, and hence,

$$k_2 \;+\; 2(4 - j) \;\leq\; k_2 + \sum_{i \geq 3} 2k_i \;\leq\; \sum_{i \geq 2} (i-1) k_i \;\;.$$

By (1) with $k_0 \leq 9$, $k_2 + 8 - 2j \leq 9 - 2j - 2$ and thus $k_2 \leq -1$, which is a contradiction.

Case 2. $k_0 \leq 7$: Let A be the set of edges of $K_7 \setminus M_j$ routed using an extremal port at exactly one end-vertex. Let B be the set of edges routed using extremal ports in the same direction at its end-vertices. Let C be the set of edges routed using extremal ports in differing directions at its end-vertices. Since, all but $2j$ ports in the drawing of $K_7 \setminus M_j$ are used, and there is at least one extremal port in each of the six directions, $|A| + |B| + 2|C| \geq 6 - 2j$. As illustrated in Figure 20(a) an edge in A or B has at least two bends, and an edge in C has at least three bends, as illustrated in Figure 20(b). Hence,



(a) (b)

Figure 20: Edges using extremal ports have at least two bends.

$$k_2 + 2 \sum_{i \geq 3} k_i \;\geq\; 6 - 2j \;\;,$$

which implies,

$$\sum_{i\geq 2}(i-1)k_i \;\geq\; 6-2j \;\; . \tag{2}$$

However, by (1) with $k_0 \leq 7$,

$$\sum_{i\geq 2}(i-1)k_i \;\leq\; 5-2j \;\; ,$$

which contradicts (2). The result follows. □

3.1 Drawings of Multigraphs

We now prove tight bounds for the number of bends in drawings of the complete multigraphs $j \cdot K_2$ for $2 \leq j \leq 6$.

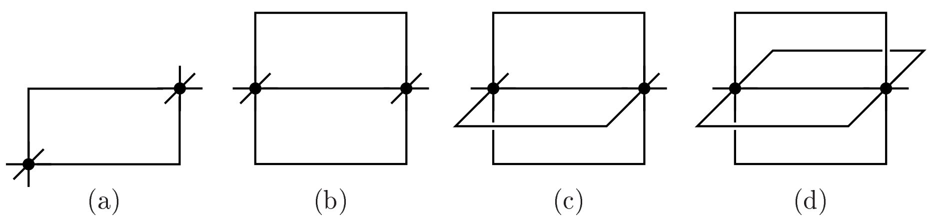


Figure 21: Bend-minimum drawings of (a) $2 \cdot K_2$, (b) $3 \cdot K_2$, (c) $4 \cdot K_2$ and (d) $5 \cdot K_2$.

We omit the proof of the following elementary result as the method is similar and simpler than the proofs of Theorems 10 and 11 for $6 \cdot K_2$ below.

Lemma 7. *For each of the graphs $j \cdot K_2$ ($2 \leq j \leq 5$), the drawings in Figure 21 have the minimum maximum number of bends per edge and the minimum total number of bends.* □

In Figure 22 we show two drawings of $6 \cdot K_2$. We now provide a formal prove of the well-known result that the maximum number of bends per edge in the drawing in Figure 22(a) is optimal.

Theorem 10. *Every drawing of $6 \cdot K_2$ has an edge with at least three bends.*

Proof. Let the vertices of $6 \cdot K_2$ be v and w. Since $6 \cdot K_2$ is 6-regular every port at v and w is used. The two vertices are either (a) collinear, (b) coplanar but not collinear, or (c) not coplanar, as illustrated in Figure 23.

In each case there is a port at v pointing away from w such that the edge using this port requires at least three bends to reach w. □

We now prove that the drawing in Figure 22(b) is bend-minimum.

Theorem 11. *Every drawing of the multigraph $6 \cdot K_2$ has at least twelve bends.*

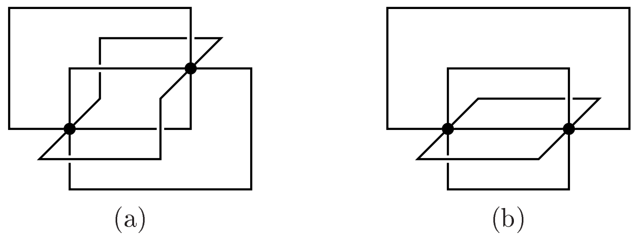


Figure 22: Drawings of $6 \cdot K_2$ with (a) a maximum of three bends per edge, and (b) a total of twelve bends.

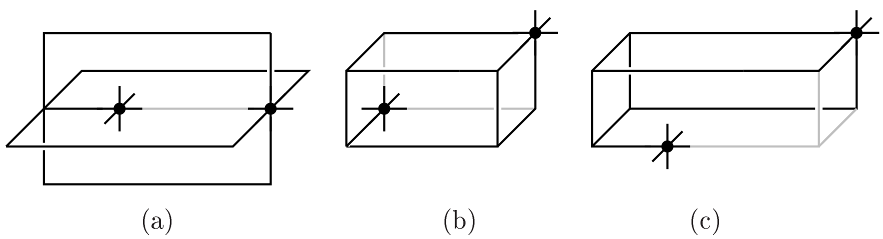


Figure 23: $6 \cdot K_2$ has a 3-bend edge.

Proof. Let the vertices of $6 \cdot K_2$ be v and w. Suppose that v and w are not coplanar. The edges using the three ports at v pointing towards w have at least two bends, and the other edges have at least three bends. Thus the drawing has at least 15 bends.

Suppose v and w are coplanar but not collinear. The edges using the two ports at v pointing towards w have at least one bend, the edges using the two opposite ports have at least two bends, and the remaining two edges have at least three bends. Thus the drawing has at least 12 bends.

Suppose v and w are collinear, and without loss of generality, that v and w lie in an X-axis parallel line, such that the X-coordinate of v is less than the X-coordinate of w. The edge using the port X_v^- has at least three bends, and the four edges using the other four ports at v pointing away from w have at least two bends. Thus the drawing has at least 11 bends. Suppose there is such a drawing with exactly 11 bends. Then there are four 2-bend edges, and one 3-bend edge. These four 2-bend edges use the $Y^\pm$ and $Z^\pm$ ports at each vertex. Therefore, the edge using the X_v^- port and the X_w^+ port has four bends, and thus the drawing has 12 bends, which is a contradiction. The result follows. □

4 Constructing Large Graphs

In this section we use the lower bounds for the number of bends in drawings of the complete graphs established in Section 3 as building blocks to construct infinite families of c-connected graphs ($2 \leq c \leq 6$) with maximum degree Δ ($2 \leq \Delta \leq 6$), and with lower bounds on the number of bends in drawings of every graph in the family.

A graph is *c-connected* ($c \geq 1$) if the removal of fewer than c vertices results in neither a disconnected graph nor the trivial graph. To establish that our graphs are c-connected we use the following characterisation due to Whitney [31], which is part of the family of results known as 'Menger's Theorem'. A graph G is c-connected if and only if for each pair u, v of distinct vertices there are at least c internally disjoint paths from u to v in G. Our proofs of connectivity are postponed until Appendix B.

We employ two methods for constructing new graphs from two given graphs. First, given graphs G and H, we define $H\langle G\rangle$ to be the graph obtained by replacing each vertex of H by a copy of G, and connecting the edges in H incident to a particular vertex in H to different vertices in the corresponding copy of G. In most cases, H is Δ-regular and G is a complete graph K_p for some $p \geq \Delta$; thus $H\langle G\rangle$ is well-defined. In other cases we shall specify which edges of H are connected to which vertices in each copy of G.

Our second method for constructing large graphs is the *cartesian product* $G \times H$ of graphs G and H. $G \times H$ has vertex set $V(H) \times V(G)$ with (v_1, w_1) and (v_2, w_2) adjacent in $G \times H$ if either $v_1 = v_2$ and $w_1 w_2 \in E(G)$, or $w_1 = w_2$ and $v_1 v_2 \in E(H)$. For example, $C_p \times C_q$ is the 4-regular $p \times q$ torus graph.

Our lower bounds for simple disconnected graphs are obtained by taking disjoint copies of K_p for $4 \leq p \leq 7$. For consistency we denote these graphs by $I_r\langle K_p\rangle$, where I_r is the r-vertex graph with no edges. Our lower bounds for disconnected multigraphs are obtained by $I_r\langle p \cdot K_2\rangle$ for $3 \leq p \leq 6$, and we use $I_r\langle L_p\rangle$ with $1 \leq p \leq 3$ to obtain lower bounds for disconnected pseudographs.

As illustrated in Figure 24, to obtain lower bounds for simple 2-connected graphs, we use $C_r\langle K_p\rangle$ for $3 \leq p \leq 6$ and $r \geq 3$, and $C_r\langle K_p \setminus M_1\rangle$ for $4 \leq p \leq 7$ and $r \geq 2$, where the non-adjacent vertices in each copy of $K_p \setminus M_1$ are incident to the edges of C_r. To obtain lower bounds for 2-connected multigraphs, we use $C_r\langle p \cdot K_2\rangle$ with $2 \leq p \leq 5$, and we use $C_r\langle L_p\rangle$ with $1 \leq p \leq 2$ to obtain lower bounds for 2-connected pseudographs.

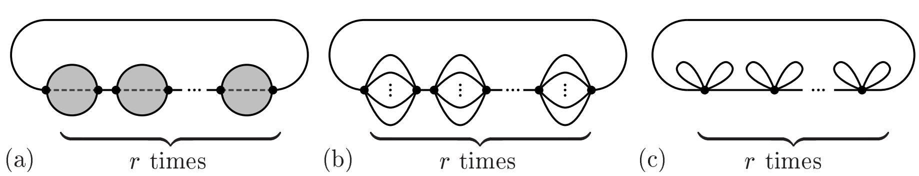


Figure 24: 2-connected graphs: (a) $C_r\langle K_p \setminus M_1\rangle$, (b) $C_r\langle p \cdot K_2\rangle$, and (c) $C_r\langle L_2\rangle$.

As illustrated in Figure 25, to obtain lower bounds for simple 3-connected graphs, we use $(C_r \times K_2)\langle K_p\rangle$ for $3 \leq p \leq 6$ and $r \geq 3$, and $(C_r \times K_2)\langle K_p \setminus M_2\rangle$ for $5 \leq p \leq 7$ and $r \geq 3$, where the non-adjacent pairs of vertices in each copy of $K_p \setminus M_2$ are incident to opposite edges of $C_r \times K_2$ where possible. To obtain lower bounds for 3-connected multigraphs, we use $C_r \times (p \cdot K_2)$ with $r \geq 3$ and

$2 \le p \le 4$. We use $C_r \times ((p \cdot K_2)\langle L_1 \rangle)$ with $1 \le p \le 2$ to obtain lower bounds for 3-connected pseudographs.

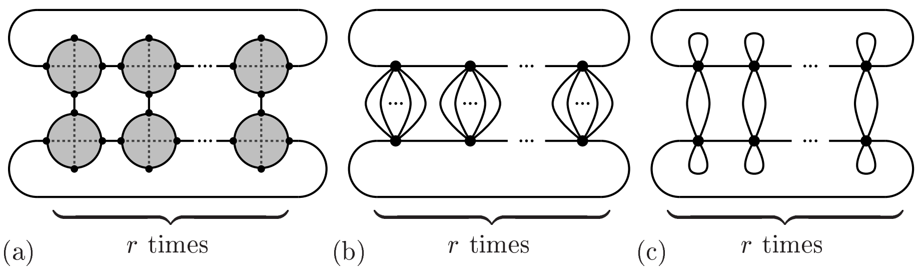


Figure 25: 3-connected graphs: (a) $(C_r \times K_2)\langle K_p \setminus M_2 \rangle$, (b) $C_r \times (p \cdot K_2)$, and (c) $C_r \times ((p \cdot K_2)\langle L_1 \rangle)$.

To obtain lower bounds for 4-connected simple graphs, we use $(C_r \times C_3)\langle K_p \rangle$ for $4 \le p \le 6$ and $r \ge 3$, and $(C_r \times C_3)\langle K_p \setminus M_2 \rangle$ for $5 \le p \le 7$ and $r \ge 3$, where the non-adjacent pairs of vertices in each copy of $K_p \setminus M_2$ are incident to opposite edges of $C_r \times C_3$, as illustrated in Figure 26(a). We use $(C_r \times C_3)\langle L_1 \rangle$ with $r \ge 3$ to obtain lower bounds for 4-connected pseudographs, as illustrated in Figure 26(b).

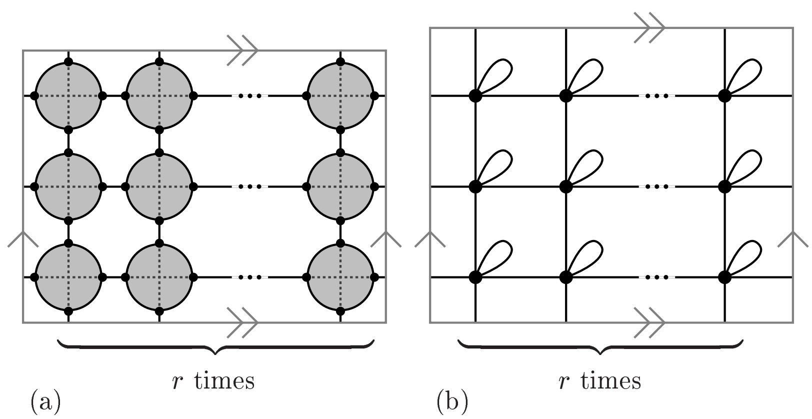


Figure 26: 4-connected graphs: (a) $(C_r \times C_3)\langle K_p \setminus M_2 \rangle$, and (b) $(C_r \times C_3)\langle L_1 \rangle$.

Let $2 \cdot C_m$ be the m-edge cycle with each edge having multiplicity 2, and let $\frac{3}{2} \cdot C_m$ for even m be the m-edge cycle with alternating edges around the cycle having multiplicity 2. To obtain lower bounds for 4-connected multigraphs, we use the 6-regular multigraph $C_r \times (2 \cdot C_3)$ for some $r \ge 3$, as illustrated in Figure 27(b), and the 5-regular multigraph $C_r \times (\frac{3}{2} \cdot C_4)$ for $r \ge 3$, as illustrated in Figure 27(c).

To obtain lower bounds for 5-connected graphs, we use $(C_r \times C_3 \times K_2)\langle K_p \rangle$ for $5 \le p \le 6$ and $r \ge 3$, and $(C_r \times C_3 \times K_2)\langle K_7 \setminus M_3 \rangle$ for $r \ge 3$, where the non-

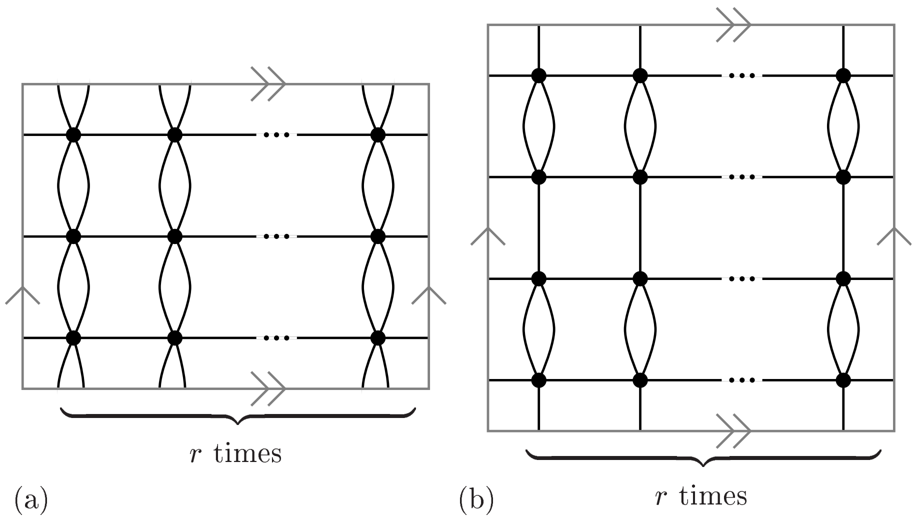


Figure 27: 4-connected graphs: (a) $C_r \times (2 \cdot C_3)$, (b) $C_r \times (\frac{3}{2} \cdot C_4)$.

adjacent pairs of vertices in each copy of $K_7 \setminus M_3$ are incident to opposite edges of $C_r \times C_3 \times K_2$ where possible, as illustrated in Figure 28(a). To obtain lower bounds for 5-connected multigraphs, we use $C_r \times C_3 \times (2 \cdot K_2)$, as illustrated in Figure 28(b).

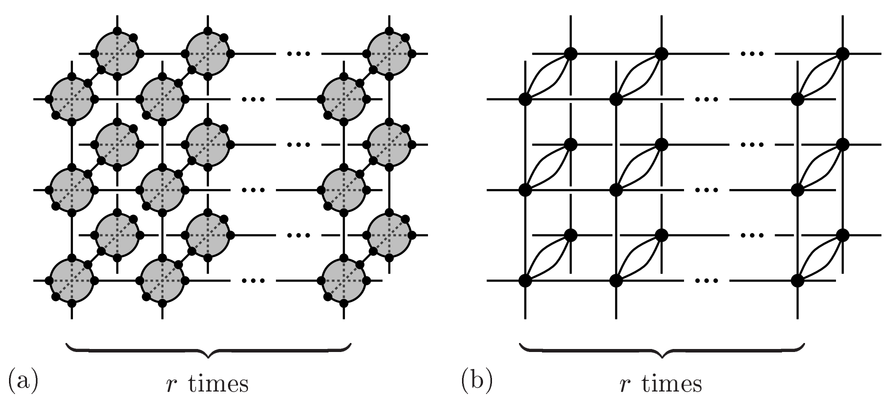


Figure 28: 5-connected graphs: (a) $(C_r \times C_3 \times K_2)\langle G \rangle$, (b) $C_r \times C_3 \times (2 \cdot K_2)$.

To obtain lower bounds for 6-connected graphs we use $(C_r \times C_3 \times C_3)\langle K_6 \rangle$ and $(C_r \times C_3 \times C_3)\langle K_7 \setminus M_3 \rangle$ for $r \geq 3$, where the non-adjacent pairs of vertices in each copy of $K_7 \setminus M_3$ are incident to opposite edges in $C_r \times C_3 \times C_3$, as illustrated in Figure 29.

In Table 3 we prove lower bounds on the number of bends in drawings of the

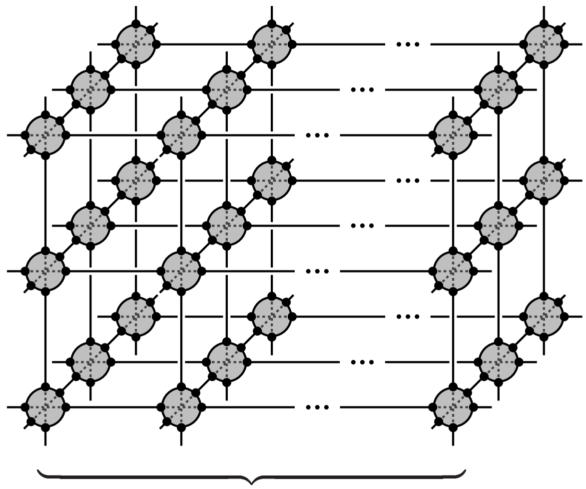


Figure 29: 6-connected 6-regular graphs $(C_r \times C_3 \times C_3)\langle K_6\rangle$ and $(C_r \times C_3 \times C_3)\langle K_7 \setminus M_3\rangle$.

above families of graphs. Each line of the table corresponds to one such family $H\langle G\rangle$ (or $H \times G$) parameterised by some value r, all of which have maximum degree Δ (shown in the first column). The third column shows the lower bounds on the number of bends in a drawing of G, as proved earlier in the paper. The fourth and fifth columns shows the number of edge-disjoint copies of G and the number of edges in $H\langle G\rangle$ (or $H \times G$), respectively. The sixth column shows the lower bound on the average number of bends per edge in $H\langle G\rangle$ (or $H \times G$) obtained by

$$\text{average } \# \text{ bends}(H\langle G\rangle \text{ or } H \times G) \;\geq\; \frac{\# \text{ bends}(G) \times \# \text{ copies}(G)}{\# \text{ edges}(H\langle G\rangle \text{ or } H \times G)} \;.$$

A line marked with a $\star$ indicates the corresponding lower bound is the best out of those for graphs with a specific connectivity and maximum degree. These 'best known' lower bounds are those listed in Table 2 in Section 1.

Table 3: Lower bounds for the average number of bends per edge.

Δ	$H\langle G\rangle$ or $H \times G$	#bends(G)	#copies(G)	#edges	avg. #bends	
Disconnected Simple Graphs						
6	$I_r\langle K_7\rangle$	20 (Thm. 9)	r	$21r$	$\frac{20}{21}$	$\star$

continued on next page

Table 3: *continued*

Δ	$H\langle G\rangle$ or $H \times G$	#bends(G)		#copies(G)	#edges	avg. #bends	
5	$I_r\langle K_6\rangle$	12	(Thm. 7)	r	$15r$	$\frac{12}{15}=\frac{4}{5}$	$\star$
4	$I_r\langle K_5\rangle$	7	(Thm. 4)	r	$10r$	$\frac{7}{10}$	$\star$
3	$I_r\langle K_4\rangle$	3	(Thm. 2)	r	$6r$	$\frac{3}{6}=\frac{1}{2}$	$\star$
2	$I_r\langle K_3\rangle$	1	(Obs. 3)	r	$3r$	$\frac{1}{3}$	$\star$
Disconnected Multigraphs							
6	$I_r\langle 6\cdot K_2\rangle$	12	(Thm. 11)	r	$6r$	$\frac{12}{6}=2$	$\star$
5	$I_r\langle 5\cdot K_2\rangle$	8	(Lem. 7)	r	$5r$	$\frac{8}{5}$	$\star$
4	$I_r\langle 4\cdot K_2\rangle$	6	(Lem. 7)	r	$4r$	$\frac{6}{4}=\frac{3}{2}$	$\star$
3	$I_r\langle 3\cdot K_2\rangle$	4	(Lem. 7)	r	$3r$	$\frac{4}{3}$	$\star$
2	$I_r\langle 2\cdot K_2\rangle$	2	(Lem. 7)	r	$2r$	$\frac{2}{2}=1$	$\star$
Disconnected Pseudographs							
6	$I_r\langle L_3\rangle$	9		r	$3r$	3	$\star$
4	$I_r\langle L_2\rangle$	6		r	$2r$	3	$\star$
2	$I_r\langle L_1\rangle$	3		r	r	3	$\star$
2-Connected Simple Graphs							
6	$C_r\langle K_6\rangle$	12	(Thm. 7)	r	$16r$	$\frac{12}{16}=\frac{3}{4}$	
6	$C_r\langle K_7\setminus M_1\rangle$	17	(Thm. 9)	r	$21r$	$\frac{17}{21}$	$\star$
5	$C_r\langle K_5\rangle$	7	(Thm. 4)	r	$11r$	$\frac{7}{11}$	
5	$C_r\langle K_6\setminus M_1\rangle$	10	(Thm. 7)	r	$15r$	$\frac{10}{15}=\frac{2}{3}$	$\star$
4	$C_r\langle K_4\rangle$	3	(Thm. 2)	r	$7r$	$\frac{3}{7}$	
4	$C_r\langle K_5\setminus M_1\rangle$	5	(Thm. 5)	r	$10r$	$\frac{5}{10}=\frac{1}{2}$	$\star$
3	$C_r\langle K_3\rangle$	1	(Obs. 3)	r	$4r$	$\frac{1}{4}$	
3	$C_r\langle K_4\setminus M_1\rangle$	2	(Thm. 2)	r	$6r$	$\frac{2}{6}=\frac{1}{3}$	$\star$
2-Connected Multigraphs							
6	$C_r\langle 5\cdot K_2\rangle$	8	(Lem. 7)	r	$6r$	$\frac{8}{6}=\frac{4}{3}$	$\star$
5	$C_r\langle 4\cdot K_2\rangle$	6	(Lem. 7)	r	$5r$	$\frac{6}{5}$	$\star$
4	$C_r\langle 3\cdot K_2\rangle$	4	(Lem. 7)	r	$4r$	$\frac{4}{4}=1$	$\star$
3	$C_r\langle 2\cdot K_2\rangle$	2	(Lem. 7)	r	$3r$	$\frac{2}{3}$	$\star$

continued on next page

Table 3: *continued*

Δ	$H\langle G\rangle$ or $H \times G$	#bends(G)	#copies(G)	#edges	avg. #bends	
2-Connected Pseudographs						
6	$C_r\langle L_2\rangle$	6	r	$3r$	2	$\star$
4	$C_r\langle L_1\rangle$	3	r	$2r$	$\frac{3}{2}$	$\star$
3-Connected Simple Graphs						
6	$(C_r \times K_2)\langle K_6\rangle$	12 (Thm. 7)	$2r$	$33r$	$\frac{2\cdot 12}{33} = \frac{8}{11}$	$\star$
6	$(C_r \times K_2)\langle K_7 \setminus M_2\rangle$	14 (Thm. 9)	$2r$	$41r$	$\frac{2\cdot 14}{41} = \frac{28}{41}$	
5	$(C_r \times K_2)\langle K_5\rangle$	7 (Thm. 4)	$2r$	$23r$	$\frac{2\cdot 7}{23} = \frac{14}{23}$	$\star$
5	$(C_r \times K_2)\langle K_6 \setminus M_2\rangle$	8 (Thm. 7)	$2r$	$29r$	$\frac{2\cdot 8}{29} = \frac{16}{29}$	
4	$(C_r \times K_2)\langle K_4\rangle$	3 (Thm. 2)	$2r$	$15r$	$\frac{2\cdot 3}{15} = \frac{2}{5}$	
4	$(C_r \times K_2)\langle K_5 \setminus M_2\rangle$	4 (Thm. 5)	$2r$	$19r$	$\frac{2\cdot 4}{19} = \frac{8}{19}$	$\star$
3	$(C_r \times K_2)\langle K_3\rangle$	1 (Obs. 3)	$2r$	$9r$	$\frac{2}{9}$	$\star$
3-Connected Multigraphs						
6	$C_r \times (4 \cdot K_2)$	6 (Lem. 7)	r	$6r$	$\frac{6}{6} = 1$	$\star$
5	$C_r \times (3 \cdot K_2)$	4 (Lem. 7)	r	$5r$	$\frac{4}{5}$	$\star$
4	$C_r \times (2 \cdot K_2)$	2 (Lem. 7)	r	$4r$	$\frac{2}{4} = \frac{1}{2}$	$\star$
3-Connected Pseudographs						
6	$C_r \times ((2 \cdot K_2)\langle L_1\rangle)$	8	r	$6r$	$\frac{4}{3}$	$\star$
5	$(C_r \times K_2)\langle L_1\rangle$	3	$2r$	$5r$	$\frac{6}{5}$	$\star$
4-Connected Simple Graphs						
6	$(C_r \times C_3)\langle K_6\rangle$	12 (Thm. 7)	$3r$	$(3 \cdot 15 + 6)r$	$\frac{3\cdot 12}{51} = \frac{12}{17}$	$\star$
6	$(C_r \times C_3)\langle K_7 \setminus M_2\rangle$	14 (Thm. 9)	$3r$	$(3 \cdot 19 + 6)r$	$\frac{3\cdot 14}{63} = \frac{2}{3}$	
5	$(C_r \times C_3)\langle K_5\rangle$	7 (Thm. 4)	$3r$	$(3 \cdot 10 + 6)r$	$\frac{3\cdot 7}{36} = \frac{7}{12}$	$\star$
5	$(C_r \times C_3)\langle K_6 \setminus M_2\rangle$	8 (Thm. 7)	$3r$	$(3 \cdot 13 + 6)r$	$\frac{3\cdot 8}{45} = \frac{8}{15}$	
4	$(C_r \times C_3)\langle K_4\rangle$	3 (Thm. 2)	$3r$	$(3 \cdot 6 + 6)r$	$\frac{3\cdot 3}{24} = \frac{3}{8}$	
4	$(C_r \times C_3)\langle K_5 \setminus M_2\rangle$	4 (Thm. 5)	$3r$	$(3 \cdot 8 + 6)r$	$\frac{3\cdot 4}{30} = \frac{2}{5}$	$\star$
4-Connected Multigraphs						
5	$C_r \times (\frac{3}{2} \cdot C_4)$	4 (Lem. 7)	r	$10r$	$\frac{4}{10} = \frac{2}{5}$	$\star$
6	$C_r \times (2 \cdot C_3)$	6 (Lem. 7)	r	$9r$	$\frac{6}{9} = \frac{2}{3}$	$\star$
4-Connected Pseudographs						
6	$(C_r \times C_3)\langle L_1\rangle$	3	$3r$	$(3 \cdot 1 + 6)r$	$\frac{9}{9} = 1$	$\star$

continued on next page

Table 3: *continued*

Δ	$H\langle G\rangle$ or $H \times G$	#bends(G)	#copies(G)	#edges	avg. #bends	
5-Connected Simple Graphs						
6	$(C_r \times C_3 \times K_2)\langle K_6\rangle$	12 (Thm. 7)	$6r$	$(6 \cdot 15 + 15)r$	$\frac{6 \cdot 12}{105} = \frac{24}{35}$	$\star$
6	$(C_r \times C_3 \times K_2)\langle K_7 \setminus M_3\rangle$	11 (Thm. 9)	$6r$	$(6 \cdot 18 + 15)r$	$\frac{6 \cdot 11}{123} = \frac{22}{41}$	
5	$(C_r \times C_3 \times K_2)\langle K_5\rangle$	7 (Thm. 4)	$6r$	$(6 \cdot 10 + 15)r$	$\frac{6 \cdot 7}{75} = \frac{14}{25}$	$\star$
5-Connected Multigraphs						
6	$C_r \times C_3 \times (2 \cdot K_2)$	2 (Lem. 7)	$3r$	$18r$	$\frac{2 \cdot 3}{18} = \frac{1}{3}$	$\star$
6-Connected Simple Graphs						
6	$(C_r \times C_3 \times C_3)\langle K_6\rangle$	12 (Thm. 7)	$9r$	$(9 \cdot 15 + 27)r$	$\frac{9 \cdot 12}{162} = \frac{2}{3}$	$\star$
6	$(C_r \times C_3 \times C_3)\langle K_7 \setminus M_3\rangle$	11 (Thm. 9)	$9r$	$(9 \cdot 18 + 27)r$	$\frac{9 \cdot 11}{189} = \frac{11}{21}$	

5 Lower Bounds for General Position Drawings

Recall that a 3-D orthogonal graph drawing is said to be in *general position* if no two vertices lie in a common grid-plane. The general position model has been used for 3-D orthogonal graph drawing by Eades *et al.* [13] and Wood [33, 36, 37], and for 3-D orthogonal box-drawing of arbitrary degree graphs by Papakostas and Tollis [21], Biedl [8] and Wood [34]. In this section we establish lower bounds for the number of bends in general position drawings of 2-connected and 4-connected graphs. The next result will be crucial for the lower bounds to follow.

Lemma 8. *If the graph G has at least k bends in every general position drawing then for every edge e of G, the graph $G \setminus e$ has at least $k - 4$ bends in every general position drawing.*

Proof. Suppose $G \setminus e$ has a general position drawing with b bends. Wood [37] proved that the edge e can be inserted into the drawing of $G \setminus e$ with at most four bends (possibly introducing edge crossings), and that the edges can be rerouted to eliminate all edge crossings without increasing the total number of bends. Thus there is a (crossing-free) general position drawing of G with $b + 4$ bends. By assumption, every general position drawing of G has at least k bends. Thus $b + 4 \geq k$ and $b \geq k - 4$. □

Clearly every edge in a general position drawing has at least two bends. Observe that if an edge is routed using an extremal port, then this edge has at least three bends, as illustrated in Figure 30.

Since all ports are used in a drawing of a 6-regular m-edge graph, a general position drawing of such a graph requires at least $2m + 6$ bends. Hence the graphs consisting of disjoint copies of K_7 provide the following lower bound.

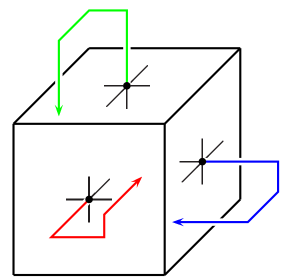


Figure 30: General position edges using extreme ports have at least three bends.

Lemma 9. *There exists an infinite family of n-vertex m-edge simple graphs, each with at least $2m + \frac{6}{7}n$ bends in every general position drawing.* □

Note that for 6-regular graphs $m = 3n$; thus the above lower bound matches the upper bound of $\frac{16}{7}m$ for the total number of bends in general position drawings established by the DIAGONAL LAYOUT & MOVEMENT algorithm [37].

To obtain a lower bound for general position drawings of 2-connected graphs, we use the 6-regular graph $C_r\langle K_7 \setminus M_1\rangle$, where the non-adjacent vertices of each $K_7 \setminus M_1$ are incident to the edges of C_r, as illustrated in Figure 24(a).

Lemma 10. *There exists an infinite family of n-vertex m-edge simple 2-connected graphs, each with at least $2m + \frac{4}{7}n$ bends in every general position drawing.*

Proof. Clearly $C_r\langle K_7 \setminus M_1\rangle$ is 2-connected. K_7 has at least $2|E(K_7)| + 6$ bends in any general position drawing. Thus by Lemma 8, a general position drawing of $K_7 \setminus M_1$ has at least $2|E(K_7)| + 6 - 4 = 2|E(K_7 \setminus M_1)| + 4$ bends. The edges of C_r each have at least two bends. Thus $C_r\langle K_7 \setminus M_1\rangle$ has at least $2m + \frac{4}{7}n$ bends. □

To obtain a lower bound for general position drawings of 4-connected graphs, we use the 6-regular graph $(C_r \times C_3)\langle K_7 \setminus M_2\rangle$ for $r \geq 3$, as illustrated in Figure 26(a).

Lemma 11. *There exists an infinite family of n-vertex m-edge simple 4-connected graphs, each with at least $2m + \frac{2}{7}n$ bends in every general position drawing.*

Proof. As proved in Appendix B, $(C_r \times C_3)\langle K_7 \setminus M_2\rangle$ is 4-connected. K_7 has at least $2|E(K_7)|+6$ bends in any general position drawing. Hence, by Lemma 8 a general position drawing of $K_7 \setminus M_2$ has at least $2|E(K_7)| + 6 - 8 = 2|E(K_7 \setminus M_2)| + 2$ bends. Edges not in a $K_7 \setminus M_2$ have at least two bends. Thus $(C_r \times C_3)\langle K_7 \setminus M_2\rangle$ has at least $2m + \frac{2}{7}n$ bends. □

6 On the 2-Bends Problem

We now look at the ramifications of the above general position lower bounds for the 2-bends problem. Edges with at most two bends can be classified as 0-bend, 1-bend, 2-bend planar or 2-bend non-planar, as illustrated in Figure 31.

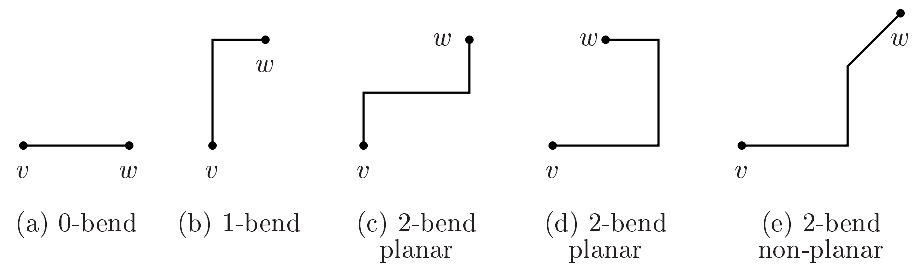


Figure 31: Edges vw with at most two bends.

In a given 2-bend drawing of a graph G, we denote the number of 0-bend edges by k_0, and the number of 2-bend planar edges by k_2'. We now describe how to transform a given 2-bend drawing into a general position drawing.

Lemma 12. *If there is a 2-bend drawing of a graph G then there exists a general position drawing of G with $2m + k_0 + k_2'$ bends.*

Proof. We show that by inserting planes and adding bends to the edge routes a given 2-bend drawing can be transformed into a drawing with a general position vertex layout and the stated number of bends. Consider a grid plane P containing k vertices ($k > 1$). As illustrated in Figure 32, replace the plane by k adjacent planes, and position each of the k vertices in a unique plane.

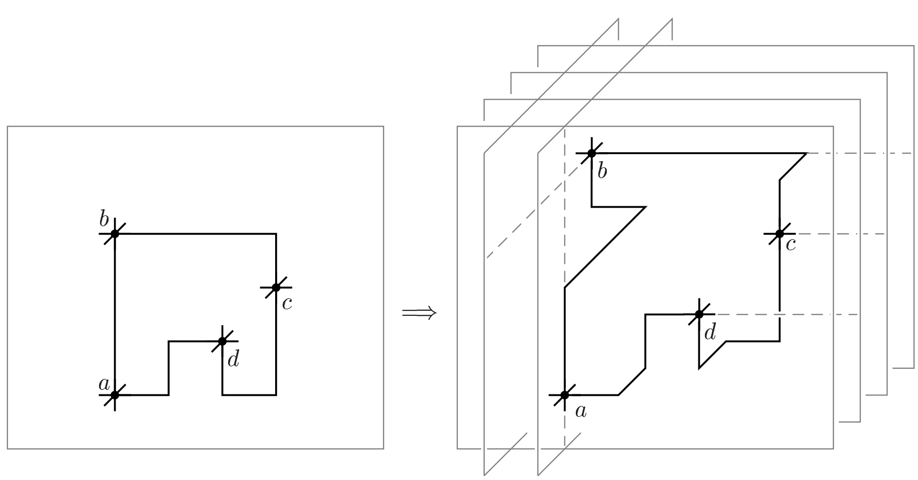


Figure 32: Removing a plane containing many vertices.

A 0-bend edge is split in the middle and replaced by the 2-bend planar edge illustrated in Figure 31(c). If the 0-bend edge has length one then an extra plane perpendicular to the 0-bend edge is also inserted.

Edge segments from an edge with at least one bend and incident to a vertex v are routed in the plane containing v. For a 1-bend edge vw in the original plane, an extra segment is inserted perpendicular to P, running between the planes containing v and w. Hence vw is replaced by a 2-bend non-planar edge (for example, edge bc in Figure 32).

For a 2-bend edge vw in the original plane, the middle segment of vw is routed arbitrarily in the plane containing v or w, and a third segment is inserted perpendicular to P, running between the planes containing v and w. Hence vw is replaced by a 3-bend non-planar edge (for example, edges ad and cd in Figure 32).

For a 2-bend non-planar edge vw incident to one of the k vertices, the segment of vw perpendicular to P is extended in the obvious manner. Similarly, an edge passing through the original plane and not incident to any of the k vertices, is extended so that it passes through all k planes.

This process is continued until there are no grid planes containing more than one vertex. Note that a 0-bend edge will initially be replaced by a 2-bend planar edge, and in a second transformation will be replaced by a 3-bend edge route (for example, edge ab in Figure 32). The resulting drawing has no crossings, has a general position vertex layout, and every edge has two bends except for the 0-bend and 2-bend planar edges in the original drawing, which now have three bends. Hence the new drawing has $2m + k_0 + k_2'$ bends. □

We now prove that for certain graphs any 2-bend drawing has many 0-bend or 2-bend planar edge routes.

Corollary 1. *There exists an infinite family of 6-regular n-vertex graphs, such that in any 2-bend drawing of any one of the graphs, $k_0 + k_2' \geq \frac{6}{7}n$.*

Proof. By Lemma 9, there exists an infinite family of graphs, each with at least $2m + \frac{6}{7}n$ bends in any general position drawing. If there is a 2-bend drawing of such a graph, then by Lemma 12 there exists a general position drawing with $2m + k_0 + k_2'$ bends. Hence $2m + k_0 + k_2' \geq 2m + \frac{6}{7}n$ and $k_0 + k_2' \geq \frac{6}{7}n$. □

The following two results are obtained using the same argument used in the proof of Corollary 1 applied with Lemma 10 and Lemma 11, respectively.

Corollary 2. *There exists an infinite family of 6-regular 2-connected n-vertex graphs, such that in any 2-bend drawing of any one of the graphs, $k_0 + k_2' \geq \frac{4}{7}n$.* □

Corollary 3. *There exists an infinite family of 6-regular 4-connected n-vertex graphs, such that in any 2-bend drawing of any one of the graphs, $k_0 + k_2' \geq \frac{2}{7}n$.* □

A natural variation of the general position model allows at most two vertices in any one grid-plane and with each vertex being coplanar with at most one other vertex. We now show that there exists graphs which do not have 2-bend drawings in this model.

Theorem 12. *There exists an infinite family of 2-connected graphs each of which does not have a 2-bend drawing with at most two vertices in any one grid-plane and with each vertex being coplanar with at most one other vertex.*

Proof. By Corollary 2 there exists an infinite family of 6-regular 2-connected n-vertex graphs, such that in any 2-bend drawing of any one of the graphs, $k_0 + k_2' \geq \frac{4}{7}n$. Assume, to the contrary, that for such a graph there is a 2-bend drawing with at most two vertices in any one grid-plane and with each vertex being coplanar with at most one other vertex. Then the number of pairs of vertices in a common grid-plane is at most $\frac{n}{2}$, and the number of planar edge routes is at most $\frac{n}{2}$; that is, $k_0 + k_1 + k_2' \leq \frac{n}{2}$. Hence $\frac{4}{7}n \leq k_0 + k_2' \leq \frac{n}{2} - k_1$, implying $k_1 < 0$, which is a contradiction, as required. □

7 Conclusion and Open Problems

In this paper we have initiated the study of lower bounds for the number of bends in 3-D orthogonal drawings of maximum degree six graphs. As well as closing the gap between the established lower and upper bounds, the following are interesting open problems not already discussed in this paper.

- The sequence of lower bounds on the number of bends in general position drawings in Section 5 suggests the following open problem. Does every 6-connected 6-regular graph have a general position drawing with at most $2m + 6$ bends?
- Are there classes of graphs (besides maximum degree five simple graphs) which admit general position 2-bend drawings? For example, it is conceivable that planar graphs with maximum degree at most six admit general position 2-bend drawings.
- In the bend-minimum drawings of K_4, K_5 and K_6 the 0-bend subgraph is a tree. Is this the case for all graphs? It is easily seen that every tree has a 0-bend drawing.
- Does every graph with maximum degree at most three have a 1-bend drawing?

Acknowledgements

The author gratefully acknowledges Therese Biedl, Antonios Symvonis, and Ben Lynn for helpful discussions. In particular, Lemmata 10, 12 and 19 were developed in conjunction with Therese Biedl, Antonios Symvonis, and Ben Lynn, respectively. Many thanks to Graham Farr for advice and encouragement. Thanks also to the anonymous referees for numerous helpful comments, in particular a simplified proof of Theorem 6.

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A Existence of Small Subgraphs

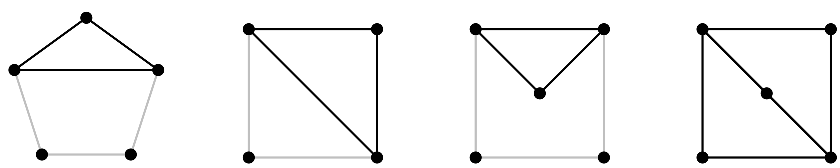
In this appendix we prove a number of results concerning the existence of cycles and other small subgraphs in graphs of a certain size.

Lemma 13. *Every 5-vertex graph with at least six edges contains C_3 or $K_{2,3}$.*

Proof. Suppose to the contrary that there exists a 6-edge 5-vertex graph G not containing C_3 or $K_{2,3}$. G contains a cycle. Let C be the cycle of maximum length in G. Then $|C| = 5$ or $|C| = 4$.

Case 1. $|C| = 5$: Since G has six edges, C has a chord, in which case G contains a 3-cycle, as illustrated in Figure 33(a).

Case 2. $|C| = 4$: If C has a chord then G contains a 3-cycle, as illustrated in Figure 33(b). Thus C does not have a chord. Hence the vertex v not in C is incident to two edges vu and vw, where u and w are in C. If u and w are adjacent in C then G contains a 3-cycle, as illustrated in Figure 33(c). Thus u and w are not adjacent in C, which implies that G contains $K_{2,3}$, as illustrated in Figure 33(d). □



(a) (b) (c) (d)

Figure 33: C_3 or $K_{2,3}$ in a 5-vertex 6-edge graph.

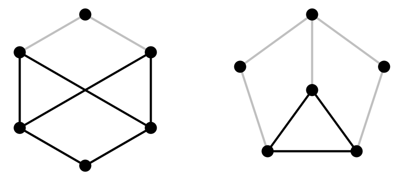
Lemma 14. *Every 6-vertex graph with at least eight edges contains C_3 or $K_{2,3}$.*

Proof. Suppose to the contrary that there exists an 8-edge 6-vertex graph G not containing C_3 or $K_{2,3}$. Let C be the longest cycle in G. Clearly $3 \leq |C| \leq 6$.

Case 1. $|C| = 6$: If $|C| = 6$ then C has at least two chords. Any two chords of C which do not induce a 3-cycle induce $K_{2,3}$, as illustrated in Figure 34(a).

Case 2. $|C| = 5$: If $|C| = 5$ then any chord of C induces a 3-cycle, and we are done. Otherwise C has no chords. Since G has at least eight edges, the vertex v not in C is adjacent to three vertices u, w and x in C. Two of u, w and x are adjacent in C, which implies G contains a 3-cycle, as illustrated in Figure 34(b).

Case 3. $|C| = 4$: If $|C| = 4$ then any chord of C induces a 3-cycle, and we are done. Otherwise C has no chords. Let v and w be the vertices not in C. Since G has at least eight edges, there are at least four edges in G incident with v or w. Even if G contains the edge vw, there are two edges from v to vertices on C, or two edges from w to vertices on C. In either case, G contains C_3 or $K_{2,3}$, as illustrated in Figure 33(c) and Figure 33(d), respectively. □



(a) (b)

Figure 34: C_3 or $K_{2,3}$ in an 8-edge 6-vertex graph.

Lemma 15. *Every 7-vertex graph with at least ten edges contains C_3 or $K_{2,3}$.*

Proof. Suppose to the contrary that there exists a 10-edge 7-vertex graph G not containing C_3 or $K_{2,3}$. Let C be the longest cycle in G. Clearly $3 \leq |C| \leq 7$.

Case 1. $|C| = 7$: If $|C| = 7$ then there are three chords of C in G. If two of these chords are incident to one vertex then, as illustrated in Figure 35(a), G contains a 3-cycle. Thus each vertex is incident to at most one chord. Hence there exists a vertex not incident to any chords of C. It is easily seen that the only configuration of three chords of C not inducing a 3-cycle is that illustrated in Figure 35(b); however in this case, G contains $K_{2,3}$.

Case 2. $|C| = 6$: Suppose the vertex not in C is v. A chord of C which does not induce a 3-cycle is between vertices at distance two in C; that is, 'opposite' vertices. As illustrated in Figure 34(a) any two such chords induce an $K_{2,3}$ subgraph. Thus the number of chords of C is at most one.

Suppose C has one chord. Then there are three edges vu, vw and vx in G incident to v. If two of u, w and x are adjacent in C then there is a 3-cycle in G, as illustrated in Figure 35(c). Otherwise, one of u, w or x is incident to the chord of C, and hence G contains $K_{2,3}$, as illustrated in Figure 35(d).

If C has no chords then v is adjacent to four vertices u, w, x and y in C. Two of u, w, x and y are adjacent in C; thus G contains a 3-cycle, as illustrated in Figure 35(c).

Case 3. $|C| = 5$: Suppose the vertices not in C are v and w. Any chord of C induces a 3-cycle, as illustrated in Figure 33(a). Thus C has no chords, and there are five edges incident to v and w. If vw is an edge of G and each of v and w are incident to two edges then G contains a 3-cycle or $K_{2,3}$, as illustrated in Figure 35(e). Otherwise at least one of v and w, say v, is adjacent to three vertices u, x and y in C. Two of u, x and y are adjacent in C. Thus G contains a 3-cycle, as illustrated in Figure 34(b).

Case 4. $|C| = 4$: Suppose the vertices not on C are u, v and w. Any chord of C induces a 3-cycle, as illustrated in Figure 33(b). Thus C has no chords, and there are at least six edges incident to u, v and w. Since u, v and w do not form a 3-cycle, there are at least four edges between u, v or w and vertices in C. Hence at least one of u, v and w, say v, is adjacent to two vertices x and y in C. So that v, x and y do not form a 3-cycle, x and y are not adjacent. In this case, G contains $K_{2,3}$, as illustrated in Figure 33(d). □

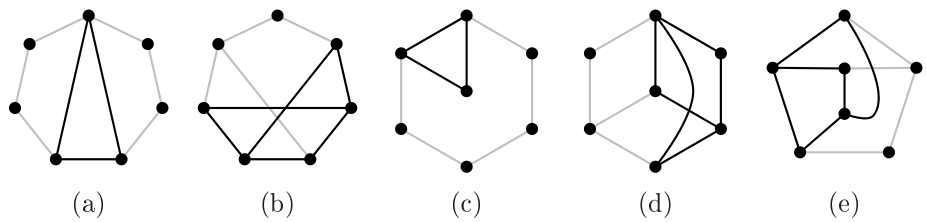


Figure 35: C_3 or $K_{2,3}$ in a 10-edge 7-vertex graph.

Lemma 16. *Every 7-vertex graph with at least eight edges contains a cycle C_k ($k \neq 4$), two chord-disjoint cycles, or $K_{2,3}$.*

Proof. Let G be a 7-vertex graph with at least eight edges. Let C be the longest cycle in G; thus $3 \leq |C| \leq 7$. If $|C| \neq 4$ then we are done, otherwise $|C| = 4$. If C has a chordal path then G either contains $K_{2,3}$ or a cycle C_k ($k \neq 4$), as illustrated in Figure 36(a) and Figure 36(b). Thus we now assume that C has no chordal path.

There are at least four edges not in C. Let X be the subgraph of G induced by the vertices not in C. Then X has at least three vertices, and the number of edges in X is at most three. We proceed by considering the number of edges in X.

Case 1. $|E(X)| = 3$: Then X is a 3-cycle, as illustrated in Figure 36(c), and we are done.

Case 2. $|E(X)| = 2$: If there are two edges in X then X is connected and there are at least two edges e_1 and e_2 between X and C. Since X is connected, e_1 and e_2 have the same end-vertex in C for C not to have a chordal path. In this case, e_1 and e_2 along with one or two of the edges in X form a cycle which is chord-disjoint from C, as illustrated in Figure 36(d).

Case 3. $|E(X)| = 1$: If there is one edge in X then there are at least three edges between X and C. If one of the vertices in X is incident to at least two edges between X and C then C has a chordal-path. Thus every vertex in X is incident to at most one edge between X and C. Since X has three vertices and there are at least three edges between X and C, each vertex in X is incident to exactly one edge between X and C. Let vw be the edge in X. For C not to have a chordal path, v and w are incident to the same vertex in C, in which case G contains a 3-cycle, as illustrated in Figure 36(d).

Case 4. $|E(X)| = 0$: If there are no edges in X then there are at least four edges between X and C. Thus one of the vertices in X is incident to at least two edges between X and C, in which case C has a chordal-path. □

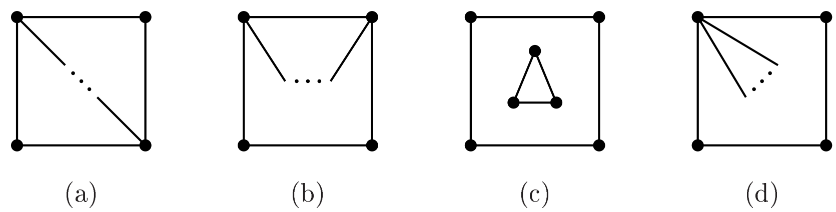


Figure 36: C_k ($k \neq 4$), $K_{2,3}$ or two chord-disjoint cycles in a 7-vertex 8-edge graph.

B Proofs of Connectivity

In this section we prove the connectivity of the graphs used to establish our main lower bounds. We first prove that the 'grid-graphs' have the desired connectivity. Of course C_r with $r \geq 3$ is 2-connected.

Observation 7. *$C_r \times K_2$ with $r \geq 3$ is 3-connected.*

Proof. Let v and w be distinct vertices of $C_r \times K_2$. As illustrated in Figure 37, if v and w are (a) in the same 'row', (b) in the same 'column', or (c) 'non-collinear', there are three internally disjoint paths between v and w in $C_r \times K_2$. Since $r \geq 3$, in case (c) we can assume that v and w are at least two columns apart. By Menger's Theorem, $C_r \times K_2$ is 3-connected. □

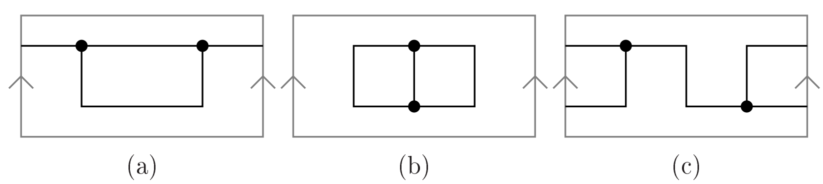


Figure 37: Three disjoint paths in $C_r \times K_2$.

Observation 8. *$C_r \times C_3$ with $r \geq 3$ is 4-connected.*

Proof. Let v and w be distinct vertices of $C_r \times C_3$. As illustrated in Figure 38, if v and w are (a) in the same 'row', (b) in the same 'column', or (c) 'non-collinear', there are four internally disjoint paths between v and w in $C_r \times C_3$. Since $r \geq 3$, in case (c) we can assume that v and w are one row apart and at least two columns apart. By Menger's Theorem, $C_r \times C_3$ is 4-connected. □

Observation 9. *$C_r \times C_3 \times K_2$ with $r \geq 3$ is 5-connected.*

Proof. Let v and w be distinct vertices of $C_r \times C_3 \times K_2$. As illustrated in Figure 39, if v and w (a) are in the same 'row', (b) are in the same 'column', or

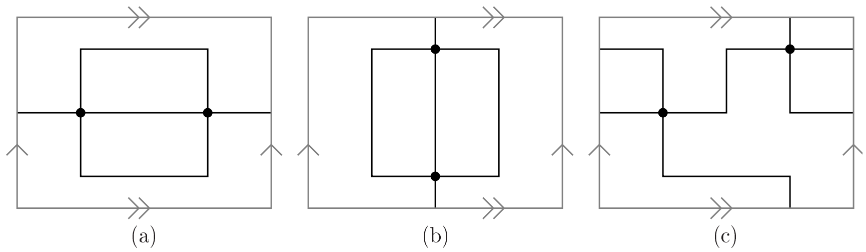


Figure 38: Four internally disjoint paths in $C_r \times C_3$.

(c) have the same 'depth', (d) have the same 'height', or (e) are 'non-coplanar', there are five internally disjoint paths between v and w in $C_r \times C_3 \times K_2$. Since $r \geq 3$, in cases (c), (d) and (e) we can assume that v and w are at least two columns apart. By Menger's Theorem, $C_r \times C_3 \times K_2$ is 5-connected. □

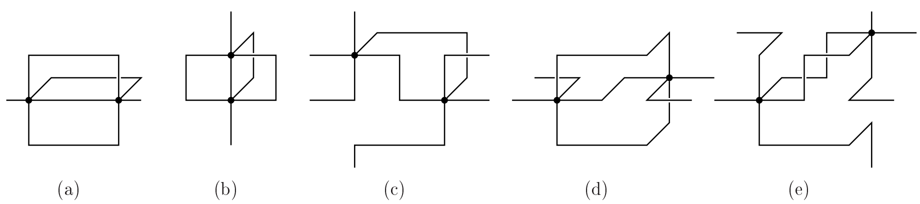


Figure 39: Five internally disjoint paths in $C_r \times C_3 \times K_2$.

Observation 10. *$C_r \times C_3 \times C_3$ with $r \geq 3$ is 6-connected.*

Proof. Let v and w be distinct vertices of $C_r \times C_3 \times C_3$. As illustrated in Figure 40, if v and w are (a) 'collinear', (b) 'coplanar' or (c) 'non-coplanar', there are six internally disjoint paths between v and w in $C_r \times C_3 \times C_3$. Since $r \geq 3$, in cases (b) and (c) we can assume that v and w are at least two columns apart. By symmetry these three cases suffice. By Menger's Theorem, $C_r \times C_3 \times C_3$ is 6-connected. □

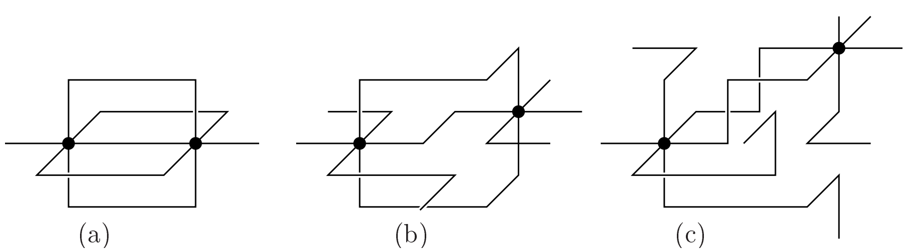


Figure 40: Six internally disjoint paths in $C_r \times C_3 \times C_3$.

The next lemma combined with the above observations proves that $C_r\langle K_p\rangle$ for $p \geq 2$ is 2-connected, that $(C_r \times K_2)\langle K_p\rangle$ for $p \geq 3$ is 3-connected, that $(C_r \times C_3)\langle K_p\rangle$ for $p \geq 4$ is 4-connected, that $(C_r \times C_3 \times K_2)\langle K_p\rangle$ for $p \geq 5$ is 5-connected, and that $(C_r \times C_3 \times C_3)\langle K_p\rangle$ for $p \geq 6$ is 6-connected.

Lemma 17. *If a graph G is c-connected for some $c \geq 2$, then $G\langle K_p\rangle$ is c-connected for all $p \geq c$.*

Proof. In each K_p subgraph H, those vertices of H adjacent to vertices not in H are called *exit vertices*. Every vertex v of H is adjacent to every exit vertex of H (except for itself). Hence there are c internally disjoint paths between any two vertices of $G\langle K_p\rangle$ that are in distinct K_p subgraphs, since G itself is c-connected. Consider vertices v and w in the same K_p subgraph H whose original vertex in G is u.

Suppose v and w are both exit vertices. Then there are $c-2$ internally disjoint vw-paths via the other exit vertices of H. Let x and y be the original vertices of G such that there are edges incident to v and w whose other end-vertices are in the subgraphs corresponding to x and y, respectively. Since G is 2-connected, there is an xy-path in G which avoids u. This path and the edge vw gives a total of c internally disjoint vw-paths.

Now suppose one of v and w, say v, is not an exit vertex. Then there are at least $c-1$ internally disjoint vw-paths via the other exit vertices. Along with the edge vw, there are at least c disjoint vw-paths.

By Menger's Theorem, $G\langle K_p\rangle$ is c-connected. □

Lemma 18. *[(a)]*

$C_r\langle K_p \setminus M_1\rangle$ is 2-connected for all $p \geq 4$,

2. *$(C_r \times K_2)\langle K_p \setminus M_2\rangle$ is 3-connected for all $p \geq 5$,*
3. *$(C_r \times C_3)\langle K_p \setminus M_2\rangle$ is 4-connected for all $p \geq 5$,*
4. *$(C_r \times C_3 \times K_2)\langle K_p \setminus M_3\rangle$ is 5-connected for all $p \geq 7$, and*
5. *$(C_r \times C_3 \times C_3)\langle K_p \setminus M_3\rangle$ is 6-connected for all $p \geq 7$.*

Proof. Each part of the lemma states that a graph of the form $G\langle K_p \setminus M_j\rangle$ is c-connected, where by the preceding observations, G is a c-connected graph, $j = \lceil \frac{c}{2} \rceil$, and $p \geq 2j+1$. Observe that $p \geq c+1$. Moreover, in each $K_p \setminus M_j$ subgraph H, if c is even then it is precisely the exit vertices in H that are matched in M_j, and for odd c, all but one of the exit vertices are matched to each other in M_j, and the one remaining exit vertex is matched with one of the (at least two) non-exit vertices. By the same argument used in Lemma 17, for any two exit vertices v and w of a $K_p \setminus M_j$ subgraph H, there is a vw-path in $G\langle K_p \setminus M_j\rangle$ not using any edges in H.

Let v be a vertex of $G\langle K_p \setminus M_j\rangle$ contained in a $K_p \setminus M_j$ subgraph H. We claim that there are c internally disjoint (possibly empty) paths from v to the exit vertices of H. If v is an exit vertex then there are at least $c-1$ other exit vertices in H, none of which are matched with v. Hence v is adjacent to each such exit vertex, and counting the empty path from v to v, the claim holds. Now suppose v is not an exit vertex. If v is adjacent to each exit vertex, which is guaranteed in the case of even c, then the claim holds. Otherwise c is odd, and v is matched with one of the exit vertices x. In this case v is adjacent to the $c-1$ remaining exit vertices, and there is a 2-path from v to x via the other non-exit vertex in H, giving a total of c internally disjoint paths from v to the exit vertices of H. This proves our claim. It follows since G is c-connected that between any two vertices in distinct $K_p \setminus M_j$ subgraphs, there are c internally disjoint paths.

Now consider two vertices v and w contained in the same $K_p \setminus M_j$ subgraph H. First suppose v and w are both exit vertices. If v and w are matched then there are $c-2$ internally disjoint vw-paths via the other exit vertices, there is at least one path between v and w via the non-exit vertices of H, and there is a path between v and w not using the edges of H. Thus there are c internally disjoint vw-paths in $G\langle K_p \setminus M_j\rangle$. Now suppose v and w are non-matched exit vertices of H. Thus $c \geq 4$. Suppose $c \in \{4, 6\}$. Let vx and wy be in M_j. By construction, v is opposite to x, and w is opposite to y (with respect to the grid-structure of G). Thus there exists a vx-path P disjoint from some wy-path Q in $G\langle K_p \setminus M_j\rangle$, not using any edges in H. Hence $P \cup \{xw\}$ and $Q \cup \{vy\}$ are internally disjoint vw-paths. There are $c-4$ internally disjoint vw-paths via the other exit vertices of H. There is one vw-path via the non-exit vertex of H, and there is the edge vw, giving a total of c internally disjoint vw-paths in $G\langle K_p \setminus M_j\rangle$. Now suppose $c = 5$. Either both of v and w are matched to other exit vertices, or one of v and w is matched with an exit vertex and the other is matched with a non-exit vertex. First suppose that v is matched with an exit-vertex x and w is matched with an exit vertex y. There are $c-4$ internally disjoint vw-paths via the other exit vertices, there is the edge vw, there are two vw-paths via the two non-exit vertices, and there is the path v–y–x–w, giving a total of c internally disjoint vw-paths. Now suppose v is matched with an exit vertex x, but w is matched with a non-exit vertex y. There is a vx-path P not using any edges in H. There are $c-3$ vw-paths via the other exit vertices, there is the edge vw, there is one vw-path via the one remaining non-exit vertex, and $P \cup \{xw\}$ forms a vw-path, giving a total of c internally disjoint vw-paths.

Now consider two vertices v and w contained in the same $K_p \setminus M_j$ subgraph H, where v is an exit vertex and w is not an exit vertex. First suppose v and w are matched, in which case c is odd. There are $c-1$ vw-paths via the other exit vertices, and there is a vw-path via the other non-exit vertex of H, giving a total of c internally disjoint vw-paths. Now suppose v and w are not matched. Let x be the vertex matched with v. Suppose x is an exit vertex. There is a vx-path P not using any edge in H, and thus $P \cup \{wx\}$ forms a vw-path. There are $c-2$ vw-paths via the other exit vertices of H, and there is the edge vw, giving a total of c internally disjoint vw-paths. If x is not an exit vertex, then there are $c-1$ vw-paths via the other exit vertices, and the edge vw gives a total of c internally disjoint vw-paths.

The final case is when v and w are both not exit vertices contained in the same $K_p \setminus M_j$ subgraph H. At most one of v and w is matched with an exit vertex. Thus there are at least $c-1$ vw-paths via the remaining exit vertices, which along with the edge vw, give c internally disjoint vw-paths.

Thus for every pair of vertices v and w of $G\langle K_p \setminus M_j\rangle$, there are c internally disjoint vw-paths. By Menger's Theorem, $G\langle K_p \setminus M_j\rangle$ is c-connected. □

The multigraphs and pseudo graphs constructed in Section 4 have the claimed connectivity, since each contains a simple subgraph that is proved in Lemmata 17 and 18 to have the same connectivity.

C Final Observation

Lemma 8 states that if a graph G has at least k bends in every general position drawing then for any edge e of G the graph $G \setminus e$ has at least $k-4$ bends in every drawing. We now prove the analogue of this result for arbitrary (non general position) drawings.

Lemma 19. *If a graph G has at least k bends in every drawing then for any edge e of G the graph $G \setminus e$ has at least $k-6$ bends in every drawing.*

Proof. Suppose there is a drawing of $G \setminus e$ with b bends. Let $e = vw$. At each of v and w there is an unused port. Regardless of the relative directions of the unused ports at v and w, by inserting at most two planes at each of v and w, we can route e with at most six bends and entirely within the inserted planes. Hence the edge route for e does not intersect any existing edge routes in the drawing of $G \setminus e$. In Figure 41 we illustrate such an edge routing in the worst case scenario with v and w non-coplanar and the unused ports at v and w pointing away from each other. Note that in many other cases less than six bends are needed. Hence G has a drawing with $b+6$ bends. By assumption, every drawing of G has at least k bends. Thus $b+6 \geq k$ and $b \geq k-6$. □

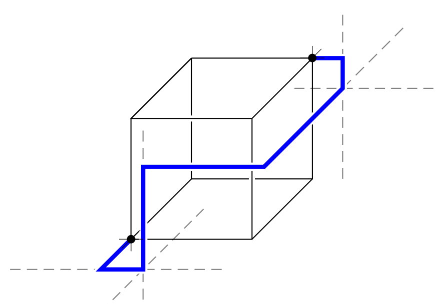


Figure 41: Inserting a 6-bend edge.

Note that this technique can also be used to provide an upper bound on the maximum number of bends per edge route in a given drawing. For example, the

REDUCE FORKS algorithm of Di Battista *et al.* [11] does not provide a bound on the maximum number of bends per edge, and in many instances, edges are routed with more than six bends [11, 33]. By replacing each edge route with more than six bends by an edge route with at most six bends, as described in the proof of Lemma 19, the algorithm can be modified to produce drawings with an upper bound on the maximum number of bends per edge. Of course, doing so may increase the volume of the drawing.