Multi-dimensional Orthogonal Graph Drawing with Small Boxes

Extended Abstract

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Abstract. In this paper we investigate the general position model for the drawing of arbitrary degree graphs in the D-dimensional ($D \geq 2$) orthogonal grid. In this model no two vertices lie in the same grid hyperplane. We show that for $D \geq 3$, given an arbitrary layout and initial edge routing a crossing-free orthogonal drawing can be determined.
We distinguish two types of algorithms. Our *layout-based* algorithm, given an arbitrary fixed layout, determines a degree-restricted orthogonal drawing with each vertex having aspect ratio two. Using a *balanced* layout this algorithm establishes improved bounds on the size of vertices for 2-D and 3-D drawings. Our *routing-based* algorithm produces 2-degree-restricted 3-D orthogonal drawings.
One advantage of our approach in 3-D is that edges are typically routed on each face of a vertex; hence the produced drawings are more truly three-dimensional than those produced by some existing algorithms.

1 Introduction

In this paper we consider orthogonal drawings of an n-vertex m-edge simple graph $G = (V, E)$ with maximum degree Δ. The *D-dimensional orthogonal grid* ($D \geq 2$) is the D-dimensional cubic lattice, consisting of *grid points* with integer coordinates, together with the coordinate-axis-parallel *grid lines* determined by these points. An integer i, $1 \leq i \leq D$, is called a *dimension*, and an integer d, $1 \leq |d| \leq D$, is called a *direction* (with the obvious interpretation).

An *orthogonal drawing* of G represents vertices $v \in V$ by pairwise non-intersecting *boxes*; i.e. sets $\{(a_1, a_2, \ldots, a_D) : l_i(v) \leq a_i \leq r_i(v), 1 \leq i \leq D\}$ for some closed integer intervals $[l_i(v), r_i(v)]$, $1 \leq i \leq D$. The graph-theoretic term 'vertex' will also refer to the corresponding box[1]. The size of a vertex v in a D-dimensional orthogonal drawing is denoted by $\alpha_1(v) \times \cdots \times \alpha_D(v)$ where $\alpha_i(v) = r_i(v) - l_i(v) + 1$.

[1] Vertices are possibly degenerate; this is the approach taken in [3, 4, 23], but not in [18]; enlarging vertices to remove this degeneracy increases the volume by a multiplicative constant.

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For each direction d we call the face of a box extremal in direction d the *d-face*. At each grid point on the d-face of a box there is a *port* in direction d. The number of ports on a box will be called its *surface*, and we shall refer to the number of grid points in a box as its *volume* (and *area* in two dimensions).

Each edge is represented by a sequence of contiguous segments of grid lines called an *edge route* possibly bent at grid points, and only intersecting its incident vertices. Edge routes are pairwise non-overlapping, and only in the 2-D orthogonal grid are edge routes allowed to cross. An orthogonal drawing with no more than b bends per edge route is called a *b-bend orthogonal drawing.* The directed graph with vertex set V consisting of the *reversal* arcs vw, wv for each edge $vw \in E$ is denoted by $G' = (V, A(G))$. An orthogonal drawing of G assigns each arc $vw \in A(G)$ a unique port at v.

An orthogonal drawing with a particular shape of box representing every vertex, e.g. point, line, square, box or cube, will be called an orthogonal *shape*-drawing for each particular shape, as illustrated in Fig. 1.

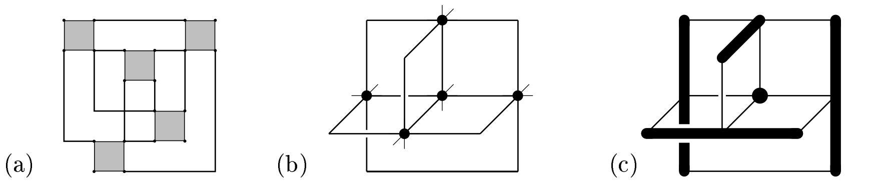

Fig. 1. Orthogonal drawings of K_5: (a) 1-bend 2-D square-drawing, (a) 2-bend 3-D point-drawing, (b) 0-bend 3-D line-drawing.

Orthogonal point-drawings have been studied extensively in two dimensions (see [7]) and to a lesser extent in three dimensions [10, 11, 18, 21]. However, D-dimensional point-drawings can only exist for graphs with maximum degree at most $2D$. Overcoming this restriction has motivated recent interest in 2-D box-drawings [1, 2, 6, 9, 12, 14, 15, 13, 17] and in 3-D box-drawings [3, 4, 18, 23]. In this paper we shall present unified results for orthogonal graph drawing in two and more dimensions.

The smallest D-dimensional box surrounding a D-dimensional orthogonal drawing is called the *bounding box.* The bounding box volume and the maximum number of bends per edge route are the most commonly proposed measures for determining the aesthetic quality of an orthogonal drawing. For box-drawings the shape and size of a vertex with respect to its degree are also considered an important measure of aesthetic quality. A vertex v is said to be *a-degree-restricted* if the surface$(v) \leq a \times \deg(v) + O(1)$. If for some constants a and d independent of the input graph, every vertex v with $\deg(v) \geq d$ is a-degree-restricted, then we say the drawing is *a-degree-restricted.* The *aspect ratio* of v is $\max_i \alpha_i(v)/ \min_i \alpha_i(v)$. Degree-restricted orthogonal drawings with bounded aspect ratio are considered aesthetically pleasing.

Algorithms for 2-D orthogonal graph drawing which follow the so-called *Topology-Shape-Metrics* approach include the algorithms of Fößmeier, Kant and Kaufmann [13, 14, 15] in the *Kadinsky* model, the GIOTTO algorithm [8], and the recent algorithm of Di Battista et al. [6] which allows the user to specify vertex sizes. This approach based on network flow techniques can be traced to the classical algorithm of Tamassia [19] for finding a bend-minimum orthogonal point-drawing which preserves a fixed planar embedding.

Algorithms which do not guarantee crossing-free drawings, even for planar graphs, include that of Even and Granot [12], Papakostas and Tollis [17] and Biedl and Kaufmann [1]. The latter two algorithms both produce 2-degree-restricted 2-D orthogonal drawings with each vertex v having aspect ratio at most $\deg(v)/2$. At the expense of an increase in area, bounded aspect ratio drawings are produced by a second algorithm in [1] and by the *layout-based* algorithm presented in this paper, which also improves the degree restriction bound to 3/2. Using a diagonal layout our algorithm produces 2-degree-restricted square-drawings. Table 1 summarizes bounds for 2-D orthogonal graph drawing.

Table 1. Upper Bounds for 2-D Orthogonal Graph Drawing

area	max bends	degree restriction	aspect ratio	reference
$(m-1) \times (\frac{m+1}{2})$	1	2	$\deg(v)/2$	[17]
$(\frac{m+n}{2}) \times (\frac{m+n}{2})$	1	2	$\deg(v)/2$	[1]
$(\frac{3m+2n}{4}) \times (\frac{3m+2n}{4})$	1	2	2	[1]
$(\frac{3m+4n+2}{4}) \times (\frac{3m+4n+2}{4})$	1	3/2	2	Theorem 6
$(\frac{3(m+n)}{4}) \times (\frac{3(m+n)}{4})$	1	2	1	Theorem 8

The trade-off between the maximum number of bends per edge route and the bounding box volume apparent in 3-D point-drawing algorithms [10, 11], is seen to a lesser extent in 3-D orthogonal box-drawings. Biedl et al. [3] constructs 1-, 2- and 3-bend 3-D orthogonal drawings of K_n with respective bounding box volumes $O(n^3)$, $O(n^3)$ and $O\left(n^{5/2}\right)$. However for arbitrary graphs the drawings are not necessarily degree-restricted.

Papakostas and Tollis [18] first established that every graph has a degree-restricted 3-D orthogonal drawing. Their bounding box volume upper bound has subsequently been improved to $O(n^3)$ by the *lifting half-edges* line-drawing algorithm of Biedl [4]. At the cost of an increase in volume, a modified technique is used to produce cube-drawings. Biedl also presents algorithms for 3-D line- and cube-drawings in general position, which is the model employed in this paper.

In [23] an algorithm which generalizes the COMPACT point-drawing algorithm of Eades et al. [10, 11] produces 6-bend 3-D orthogonal line-drawings of multigraphs with bounding box volume $O(m^2/\sqrt{n})$, which is the best known bounding box volume upper bound for graphs with $O(n^{7/4})$ edges.

Our *layout-based* algorithm improves the best known bound for the degree-restriction of vertices in bounded aspect ratio 3-D orthogonal drawings. Using a diagonal vertex layout the algorithm produces cube-drawings. Our *routing-based* algorithm produces 2-degree-restricted orthogonal drawings. Table 2 summarizes the bounds for 3-D orthogonal graph drawings.

Table 2. Upper Bounds for 3-D Degree-Restricted Orthogonal Drawings

max bends	volume	degree restriction	aspect ratio	method
2	$O(m^3)$	6	?	incremental [18]
2	$O(n^3)$	2	$\deg(v)/2$	lifting $\frac{1}{2}$-edges [4]
2	$O(n^2m)$	6	1	lifting $\frac{1}{2}$-edges [4]
2	$O(n^2m)$	2	$\deg(v)/2$	routing-based [4]
2	$O\big((nm)^{3/2}\big)$	6	1	routing-based [4]
5	$O(m^2)$	2	$\deg(v)/2$	compact [23]
6	$O(m^2/\sqrt{n})$	2	$\deg(v)/2$	compact [23]
2	$O\big((nm)^{3/2}\big)$	10/3	2	layout-based (Theorem 7)
2	$O\big((nm)^{3/2}\big)$	4	1	layout-based (Theorem 8)
2	$O(n^2m)$	2	$\deg(v)/2$	layout-based (Theorem 9)
2	$O(\Delta(nm)^{3/2})$	2	$\deg(v)/4$	routing-based (Theorem 10)

We shall present our algorithms in the following three stages (closely related to the so-called *three-phase method* [1, 2, 4]):

Vertex Layout: Determine the relative positions of the vertices.
Edge Routing: Determine the shape of each edge route.
Port Assignment: Construct vertex boxes and assign ports to arcs.

In Sec. 2 of this paper we introduce the general position model for orthogonal drawing and describe in detail the 'Port Assignment' stage of our algorithm. Our vertex layout methods are presented in Sec. 3. We distinguish two types of algorithms for the drawing of graphs in the general position model. Our *layout-based* method, described in Sec. 4, determines the edge routing with respect to a given vertex layout. This is the first algorithm for constructing orthogonal drawings in 2, 3 or more dimensions. We establish results for fixed, balanced and diagonal layouts. In Sec. 5 we discuss *routing-based* algorithms, where a layout is determined with respect to a given routing. See [22] for details of omitted proofs.

In a *vertex ordering* $(v_1, v_2, \ldots, v_n)$ of G, if $v_i v_j \in E$ $(i < j)$ we say v_j is a *successor* of v_i and v_i is a *predecessor* of v_j. The number of predecessors and successors of v_i are denoted $p(v_i)$ and $s(v_i)$ respectively. For directed graphs we only count the outgoing edges at v_i in $p(v_i)$ and $s(v_i)$.

2 The General Position Model

A D-dimensional orthogonal drawing is in *general position* if no grid hyperplane intersects any two vertices. This model has been used for 2-D orthogonal box-drawing in [1, 17], for 3-D point-drawing in [10, 11, 21] and for 3-D box-drawing in [4, 18, 22]. It is particularly useful in $D \geq 3$ dimensions since, as we shall prove, edge route intersections can always be eliminated.

The relative coordinates of the vertices in a general position D-dimensional orthogonal drawing of G are represented by D vertex orderings, called a *layout* of G. We write $v <_i w$, $1 \leq i \leq D$, if v is before w in the i-ordering. Suppose $V_i(v) = \{w : w <_i v\}$. Then v has a minimum i-coordinate of 0 if $V_i(v) = \emptyset$, and of $\sum_{w \in V_i(v)} \alpha_i(w)$ otherwise. We denote the number of successor and predecessors of v in the i-ordering by $s_i(v)$ and $p_i(v)$ respectively.

The assignment of ports to arcs is represented by a colouring of $A(G)$ with colours $\{1, 2, \ldots, D\}$ (or $\{X, Y\}$ and $\{X, Y, Z\}$ in 2 and 3 dimensions). An arc vw coloured i is assigned a port on the $(+i)$-face of v if $v <_i w$ and on the $(-i)$-face of v if $w <_i v$. The maximum of the number of arcs routed on the $(+i)$-face and on the $(-i)$-face of a vertex v is denoted $N_i(v)$.

All edge routes used by our algorithm have precisely $D-1$ bends, and thus for each edge vw, the ports assigned to the arcs vw and wv must be perpendicular; i.e. vw and wv are coloured differently. A colouring of $A(G)$ with this property is called a *routing* of G. Suppose the arcs vw and wv are respectively coloured i and j $(i < j)$. A $(D-1)$-bend edge route vw consists of consecutive grid-line segments between hyperplanes unique to v and w, respectively parallel to the following sequence of dimensions: $i \to (i-1) \to \ldots \to 1 \to (i+1) \to (i+2) \to \ldots \to (j-1) \to D \to (D-1) \to \ldots \to j$. We have the following upper bound for the volume of the bounding box.

Theorem 1. *A d-degree-restricted D-dimensional general position orthogonal drawing with each vertex having aspect ratio a has bounding box volume at most*

$$a \left(n^{D-2} \left(\frac{d}{D} m + \frac{O(1)}{2D} n \right) \right)^{D/(D-1)}$$

2.1 Determining Vertex Size

For each vertex v, we wish to determine positive integers $\alpha_i(v)$, $1 \leq i \leq D$, to minimize the surface(v); i.e.

$$\text{minimize} \sum_{i=1}^{D} \left(\prod_{\substack{1 \leq j \leq D \\ j \neq i}} \alpha_j(v) \right) \text{ such that } \forall i \prod_{\substack{1 \leq j \leq D \\ j \neq i}} \alpha_j(v) \geq N_i(v). \qquad (1)$$

A solution to (1) has surface$(v) \geq 2\sum_i N_i(v)$. We define κ_D to be the minimum k such that for every D-dimensional vertex v there is a solution to (1) with surface$(v) \leq k(2\sum_i N_i(v)) + O(1)$. A real-valued solution to (1) is given by

$$r_i(v) = \left(\left(\prod_{\substack{1 \leq j \leq D \\ j \neq i}} N_j(v) \right) \Big/ \, N_i(v)^{D-2} \right)^{1/(D-1)} .$$

Lemma 1. *(a)* $\kappa_2 = 1$*, (b)* $\kappa_3 \leq 2$ *and (c) if* $r_i(v) \geq 1$ *for all* i*,* $1 \leq i \leq D$*, then* $\kappa_D < 2^D$*.*

Proof. (Outline) For $D = 2$ (1) is trivial: simply set $\alpha_X(v) = N_Y(v)$ and $\alpha_Y(v) = N_X(v)$. An integer-valued solution can obviously be obtained by setting $\alpha_i(v) = \lceil r_i(v) \rceil$. In the case of $D = 3$, if $r_i(v) \geq 2$ for all i, $1 \leq i \leq 3$, then this method determines a solution with $\kappa_3 \leq 2$. If for some i, $r_i(v) < 2$ then a case-by-case analysis establishes there is a solution with $\kappa_3 \leq 2$. For $D \geq 3$, if $r_i(v) \geq 1$ then $\alpha_i(v) < 2r_i(v)$ and $\kappa_D < 2^D$.

2.2 Port Assignment

For each face of a vertex we group the edges to be routed on this face according to the direction of their second segment. By the edge routing described in Sect. 2 there are four possible directions for the second segment. For each of the four groupings, we assign sufficiently many ports so that corresponding edges within different groups cannot intersect (see [22] for details). Within a grouping, ports are assigned to arcs vw in increasing order of the length of the first segment of the edge route from v to w, as illustrated in Fig. 2 (see [22] for details). on

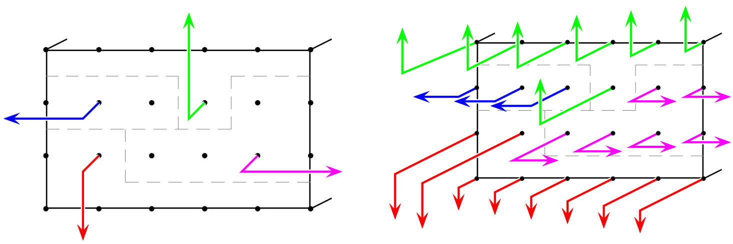

Fig. 2. Determining port assignments on a face.

Since a grid point on an edge route vw has at most one coordinate not unique to v or w, edge routes can only intersect if they are incident to a common vertex. For directions i and j, an (i,j)-*section* at a vertex v consists of the arcs vw where the first and second segments of the edge vw are in directions i and j and in the same ij-hyperplane. An edge vw is in exactly one section at v and one section at

w. Edge route intersections can only occur between edges in the same section. To eliminate edge crossings within an (i,j)-section (assume $i, j > 0$ – the other cases are easily inferred), choose the unrerouted arc vw such that w has maximum i coordinate (alternately with maximum j coordinate), and assign to vw the unassigned i-port in the section with minimum j-coordinate (i-coordinate), as illustrated in Fig. 3.

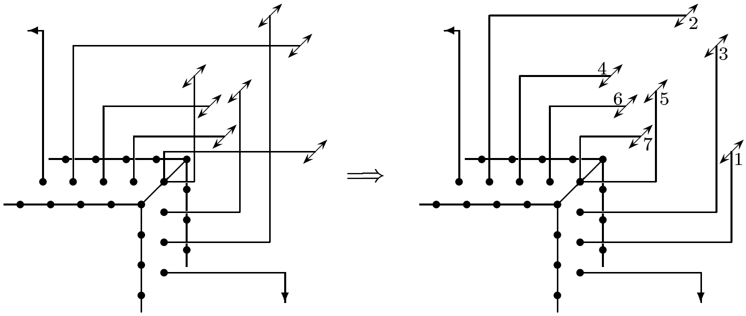

Fig. 3. Rerouting intersecting edge routes within a section in the order shown.

3 Balanced Graph Layout

For a given vertex ordering $(v_1, v_2, \ldots, v_n)$ of G, we say a vertex v is *positive* if $s(v) > p(v)$, *negative* if $p(v) > s(v)$ and *balanced* if $s(v) = p(v)$. For positive vertices v and for $k > 0$ (respectively, $k < 0$) v^k denotes the k^{th} successor (predecessor) of v to the right (left) of v in the ordering. For negative v and for $k > 0$ (respectively $k < 0$) v^k denotes the k^{th} predecessor (successor) of v to the left (right) of v in the ordering. Two adjacent vertices v_i and v_j $(i < j)$ are *opposite* if v_i is positive and v_j is negative. A vertex v has *cost* $c(v) = |s(v) - p(v)|$. We conjecture that determining a vertex ordering of a given graph with minimum total cost, the *balanced ordering problem*, is NP-hard.

We say a vertex ordering is *locally balanced* if moving any one vertex within the ordering does not reduce the total cost.

Lemma 2. *For every vertex v in a locally balanced vertex ordering, each vertex v^i, $1 \leq i \leq \lfloor c(v)/2 \rfloor$, is not opposite to v, as otherwise v could move past v^i and reduce the total cost.*

The following *median placement* heuristic for the balanced ordering problem will form the basis of our graph layout methods: Given a vertex ordering $(v_1, v_2, \ldots, v_n)$ of G, called the *insertion ordering*, for $i = 1, 2, \ldots, n$, insert v_i into the *current ordering* mid-way between its already inserted neighbours, i.e. between the predecessors of v_i in the insertion ordering.

Theorem 2. *The median placement algorithm determines in $O(m + n)$ time a vertex ordering of an undirected graph G with total cost at most $m + n$.*

Theorem 3. *For an acyclic (di)graph G, using a reverse topological ordering of G as the insertion ordering in the median placement heuristic, determines a vertex ordering of G with minimum total cost in $O(m+n)$ time.*

In a D-dimensional general position layout we define the cost of v to be the average cost of v over the D orderings. The following algorithm, based on a technique of [1], determines a 2-D *balanced* layout of a given graph: Arbitrarily order the vertices $(v_1, v_2, \dots, v_n)$. Determine the X- and Y-orderings using the median placement heuristic with insertion orderings $(v_1, v_2, \dots, v_n)$ and $(v_n, v_{n-1}, \dots, v_1)$ respectively. To determine a D-dimensional *balanced* layout calculate a 2-D balanced layout and set the i-ordering, $1 \leq i \leq D$, equal to the X/Y-ordering for odd/even i.

Theorem 4. *The above algorithm determines a D-dimensional general position layout in $O(D(m+n))$ time such that for each vertex v,*

$$c(v) \leq 1 + \frac{\lceil D/2 \rceil}{D} \deg(v) \ .$$

This bound is tight within a small additive constant in the case of a D-dimensional layout of K_n if D is even. For odd D, it is an open problem to determine tight bounds for $\max_v c(v)$ in a D-dimensional layout.

4 Layout-Based Algorithms

We now we describe an algorithm which, given a D-dimensional layout of a graph G, determines a routing of G with bounds on the number of edges routed on each face. To represent the colouring of $A(G)$ we vertex-colour a graph $H = (A(G), E_H)$. So that reversal arcs receive different colours, for each edge vw of G, we add the edge $\{vw, wv\}$ to E_H, called an *r-edge*. For each vertex v and each orthant o relative to v, we partition the arcs $\{vw : w \in o\}$ into $\lceil |\{vw : w \in o\}| / D \rceil$ sets each of size at most D, and add a clique (consisting of *c-edges*) to H between vertices corresponding to arcs in the same partition, as in Fig. 4. The vertices of H corresponding to arcs in a partition with size less than D are said to be *leftover*. For $D = 2$, we add edges to H, called *l-edges*, between certain leftover vertices (see [22]).

Theorem 5. *Every fixed D-dimensional layout ($D \geq 2$) of G admits a degree-restricted orthogonal drawing of G that can be determined in $O(D(m+n))$ time such that:*

- *Each edge route has $D-1$ bends.*
- *The aspect ratio of each vertex tends to 2 (for large degree vertices).*
- *The bounding box volume is $O\left(n^{D(D-2)/(D-1)}\, m^{D/(D-1)}\right)$.*

In the case of $D = 2$ the vertices are 2-degree-restricted and the bounding box is at most $(m+3n/4) \times (m+3n/4)$. For $D = 3$ the vertices are 4-degree-restricted and the bounding box volume is at most $4\sqrt{2}(nm)^{3/2}$.

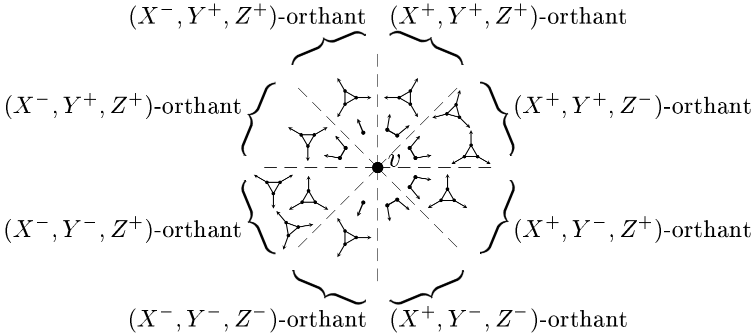

Fig. 4. Partitioning of $\{vw \in A(G)\}$ and construction of H for $D = 3$.

Proof. (Outline) For $D = 2$, a cycle in H consists of alternating r- and (c or l)-edges and is therefore of even length. Since $\Delta(H) \leq D$ and the complete graph $K_{D+1} \nsubseteq H$, by Brooks' Theorem, H is D-colourable in $O(|E_H|) = O(mD)$ time. The vertex-colouring of H determines a routing of G such that for each orthant o relative to a vertex v and in each partition of $\{vw : w \in o\}$, there is at most one arc vw coloured i, $1 \leq i \leq D$. It follows that,

$$N_i(v) \ \leq \ \frac{1}{D} \max\{s_i(v), p_i(v)\} + O(1). \tag{2}$$

Since $\deg(v)/2 \leq \max\{s_i(v), p_i(v)\} \leq \deg(v)$ our aspect bound follows. Also,

$$2\sum_{i=1}^{D} N_i(v) \ \leq \ \deg(v) + c(v) + O(1),$$

and hence

$$\text{surface}(v) \ \leq \ \kappa_D\,(\deg(v) + c(v)) + O(1).$$

Since $c(v) \leq \deg(v)$ and for sufficiently large $\deg(v)$, we have $r_i(v) \geq 1$, $1 \leq i \leq D$, it follows from Lemma 1(c) that $\kappa_D \leq 2^D$, and the drawing is degree-restricted. Since $\kappa_2 = 1$, a 2-D drawing is 2-degree-restricted. The area bound follows since

$$\sum_v \alpha_X(v) \ = \ \sum_v N_Y(v) \ \leq \ \frac{1}{4}\Big(3n + \sum_v (\deg(v) + c_Y(v))\Big) \ \leq \ \frac{3n}{4} + m.$$

By Lemma 1(b), $\kappa_3 = 2$, so a 3-D drawing is 4-degree-restricted. Our volume bounds follow from Theorem 1. $\square$

Using the balanced layout algorithm described in Sec. 3 and the above routing algorithm we obtain the following results for 2-D and 3-D orthogonal drawings.

Theorem 6. *Every graph has a 2-D orthogonal drawing that can be determined in $O(m+n)$ time such that:*

- *Each edge route has* 1 *bend.*
- *Each vertex is* $\frac{3}{2}$*-degree-restricted.*
- *The aspect ratio of each vertex tends to 2 (for large degree vertices).*
- *The bounding box area is* $\left(\frac{3m+4n+2}{4}\right) \times \left(\frac{3m+4n+2}{4}\right)$.

Theorem 7. *Every graph has a 3-D orthogonal drawing that can be determined in $O(m+n)$ time such that:*

- *Each edge route has* 2 *bends.*
- *Each vertex is* $\frac{10}{3}$*-degree-restricted.*
- *The aspect ratio of each vertex tends to 2 (for large degree vertices).*
- *The bounding box volume is at most* $2.34(nm)^{3/2} + O(n^3)$.

Using a *diagonal* vertex layout with corresponding vertex ordering determined by the medium placement heuristic we obtain the following results.

Theorem 8. *Every graph has a D-dimensional degree-restricted hypercube-drawing $(D \geq 2)$ which can be determined in $O(D(m+n))$ time such that:*

- *Each edge route has* $D-1$ *bends.*
- *The bounding box volume is at most* $\left((2n)^{D-2}\left(\frac{(2D-1)n+3m}{2D}\right)\right)^{D/(D-1)}$

Theorem 9. *Every graph has a D-dimensional line-drawing $(D \geq 3)$ that can be determined in $O(D(m+n))$ time such that:*

- *Each edge route has* $D-1$ *bends.*
- *Each vertex is a* 2*-degree-restricted D-axis parallel line.*
- *The bounding box volume is at most* $n^{D-1}\left(\frac{(2D-3)n+3m}{2(D-1)}\right)$

5 Routing-Based Algorithms

Given a routing of G, we determine the i-ordering, $1 \leq i \leq D$, of a layout of G by applying the median placement heuristic to the subgraph of G', denoted $G'[i]$, induced by the arcs coloured i. If each $G'[i]$ is acyclic then we say the routing is *acyclic*, and by Theorem 3 minimum cost orderings can be determined.

To determine a 2-colour acyclic routing of G, start with a vertex ordering $(v_1, v_2, \ldots, v_n)$ of G, and for each edge $v_iv_j \in E$ $(i < j)$ colour the arcs v_iv_j X and v_jv_i Y. Clearly, $G'[X]$ and $G'[Y]$ are both acyclic; the topological orderings of $G'[X]$ and $G'[Y]$ are respectively $(v_1, v_2, \ldots, v_n)$ and $(v_n, v_{n-1}, \ldots, v_1)$. This approach is used in [1] and in [4] for determining the routing and the X- and Y-orderings of a 3-D layout; each vertex is then represented by a line parallel to the Z-axis. The main criticism of this method is that the drawings are inherently two-dimensional.

We now describe a new method for determining a 3-colour acyclic routing. Firstly, determine a locally balanced vertex ordering $(v_1, v_2, \ldots, v_n)$ (see Sec. 3).

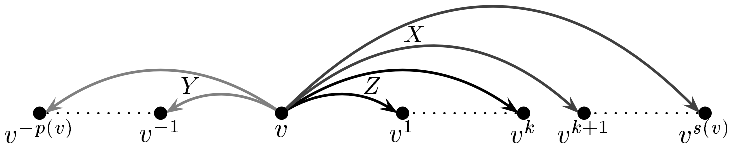

Fig. 5. Routing arcs at a positive vertex v; $k = \lfloor c(v)/2 \rfloor$.

For each vertex v_i, colour the arcs $v_i(v_i)^k$, $1 \leq k \leq \lfloor c(v_i)/2 \rfloor$, with colour Z. Remaining arcs v_iv_j are coloured X if $i < j$ and Y if $j < i$, as in Fig. 5. Clearly $G'[X]$ and $G'[Y]$ are acyclic. By Lemma 2 a positive vertex v_i cannot have an incoming arc $v_jv_i \in G'[Z]$ with $i < j$. Similarly for negative vertices. Hence $G'[Z]$ is also acyclic.

Theorem 10. *Every graph has 3-D orthogonal drawing that can be determined in $O(m+n)$ time such that,*

- *Each edge route has* 2 *bends.*
- *Each vertex is 2-degree-restricted and has aspect ratio at most* $\deg(v)/4$.
- *The bounding box volume is $O(\Delta(nm)^{3/2})$.*

Proof. (Outline) For each vertex v, $2\sum_i N_i(v) = \deg(v) + O(1)$. Since $\kappa_3 = 2$, it follows that surface$(v) \leq 2\deg(v) + O(1)$, and v is 2-degree-restricted. A vertex v has maximum aspect ratio if, in the locally balanced vertex ordering, $c(v) = 0$, $s(v) = 0$ or $p(v) = 0$, in which case v is a line of length $\deg(v)/4$. Applying Theorem 1 we obtain a bounding box volume bound of $O(\Delta(nm)^{3/2})$. □

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References

[1] T. Biedl and M. Kaufmann. Area-efficient static and incremental graph drawings. In R. Burkhard and G. Woeginger, editors, *Proc. Algorithms: 5th Annual European Symp. (ESA'97)*, volume 1284 of *Lecture Notes in Comput. Sci.*, pages 37–52, Berlin, 1997. Springer.

[2] T. Biedl, B. Madden, and I.G. Tollis. The three-phase method: A unified approach to orthogonal graph drawing. In Di Battista [5], pages 391–402.

[3] T. Biedl, T. Shermer, S. Whitesides, and S. Wismath. Orthogonal 3-D graph drawing. In Di Battista [5], pages 76–86.

[4] T.C. Biedl. Three approaches to 3D-orthogonal box-drawings. In Whitesides [20], pages 30–43.

[5] G. Di Battista, editor. *Proc. Graph Drawing: 5th International Symp. (GD'97)*, volume 1353 of *Lecture Notes in Comput. Sci.*, Berlin, 1998. Springer.

[6] G. Di Battista, W. Didimo, M. Patrignani, and M. Pizzonia. Orthogonal and quasi-upward drawings with vertices of arbitrary size. In J. Kratochvil, editor, *Proc. Graph Drawing: 7th International Symp. (GD'99)*, Lecture Notes in Comput. Sci., Berlin. Springer. to appear.

[7] G. Di Battista, P. Eades, R. Tamassia, and I.G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs.* Prentice-Hall, New Jersey, 1999.

[8] G. Di Battista, A. Garg, G. Liotta, R. Tamassia, E. Tassinari, and F. Vargiu. An experimental comparison of four graph drawing algorithms. *Comput. Geom.*, 7(5-6):303–325, 1997.

[9] W. Didimo and G. Liotta. Computing orthogonal drawings in a variable embedding setting. In K.-Y. Chwa and O. H. Ibarra, editors, *Proc. Algorithms and Computation (ISAAC'98)*, volume 1533 of *Lecture Notes in Comput. Sci.*, pages 79–88, Berlin, 1998. Springer.

[10] P. Eades, A. Symvonis, and S. Whitesides. Two algorithms for three dimensional orthogonal graph drawing. In North [16], pages 139–154.

[11] P. Eades, A. Symvonis, and S. Whitesides. Three dimensional orthogonal graph drawing algorithms. 1998. submitted.

[12] S. Even and G. Granot. Grid layouts of block diagrams — bounding the number of bends in each connection. In R. Tamassia and I.G. Tollis, editors, *Proc. Graph Drawing: DIMACS International Workshop (GD'94)*, volume 894 of *Lecture Notes in Comput. Sci.*, pages 64–75, Berlin, 1995. Springer.

[13] U. Fößmeier, G. Kant, and M. Kaufmann. 2-visibility drawings of planar graphs. In North [16], pages 155–158.

[14] U. Fößmeier and M. Kaufmann. Drawing high degree graphs with low bend numbers. In F.J. Brandenburg, editor, *Proc. Graph Drawing: Symp. on Graph Drawing (GD'95)*, volume 1027 of *Lecture Notes in Comput. Sci.*, pages 254–266, Berlin, 1996. Springer.

[15] U. Fößmeier and M. Kaufmann. Algorithms and area bounds for nonplanar orthogonal drawings. In Di Battista [5], pages 134–145.

[16] S. North, editor. *Proc. Graph Drawing: Symp. on Graph Drawing (GD'96)*, volume 1190 of *Lecture Notes in Comput. Sci.*, Berlin, 1997. Springer.

[17] A. Papakostas and I.G. Tollis. Orthogonal drawing of high degree graphs with small area and few bends. In F. Dehne, A. Rau-Chaplin, J.-R. Sack, and R. Tamassia, editors, *Proc. Algorithms and Data Structures: 5th International Workshop (WADS'97)*, volume 1272 of *Lecture Notes in Comput. Sci.*, pages 354–367, Berlin, 1997. Springer.

[18] A. Papakostas and I.G. Tollis. Incremental orthogonal graph drawing in three dimensions. In Di Battista [5], pages 52–63.

[19] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.*, 16(3):421–443, June 1987.

[20] S. Whitesides, editor. *Proc. Graph Drawing: 6th International Symp. (GD'98)*, volume 1547 of *Lecture Notes in Comput. Sci.* Springer, 1998.

[21] D.R. Wood. An algorithm for three-dimensional orthogonal graph drawing. In Whitesides [20], pages 332–346.

[22] D.R. Wood. Multi-dimensional orthogonal graph drawing in the general position model. Technical Report 99/38, School of Computer Science and Software Engineering, Monash University, Australia, 1999. (available at `http://www.csse.monash.edu.au/publications/`).

[23] D.R. Wood. A new algorithm and open problems in three-dimensional orthogonal graph drawing. In R. Raman and J. Simpson, editors, *Proc. Australasian Workshop on Combinatorial Algorithms (AWOCA'99)*, pages 157–167. Curtin University of Technology, Perth, 1999.