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Degree constrained book embeddings ☆

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Abstract

A *book embedding* of a graph consists of a linear ordering of the vertices along a line in 3-space (the *spine*), and an assignment of edges to half-planes with the spine as boundary (the *pages*), so that edges assigned to the same page can be drawn on that page without crossings. Given a graph $G = (V, E)$, let $f : V \to \mathbb{N}$ be a function such that $1 \leqslant f(v) \leqslant \deg(v)$. We present a Las Vegas algorithm which produces a book embedding of G with $O(\sqrt{|E| \cdot \max_v \lceil \deg(v)/f(v) \rceil})$ pages, such that at most $f(v)$ edges incident to a vertex v are on a single page. This result generalises that of Malitz [*J. Algorithms* 17 (1) (1994) 71–84].

Keywords: Graph; Book embedding; Book thickness; Page number; Page degree; Pushdown graph; Las Vegas algorithm; Multilayer VLSI

1. Introduction

This paper describes a Las Vegas algorithm for producing a book embedding of a graph, given constraints on the number of edges incident to each vertex which can be assigned to a single page. All graphs are undirected and simple. We denote

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the number of vertices of a graph $G = (V, E)$ by $n = |V|$, the number of edges of G by $m = |E|$, and the maximum degree of G by $\Delta(G)$, or Δ if the graph in question is clear.

Book embeddings, first introduced by Bernhart and Kainen [1], are a graph layout style with numerous applications (see [4]). A *book* consists of a line in 3-space, called the *spine*, and a number of *pages*, each a half-plane with the spine as boundary. A *book embedding* (π, ρ) of a graph consists of a linear ordering π of the vertices, called the *spine ordering*, along the spine of a book and an assignment ρ of edges to pages so that edges assigned to the same page can be drawn on that page without crossings. That is, for any two edges vw and xy, if $v <_\pi x <_\pi w <_\pi y$ then $\rho(vw) \neq \rho(xy)$. The *book thickness* or *page number* of a graph G is the minimum number of pages in a book embedding of G.

Determining the book thickness of a graph is $\mathcal{NP}$-hard, even with a fixed spine ordering [11]. A number of results establish upper bounds on the book thickness of certain classes of graphs [1,6,7,10,18], such as the celebrated theorem of Yannakakis [23] that every planar graph has book thickness at most four. For graphs with genus γ, Malitz [14] proved that the book thickness is $O(\sqrt{\gamma})$. Since $\gamma \leqslant m$, the book thickness is $O(\sqrt{m})$, a result proved independently by the same author [15]. While the proofs of Malitz are probabilistic, Shahrokhi and Shi [20] describe a deterministic algorithm, which given a vertex k-colouring of a graph G, computes a book embedding of G with $O(\sqrt{km})$ pages.

Note that a book embedding may assign all of the edges incident to a vertex to a single page. In this paper we study book embeddings where the number of edges incident to a vertex on a single page is constrained. (A similar approach is taken for the graph-theoretic thickness by Bose and Prabhu [3], and for edge colouring by Hakimi and Kariv [12].) We define the *page degree* of a vertex v in a particular book embedding to be the maximum number of edges incident to v on a single page. A *constraint function* of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ such that $1 \leqslant f(v) \leqslant \deg(v)$ for all vertices $v \in V$. For some constraint function f of G, a *degree-f* book embedding of G is one in which the page degree of every vertex v is at most $f(v)$. If for all vertices $v \in V$, $f(v) = c$ for some constant c, a degree-f book embedding is simply called a *degree-c* book embedding.

Galil, et al. [8,9] refer to a graph which admits a degree-1 book embedding with k pages as a *k-pushdown* graph. Motivated by problems in computational complexity, they established lower bounds on the size of a separator in 3-pushdown graphs. Implicit in the work of Biedl, et al. [2] is a degree-1 book embedding of the complete graph K_n with n pages. In this paper we consider the following problem: given a graph $G = (V, E)$ and an arbitrary constraint function f of G, produce a degree-f book embedding of G with few pages. Define

$$Q_f(G) = \max_{v \in V} \left\lceil \frac{\deg(v)}{f(v)} \right\rceil.$$

Obviously $Q_f(G)$ is a lower bound on the number of pages in a degree-f book embedding of G.

Consider the following naive method to produce a degree-f book embedding of a graph $G = (V, E)$. Take a book embedding of G with pages labeled $\{1, 2, \ldots, P\}$, and construct an auxiliary graph H with vertex-set $V \times \{1, 2, \ldots, P\}$ and an edge $\{(v, i), (w, i)\}$ for each edge $vw \in E$ assigned to page i. Then apply Theorem 3 of Hakimi and Kariv [12] to determine a (non-proper) edge-colouring of H with at most $f(v)$ edges incident to each vertex (v, i) of H, and with at most $Q_f(G) + 1$ colours. Combining this edge colouring with the original book embedding gives a degree-f book embedding of G with at most $P \cdot (Q_f(G) + 1)$ pages. If for instance the original book embedding of G is determined by the above-mentioned algorithm of Malitz [15] then the number of pages in the degree-f book embedding is $O(\sqrt{m}\, Q_f(G))$. In this paper we establish the following result.

Theorem 1. *Let f be a constraint function of a connected graph $G = (V, E)$ with m edges. Then there exists a degree-f book embedding of G with $O(\sqrt{m\, Q_f(G)})$ pages.*

Thus our result represents an improvement over the naive method by a factor of $\Omega(\sqrt{Q_f(G)})$. Theorem 1, and its proof, generalises the above-mentioned bound of $O(\sqrt{m})$ on the book thickness due to Malitz [15], which in turn is based on ideas of Chung, et al. [4]. In particular we describe a Las Vegas algorithm which, with high probability, determines the desired degree-f book embedding in $O(m \log^2 n \log\log m)$ time. See [17] for information about Las Vegas algorithms. Note that Theorem 1 has recently been applied to produce multilayer VLSI constructions with improved volume bounds [22].

2. Preliminary results

The following definitions are from [15]. A *2-coloured bipartite graph* is a bipartite graph $G = (V_L, V_R; E)$ whose vertices have been coloured LEFT and RIGHT such that adjacent vertices are coloured differently. For some edge $e \in E$, $L(e)$ refers to the end-vertex of e in V_L, and $R(e)$ refers to the end-vertex of e in V_R. A *canonical ordering* of a 2-coloured bipartite graph $G = (V_L, V_R; E)$ is a linear ordering of the vertices of G such that all LEFT vertices precede all RIGHT vertices.

Let π be a canonical ordering of a 2-coloured bipartite graph $G = (V_L, V_R; E)$. Two edges vw and xy are said to *cross* if $v <_\pi x <_\pi w <_\pi y$. Two edges are *disjoint* if they have no common endpoint and they do not cross. Two edges *intersect* if they have a common endpoint or they cross. For (traditional) book embeddings the number of pairwise crossing edges provides a lower bound on the

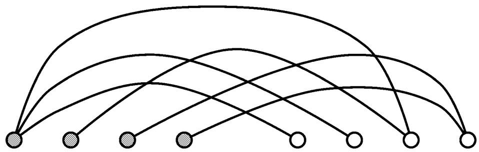

Fig. 1. A completely intersecting canonical ordering of a graph.

number of pages, whereas for degree-1 book embeddings the number of pairwise intersecting edges plays the same role. G is *completely intersecting* with respect to π if E can be labeled $e_1, e_2, \ldots, e_k$ such that

$$L(e_1) \leqslant_\pi L(e_2) \leqslant_\pi \cdots \leqslant_\pi L(e_k) \quad \text{and}$$
$$R(e_1) \leqslant_\pi R(e_2) \leqslant_\pi \cdots \leqslant_\pi R(e_k).$$

Intuitively, G is completely intersecting with respect to π, if in a degree-1 book embedding with π as the spine ordering, every edge must be placed on a unique page, as illustrated in Fig. 1.

Lemma 1. *If a 2-coloured bipartite graph G is completely intersecting with respect to some canonical ordering then G is a forest.*

Proof. Let π be a canonical ordering of G. Suppose to the contrary that G is not a forest and G is completely intersecting with respect to π. Then G contains a cycle $(v_1, w_1, v_2, w_2, \ldots, v_k, w_k, v_{k+1})$ with $v_1 = v_{k+1}$ for some $k \geqslant 2$. Without loss of generality we can assume that v_1 is the leftmost vertex. We proceed by induction on i with the following induction hypothesis: "for every $i \geqslant 1$, $v_i <_\pi v_{i+1}$ and $w_i <_\pi w_{i+1}$."

To prove the basis of the induction, observe that if $w_2 <_\pi w_1$ then v_1w_1 does not intersect v_2w_2; hence $w_1 <_\pi w_2$. By our initial assumption, $v_1 <_\pi v_2$. Suppose that $v_1 <_\pi \cdots <_\pi v_i$ and $w_1 <_\pi \cdots <_\pi w_i$. If $v_{i+1} <_\pi v_i$ then $v_{i+1}w_i$ does not intersect v_iw_{i-1}; thus $v_i <_\pi v_{i+1}$. If $w_{i+1} <_\pi w_i$ then v_iw_i does not intersect $v_{i+1}w_{i+1}$; thus $w_i <_\pi w_{i+1}$. Therefore the inductive hypothesis holds, which is a contradiction as it implies that $v_1 <_\pi v_{k+1}$ and $v_1 = v_{k+1}$. □

Note that Lemma 1 can be strengthened to say a completely intersecting graph is a forest of caterpillars. The next lemma for completely intersecting sets of edges, is the analogue of Lemma 2.2 in [15] for completely crossing sets of edges. Generalising a result of Tarjan [21], it says that book thickness can be determined efficiently if the spine ordering is a canonical ordering of a bipartite graph.

Lemma 2. *Let π be a given canonical ordering of a 2-coloured bipartite graph $G = (V_L, V_R; E)$ with m edges and n vertices. If at most k edges are completely intersecting with respect to π, then a k-page degree-1 book embedding of G with spine ordering π can be determined in $O(m \log\log n)$ time.*

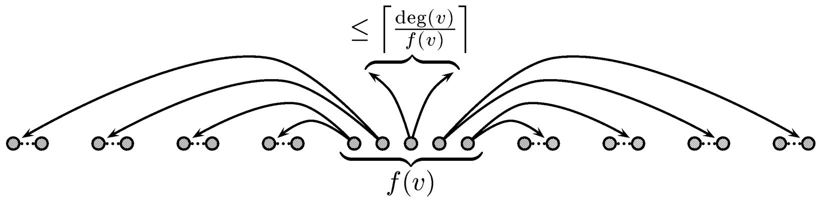

Fig. 2. Constructing π_f.

Proof. Define a poset $(E, \preceq)$ as follows. For all $e_1, e_2 \in E$ let

$$e_1 \preceq e_2 \stackrel{\text{def}}{=} e_1 = e_2 \quad \text{or} \quad \big(L(e_2) <_\pi L(e_1) \quad \text{and} \quad R(e_1) <_\pi R(e_2)\big).$$

It is a simple exercise to check that $\leqslant$ is reflexive, transitive and antisymmetric, and thus is a partial order. Two edges are incomparable under $\preceq$ if and only if they intersect. Thus an antichain is a completely intersecting set of edges, and a chain is a set of pairwise disjoint edges. By Dilworth's Theorem [5] there is a decomposition of E into k chains where k is the size of the largest antichain. That is, there is a k-page degree-1 book embedding of G with spine ordering π. The time complexity can be achieved using a dual form of the algorithm by Heath and Rosenberg [13, Theorem 2.3]. □

Note that an equivalent result to Lemma 2 with a more lengthy proof is given by Malucelli and Nicoloso [16]. To enable Lemma 2 to be extended to degree-f book embeddings, consider the following construction. Let π be a linear ordering of the vertices of a graph $G = (V, E)$, and let f be a constraint function of G. We define a graph $G_{\pi,f}$ and a linear ordering π_f of $G_{\pi,f}$ as follows (see Fig. 2). For each vertex $v \in V$, replace v by $f(v)$ consecutive vertices in π_f, which we call *sub-vertices* of v. Let α_v and β_v be the unique integers such that

$$\alpha_v + \beta_v = f(v) \quad \text{and} \quad \alpha_v \left\lceil \frac{\deg(v)}{f(v)} \right\rceil + \beta_v \left\lfloor \frac{\deg(v)}{f(v)} \right\rfloor = \deg(v).$$

To each of the α_v leftmost sub-vertices of v, connect $\lceil \deg(v)/f(v) \rceil$ edges incident to v, and to each of the β_v rightmost sub-vertices of v, connect $\lfloor \deg(v)/f(v) \rfloor$ edges incident to v, such that no two edges cross. Note that the choice of sub-vertices which are incident to $\lceil \deg(v)/f(v) \rceil$ or $\lfloor \deg(v)/f(v) \rfloor$ edges is not important. We simply want $G_{\pi,f}$ to be uniquely determined by π and f.

Lemma 3. *Let f be a constraint function, and let π be a given canonical ordering of a 2-coloured bipartite graph $G = (V_L, V_R; E)$ with m edges and n vertices. If at most k edges of $G_{\pi,f}$ are completely intersecting with respect to π_f, then a k-page degree-f book embedding of G with spine ordering π can be determined in $O(m \log\log(\sum_v f(v)))$ time.*

Proof. Apply Lemma 2 to $G_{\pi,f}$ with spine ordering π_f, to obtain a degree-1 book embedding (π_f, ρ) of $G_{\pi,f}$ with at most k pages. In (π_f, ρ), the page degree of a sub-vertex is at most one. Thus, in the book embedding (π, ρ) of G, the page degree of v is at most $f(v)$; that is, (π, ρ) is a degree-f book embedding of G. The time bound follows from Lemma 2 and that $G_{\pi,f}$ has $\sum_v f(v)$ vertices. □

3. Main result

To prove Theorem 1 we will need the following lemma.

Lemma 4. *Let f be a constraint function and let π be a random canonical ordering of a 2-coloured forest $T = (V_L, V_R; E)$ with $n = |V_L \cup V_R|$ vertices and $m = |E|$ edges. The probability that $T_{\pi,f}$ is completely intersecting with respect to π_f is at most*

$$\frac{2^n\,(Q_f(T))^m}{m!}.$$

Proof. The probability that $T_{\pi,f}$ is completely intersecting with respect to π_f is the number of canonical orderings π of T for which $T_{\pi,f}$ is completely intersecting with respect to π_f, divided by the number of canonical orderings of T. If $T_{\pi,f}$ is completely intersecting with respect to π_f then all edges incident to a vertex v must be incident to the same sub-vertex of v in π_f, and thus, T is completely intersecting with respect to π. (Note that this implies that $\Delta(T) \leqslant Q_f(T)$.) Thus, the desired probability is at most the number of canonical orderings π of T in which T is completely intersecting, divided by the number of canonical orderings of T.

We first bound the number of canonical orderings of T for which T is completely intersecting. Initially suppose T is connected; that is, $n = m + 1$. For some fixed ordering $(v_1, v_2, \ldots, v_l)$ of V_L, an ordering of V_R which makes T completely intersecting must be of the form

$$\{R(e)\colon v_1 \in e\}, \quad \{R(e)\colon v_2 \in e\}, \quad \ldots, \quad \{R(e)\colon v_l \in e\}.$$

Similarly, if $(w_1, w_2, \ldots, w_r)$ is a fixed ordering of V_R, then an ordering of V_L which makes T completely intersecting must be of the form

$$\{L(e)\colon w_1 \in e\}, \quad \{L(e)\colon w_2 \in e\}, \quad \ldots, \quad \{L(e)\colon w_l \in e\}.$$

The vertices within each set $\{R(e)\colon v_i \in e\}$ and $\{L(e)\colon w_i \in e\}$ possibly can be permuted. Thus the number of canonical orderings of T which are completely intersecting is at most $\prod_x \deg_T(x)!$.

We claim that $\prod_x \deg_T(x)! \leqslant \Delta(T)^m$. To prove this claim, we proceed by induction on m. The basis of the induction with $m = 1$ is trivial. Suppose for all

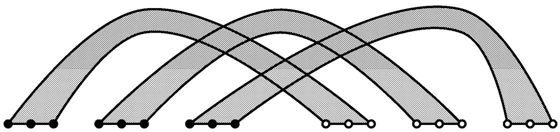

Fig. 3. Completely intersecting canonical ordering of the connected components of T.

connected trees $T' = (V', E')$ with $|E'| < m$ that $\prod_{x \in V'} \deg_{T'}(x)! \leqslant \Delta(T')^{|E'|}$. Let v be a leaf of T incident to the edge vw. Let $T' = (V', E') = T \setminus \{vw\}$. Since $\deg_{T'}(w) = \deg_T(w) - 1$, and by the inductive hypothesis applied to T',

$$\prod_{x \in V} \deg(x)! = \deg(w) \prod_{x \in V'} \deg_{T'}(x) \leqslant \deg(w) \cdot \Delta(T')^{m-1} \leqslant \Delta(T)^m. \quad (1)$$

Thus the claim is proved.

Now suppose T is disconnected. Then T has $n - m$ connected components. Let $E_1, E_2, \ldots, E_{n-m}$ be the edge sets of the connected components of T. For T to be completely intersecting, the LEFT vertices in each connected component must be consecutive in the ordering, and similarly for the RIGHT vertices. Within V_L, the components can be ordered $(n - m)!$ different ways. For a fixed ordering of the connected components of V_L, for T to be completely intersecting, the components of V_R must be ordered the same way, as illustrated in Fig. 3.

By (1), the number of canonical orderings which are completely intersecting is at most

$$(n - m)! \prod_{i=1}^{n-m} \Delta(T)^{|E_i|} \leqslant (n - m)! \Delta(T)^m.$$

The number of canonical orderings of T is $|V_L|! \cdot |V_R|!$. Thus, the probability that a random canonical ordering of T is completely intersecting is at most

$$\frac{(n - m)! \Delta(T)^m}{|V_L|! \cdot |V_R|!} \leqslant \frac{(n - m)! \Delta(T)^m}{\lceil \frac{n}{2} \rceil ! \lfloor \frac{n}{2} \rfloor !} \leqslant \frac{2^n (n - m)! \Delta(T)^m}{n!} \leqslant \frac{2^n \Delta(T)^m}{m!},$$

where the final three inequalities follow from well-known and easily proved facts concerning factorials. The result holds, since as noted earlier $\Delta(T) \leqslant Q_f(T)$. □

Proof of Theorem 1. Let $n' = |V|$, and denote $Q_f(G)$ by Q. Since G is connected, $m \geqslant n' - 1$. If $m = n' - 1$ then G is a tree. By considering a pre-order traversal of G, it is easily seen that G has a book embedding (π, ρ) with one page [4]. The graph $G_{\pi,f}$ is a forest with maximum degree Q, and thus has a edge-colouring χ with Q colours. A book embedding (π, χ) of G is a degree-f book embedding of G with $Q \leqslant \sqrt{\Delta\, Q} \leqslant \sqrt{m\, Q}$ pages. Thus the result is proved for trees.

Now assume $m \geqslant n'$. Let $n = 2^{\lceil \log n' \rceil}$, and add $n - n'$ isolated vertices to G. (Unless stated otherwise all logarithms are base 2.) G now has n vertices, with n a power of 2. Clearly, $n \leqslant 2n'$, and $n \leqslant 2m$.

Let π be a random linear ordering of V. For each j, $1 \leqslant j \leqslant \log n$, divide the linear ordering π into 2^j sections each with the same number of vertices, and label the sections from left to right L, R, L, R, etc. The edges whose endpoints are in adjacent L–R sections (but not adjacent R–L sections) are called j*-level* edges. Note that every edge of G appears in a unique level, and edges in adjacent L–R sections in some j-level are canonically ordered by π.

For each j, $1 \leqslant j \leqslant \log n$, let A_k^j be the event that there exists a k-edge 2-coloured subgraph T of G such that:

- T consists solely of j-level edges,
- T is canonically ordered with respect to π, and
- $T_{\pi,f}$ is completely intersecting with respect to π_f.

By Lemma 1, such a subgraph T is a forest. The probability that A_k^j occurs

$$\mathbf{P}\{A_k^j\} < \underbrace{\binom{m}{k}2^k}_{(1)} \cdot \underbrace{2^{j-1}}_{(2)} \cdot \underbrace{\binom{\frac{n}{2^j}}{l}\binom{\frac{n}{2^j}}{r}\frac{l!r!(n-l-r)!}{n!}}_{(3)} \cdot \underbrace{\frac{2^{l+r}Q^k}{k!}}_{(4)},$$

where:

(1) is an upper bound on the number of k-edge 2-coloured forests T with no isolated vertices (since a bipartite graph with k connected components has 2^k vertex 2-colourings);
(2) is the number of pairs of adjacent L–R sections in the j-level;
(3) is an upper bound on the probability that π canonically orders T in the fixed pair of adjacent j-level sections, where T has l LEFT vertices and r RIGHT vertices; and
(4) is the probability that T is completely intersecting, by Lemma 4 and since $Q_f(T) \leqslant Q$.

Since $\binom{a}{b} \leqslant a^b/b!$,

$$\mathbf{P}\{A_k^j\} < \frac{(2m)^k}{k!} \cdot 2^{j-1} \cdot \left(\frac{n}{2^j}\right)^{l+r} \frac{(n-l-r)!}{n!} \cdot \frac{2^{l+r}Q^k}{k!}.$$

The special case of $n = l + r$ can be handled easily. We henceforth assume $l + r < n$. The version of Stirling's formula due to Robbins [19] states that for all $n \geqslant 1$, $n! = \sqrt{2\pi n}(n/\mathrm{e})^n \mathrm{e}^{r_n}$, where $1/(12n+1) < r_n < 1/(12n)$ and e is the base of the natural logarithm. Thus,

$$\mathbf{P}\{A_k^j\} < (2m)^k \cdot 2^{j-1} \cdot \left(\frac{n}{2^j}\right)^{l+r} \sqrt{\frac{n-l-r}{n}}\left(\frac{n-l-r}{\mathrm{e}}\right)^{n-l-r}\left(\frac{\mathrm{e}}{n}\right)^n \cdot \frac{2^{l+r}Q^k \mathrm{e}^{2k} r}{k^{2k+1}},$$

where the error term $r = e^{1/12(n-l-r)}e^{-1/(12n+1)}e^{-2/(12k+1)} < e^4$.

Now, $n - l - r < n$. By elementary properties of a forest, $k + 1 \leqslant l + r \leqslant 2k$. Since $l + r \leqslant 2n/2^j$, we have $k \leqslant n/2^{j-1}$, and hence $2^{j-1} \leqslant n/k \leqslant 2m/k$. Thus,

$$\begin{aligned}\mathbf{P}\{A_k^j\} &< (2m)^{k+1} \cdot \left(\frac{1}{2^j}\right)^{k+1} n^{(l+r)+(n-l-r)-n} \cdot e^{-(n-l-r)+n+2k+4} \\ &\quad \cdot \frac{2^{2k}Q^k}{k^{2(k+1)}} \\ &< \left(\frac{8e^4 mQ}{2^j k^2}\right)^{k+1}.\end{aligned}$$

Define $k_j = 4e^2\sqrt{mQ/2^j}$. Since $m \geqslant n/2$ and $Q \geqslant 1$,

$$\mathbf{P}\{A_{k_j}^j\} < \left(\frac{1}{2}\right)^{1+4e^2\sqrt{mQ/2^j}} < \frac{1}{2}\left(\frac{1}{2}\right)^{2\sqrt{2}e^2\sqrt{n/2^j}}.$$

Consider the event that $A_{k_j}^j$ occurs for some j, $1 \leqslant j \leqslant \log n$.

$$\mathbf{P}\left\{\bigcup_{j=1}^{\log n} A_{k_j}^j\right\} < \frac{1}{2}\sum_{j=1}^{\log n}\left(\frac{1}{2}\right)^{2\sqrt{2}e^2\sqrt{n/2^j}}.$$

By induction on N, the following can be easily proved.

$$\forall a > 1,\ \forall b \geqslant \frac{1 - \log_a(a-1)}{\sqrt{2} - 1}, \quad \sum_{j=1}^{N}\left(\frac{1}{a}\right)^{b\sqrt{2^{N-j}}} < \left(\frac{1}{a}\right)^{b-1}.$$

Applying this fact with $N = \log n$, $a = 2$ and $b = 2\sqrt{2}e^2 > 1/(\sqrt{2} - 1)$,

$$\mathbf{P}\left\{\bigcup_{j=1}^{\log n} A_{k_j}^j\right\} < \frac{1}{2}\left(\frac{1}{2}\right)^{2\sqrt{2}e^2 - 1} = \left(\frac{1}{2}\right)^{2\sqrt{2}e^2}.$$

Thus,

$$\begin{aligned}\mathbf{P}\left\{\bigcap_{j=1}^{\log n} \overline{A_{k_j}^j}\right\} &= \mathbf{P}\left\{\overline{\bigcup_{j=1}^{\log n} A_{k_j}^j}\right\} = 1 - \mathbf{P}\left\{\bigcup_{j=1}^{\log n} A_{k_j}^j\right\} > 1 - \left(\frac{1}{2}\right)^{2\sqrt{2}e^2} \\ &> 0.99999.\end{aligned}$$

This says that for the random linear ordering π, with (very high) positive probability, $A_{k_j}^j$ does not occur for all j, $1 \leqslant j \leqslant \log n$. Therefore, there exists a linear ordering π' of V such that $A_{k_j}^j$ does not occur for all j. That is, in each pair of adjacent L–R sections in the j-level, there is no completely intersecting subgraph in π'_f with at least k_j edges. For each pair of adjacent L–R sections in level j, apply Lemma 3 to the subgraph of $G_{\pi',f}$ consisting of j-level edges with endpoints in that pair of sections (using the canonical ordering π'_f). By using the same set of pages for j-level edges, we obtain a degree-f book embedding of G with spine ordering π', and with the number of pages at most

$$\sum_{j=1}^{\log n} k_j = 4\mathrm{e}^2\sqrt{mQ}\sum_{j=1}^{\log n}\sqrt{\frac{1}{2^j}} < \frac{4\mathrm{e}^2\sqrt{mQ}}{\sqrt{2}-1} < 72\sqrt{m\,Q}. \qquad \square$$

Example 1. Let $G = (V, E)$ be a graph with average degree $d = 2m/n$. By Theorem 1, G has a book embedding with $O(m/\sqrt{n})$ pages, such that every vertex $v \in V$ has page degree at most $\deg(v)/d$.

Corollary 1. *Let f be a constraint function of connected graph $G = (V, E)$ with n vertices and m edges. There is a Las Vegas algorithm which will compute, with high probability, a degree-f book embedding of G with $O(\sqrt{m\,Q_f(G)})$ pages in $O(m \log^2 n \log\log m)$ time.*

Proof. Consider the following Las Vegas algorithm to compute the book embedding whose existence is proved in Theorem 1. First, add $2^{\lceil \log |V| \rceil} - |V|$ isolated vertices to G. Then repeat the following step at most $\log n$ times. Choose a random linear ordering π of V, and embed each set of j-level edges in its own set of pages (using Lemma 3 applied to $G_{\pi,f}$ as described in the proof of Theorem 1). If the total number of pages is at most $72\sqrt{m\,Q_f(G)}$ then halt, otherwise repeat.

The time taken for each iteration within each j-level is $O(m \log\log(\sum_v f(v)))$ by Lemma 3. Since $f(v) \leqslant \deg(v)$, $\sum_v f(v) \in O(m)$, and the time taken for each iteration is $O(m \log n \log\log m)$. At each iteration of the above algorithm, we say the algorithm *fails* if the randomly chosen linear ordering π does not admit a degree-f book embedding with at most $72\sqrt{m Q_f(G)}$ pages. The probability of failure is at most $2^{-2\sqrt{2}\mathrm{e}^2}$. The probability of failure every iteration is at most $2^{-2\sqrt{2}\mathrm{e}^2 \log n} = n^{-2\sqrt{2}\mathrm{e}^2} \to 0$ as $n \to \infty$. Thus, with probability tending to 1 as $n \to \infty$, the above algorithm will determine a degree-f book embedding of G with at most $72\sqrt{m Q_f(G)}$ pages in $O(m \log^2 n \log\log m)$ time. $\square$

Note that Theorem 1 with the constraint function $f(v) = \deg(v)$ is the same result proved by Malitz [15], and the above proof is based on Malitz's idea of defining j-levels and applying Dilworth's Theorem to a partial ordering of the edges in each level. However, our proof differs in two respects. First, we do not assume that $j \leqslant k$, as is the case in [15, p. 76] (also see [14, p. 92]). Furthermore, we do not use a book embedding of the complete graph $K_{\sqrt{n}}$ for levels $j = \frac{1}{2}\log n + 1, \frac{1}{2}\log n + 2, \ldots, \log n$.

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