A NOTE ON COLOURING THE PLANE GRID*

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Let n be a positive integer. The $n \times n$ *grid* is the set of points in the plane $\{(x, y) : 1 \leq x, y \leq n\}$. Let $k(n)$ denote the minimum number of colours in a colouring of the points of the $n \times n$ grid such that no three collinear points are monochromatic. The determination of $k(n)$ is a natural generalisation of the *no-three-in-line* problem [1–10], which asks for the maximum number of points in the $n \times n$ grid with no three points collinear. Since no three points in a single row or column can receive the same colour, $k(n) \geq \lceil \frac{n}{2} \rceil$. By the example shown in Figure 1, $k(4) = 2$.

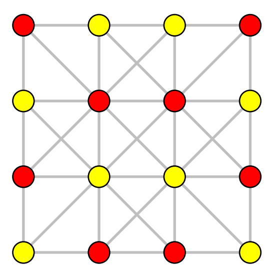

Figure 1: A 2-colouring of the 4×4 grid with no three collinear monochromatic points.

Theorem 1. *Let n be a positive integer. Let p be the minimum prime such that $p \geq n$. Then $k(n) \leq n + p - 1 \leq 3n - 2$.*

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Proof. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0 \ .$$

Let $V_i = \{(x, (x^2 \bmod p) + i) : 1 \leq x \leq n\}$ for each integer i. For all distinct $1 \leq x_1, x_2, x_3 \leq n$,

$$\begin{aligned} &\begin{vmatrix} 1 & x_1 & (x_1^2 \bmod p) + i \\ 1 & x_2 & (x_2^2 \bmod p) + i \\ 1 & x_3 & (x_3^2 \bmod p) + i \end{vmatrix} \\ \equiv &\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} \\ \equiv &(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \pmod{p} \ , \end{aligned}$$

which is nonzero since p is a prime and the x_i's are distinct modulo p. Hence no three points in each V_i are collinear. (This construction of points whose test determinant is congruent to a Vandermonde determinant is due to Erdős [5].)

Clearly $V_{i_1} \cap V_{i_2} = \emptyset$ for distinct i_1 and i_2. Each point (x, y) in the $n \times n$ grid is in V_i where $i = y - (x^2 \bmod p)$. Since $2 - p \leq y - (x^2 \bmod p) \leq n$, the set $\{V_i : 2 - p \leq i \leq n\}$ contains a partition of the points into $n + p - 1$ colour classes such that no three collinear points are monochromatic. By Bertrand's postulate $p \leq 2n - 1$, and the number of colours is at most $3n - 2$. □

Note that $k(n) \leq 3n - 6$ for $n \geq 3$ follows from the stronger form of Bertrand's postulate and the construction in Figure 1 for $n \in \{3, 4\}$. By Theorem 1 and the prime number theorem we have:

Theorem 2. *For all $\epsilon > 0$, there exists N_ϵ such that $k(n) \leq (2 + \epsilon)n$ for all $n > N_\epsilon$.* □

We conclude with the following questions:

1. What is the minimum constant c such that $k(n) \leq cn$ for all n? We know that $\frac{1}{2} \leq c \leq 3$.

2. What is the minimum constant c such that for all $\epsilon > 0$, there exists N_ϵ such that $k(n) \leq (c + \epsilon)n$ for all $n > N_\epsilon$? We know that $\frac{1}{2} \leq c \leq 2$.

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