A Polynomial Bound for Untangling Geometric Planar Graphs ⋆

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1 Introduction

This paper considers the following problem: Given a drawing of a planar graph G possibly with crossings, redraw G with straight edges and no crossings, while keeping as many vertices as possible fixed. More formally, consider a geometric graph G with vertex set $V(G) = \{p_1, \ldots, p_n\}$. A crossing-free geometric graph H with vertex set $V(H) = \{q_1, \ldots, q_n\}$ is an *untangling* of G if for all $i, j \in \{1, 2, \ldots, n\}$, q_i is adjacent to q_j in H if and only if p_i is adjacent to p_j in G. If $p_i = q_i$ then p_i is *fixed*, otherwise p_i is *free*. Of course only geometric planar graphs can be untangled. For a geometric planar graph G, let $\mathsf{fix}(G)$ be the maximum number of fixed vertices in an untangling of G. By the Fáry-Wagner Theorem, $\mathsf{fix}(G)$ is well defined. Pach and Tardos [3] asked whether $\mathsf{fix}(G) \geq n^{\varepsilon}$ for some $\varepsilon > 0$. Recently, Spillner and Wolff [4] showed that $\mathsf{fix}(G) \geq \Omega(\sqrt{\log n / \log\log n})$. This paper answers the question of Pach and Tardos [3] in the affirmative. See the full version of this paper (arXiv:0710.1641) for more results and references.

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Theorem 1.1 *Every n-vertex geometric planar graph G can be untangled while keeping at least $(n/3)^{1/4}$ vertices fixed. That is,* $\mathsf{fix}(G) \geq (n/3)^{1/4}$.

2 Canonical Orderings and Frames

To prove Theorem 1.1 we may assume that G is an edge-maximal geometric planar graph. Let $\mathcal{E}$ be an embedded planar graph isomorphic to G. So each face of $\mathcal{E}$ is bounded by a 3-cycle. Let x, y and z be the vertices on the outer face. de Fraysseix *et al.* [1] proved that $\mathcal{E}$ has a vertex ordering $\sigma = (v_1 := x, v_2 := y, v_3, \ldots, v_n := z)$, called a *canonical ordering*, with the following properties for each $i \in \{3, 4, \ldots, n\}$, where G_i is the embedded subgraph of $\mathcal{E}$ induced by $\{v_1, v_2, \ldots, v_i\}$, and C_i is the subgraph of $\mathcal{E}$ induced by the edges on the boundary of the outer face of G_i (see Figure 1(a)):
(a) C_i is a cycle containing xy; (b) G_i is 2-connected and *internally* 3-*connected* (that is, removing any two interior vertices of G_i does not disconnect it); (c) v_i is a vertex of C_i with at least two neighbours in C_{i-1}, and these neighbours are consecutive on C_{i-1}.

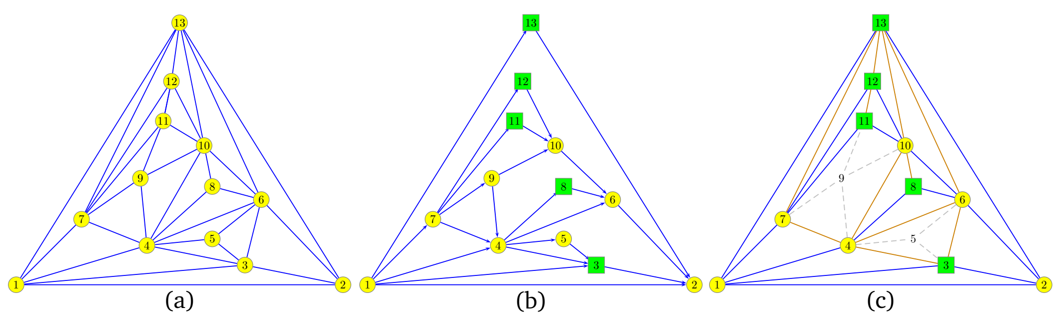

Fig. 1. (a) A canonical ordering of $\mathcal{E}$. (b) The frame $\mathcal{F}$; the vertices in S, which form a largest antichain in $\preceq$, are drawn as squares. (c) The graph $\mathcal{H}$; the vertices in $R \subseteq S$ are drawn as squares.

We now introduce a new combinatorial structure. The *frame* $\mathcal{F}$ of $\mathcal{E}$ is the oriented subgraph of $\mathcal{E}$ (see Figure 1(b)) with vertex set $V(\mathcal{F}) := V(\mathcal{E})$, where xy is an arc of $\mathcal{F}$, and for each $i \in \{3, 4, \ldots, n\}$, if p and p' are the first and the last neighbours of v_i along the path in C_{i-1} from x to y not containing the edge xy, then pv_i and $v_i p'$ are arcs in $\mathcal{F}$. We call p the *left predecessor* of v_i and p' the *right predecessor* of v_i. $\mathcal{F}$ defines a partial order $\preceq$ on $V(\mathcal{F})$, where $v \prec w$ whenever there is a directed path from v to w in $\mathcal{F}$. The following two lemmas and Dilworth's Theorem readily imply Theorem 1.1.

Lemma 2.1 *If $\preceq$ has a chain of ℓ vertices, then G can be untangled while keeping $\sqrt{\ell/3}$ vertices fixed.*

Lemma 2.2 *If $\preceq$ has an antichain of t vertices, then G can be untangled while keeping $\sqrt{t}$ vertices fixed.*

3 Big Chain—Proof of Lemma 2.1

A *chord* of a cycle C is an edge that has both endpoints in C, but itself is not an edge of C. A cycle C in an embedded planar graph $\mathcal{E}$ is *externally chordless* if each chordof C is embedded inside of C in $\mathcal{E}$. Spillner and Wolff [4] proved the following result.

Theorem 3.1 ([4]) *Let G be a geometric planar graph and $\mathcal{E}$ an embedded planar graph isomorphic to G. If $\mathcal{E}$ has an externally chordless cycle on ℓ vertices, then G can be untangled while keeping at least $\sqrt{\ell/3}$ vertices fixed.*

Proof. [of Lemma 2.1]. If $\ell \leq 2$ then the claim is trivial. Assume that $\ell \geq 3$. Now $\preceq$ has a maximal chain of size $\ell' \geq \ell$. This chain corresponds to a path P from x to y in $\mathcal{F}$ not including xy. Let C be the cycle consisting of P plus the edge xy. We claim that C is externally chordless in $\mathcal{E}$. Consider a chord v_iv_j of C. Without loss of generality, $i < j$. Thus v_i is in G_{j-1} and v_iv_j is an edge of G_j. The neighbours of v_j in G_{j-1} appear consecutively along the boundary C_{j-1} of G_{j-1}. Let $x_1, \dots, x_d$ be the neighbours of v_j in left-to-right order on C_{j-1}. Thus x_1v_j and v_jx_d are arcs in $\mathcal{F}$. Let uv_j and v_jw be the incoming and outgoing arcs in P at v_j. Then the counterclockwise order of edges incident to v_j in $\mathcal{E}$ is $(u, \dots, x_1, \dots, x_d, \dots, w, \dots)$. In particular, each edge v_jx_ℓ is contained in the closure of the interior of C. Now $v_i = x_\ell$ for some $\ell \in \{1, 2, \dots, d\}$. Thus v_iv_j is an internal chord of C, and C is externally chordless. Hence $\mathcal{E}$ contains an externally chordless cycle on ℓ' vertices, and the result follows from Theorem 3.1. □

4 Big Antichain—Proof of Lemma 2.2

Our goal is to untangle G while keeping a large set R of vertices fixed. The following geometric lemma simplifies this task by allowing us to concentrate on the case in which all the vertices in R are on the y-axis. The proof is based on a straight-forward perturbation scheme.

Lemma 4.1 *Let $\overline{G}$ be an untangling of some geometric planar graph G. Let R be a set of vertices of G such that each vertex of R is on the y-axis in $\overline{G}$ and has the same y-coordinate in $\overline{G}$ as in G. Then there exists an untangling $\overline{G'}$ of G in which the vertices in R are fixed.*

Let S be the set of vertices that comprise a largest antichain in $\preceq$, as illustrated in Figure 1(b). Now consider the given geometric graph G. Assume, by a suitable rotation, that no pair of vertices of G have the same y-coordinate. Let R be a largest subset $R \subseteq S$ such that the y-coordinates of the vertices of R are either monotonically increasing or monotonically decreasing when considered in the order given by σ. By the Erdős-Szekeres Theorem, $|R| \geq \sqrt{|S|}$. Without loss of generality, R is monotonically increasing. In what follows, we untangle G while keeping R fixed.

For each vertex $v \in V(\mathcal{F})$, define $\mathsf{Lroof}(v)$ and $\mathsf{Rroof}(v)$ to be the following directed paths in $\mathcal{F}$. First define $\mathsf{Lroof}(v_1) := \mathsf{Rroof}(v_1) := \emptyset$ and $\mathsf{Lroof}(v_2) := \mathsf{Rroof}(v_2) := \emptyset$. Now for each $i \in \{3, \ldots, n\}$, define $\mathsf{Lroof}(v_i)$ and $\mathsf{Rroof}(v_i)$ recursively by $\mathsf{Lroof}(v_i) := \mathsf{Lroof}(p) \cup \{pv_i\}$ and $\mathsf{Rroof}(v_i) := \{v_i p'\} \cup \mathsf{Rroof}(p)$, where p and p' respectively are the left and right predecessors of v_i. Let $\mathsf{roof}(v_i) := \mathsf{Lroof}(v_i) \cup \mathsf{Rroof}(v_i)$. Let $\mathcal{H}$ be the subgraph of $\mathcal{E}$ induced by $V(\mathcal{H}) := \cup\{\mathsf{roof}(w) : w \in R\}$, as illustrated in Figure 1(c).

Lemma 4.2 *The geometric planar graph $G[V(\mathcal{H})]$ can be untangled such that each vertex of R is on the y-axis and has the same y-coordinate in the untangling as in $G[V(\mathcal{H})]$. Moreover, all the internal faces of the untangling are star-shaped and the path on its outer face from x to y not containing xy is strictly x-monotone.*

Before proving Lemma 4.2, we show that it implies Lemma 2.2 when coupled with the following theorem by Hong and Nagamochi [2].

Theorem 4.3 ([2]) *Consider a 3-connected embedded planar graph $\mathcal{E}$, with outer facial cycle C. For every star-shaped geometric cycle $\overline{C}$ and isomorphic mapping from $V(C)$ to $V(\overline{C})$, there is a crossing-free geometric graph $\overline{\mathcal{E}}$ isomorphic to $\mathcal{E}$ with $\overline{C}$ as its outer face and respecting the vertex mapping.*

Proof. [of Lemma 2.2.] There is a maximal antichain S in $\preceq$ of size $t' \geq t$. Thus $|R| \geq \sqrt{t}$, and by Lemma 4.2, $G[V(\mathcal{H})]$ can be untangled such that the vertices of R are on the y-axis and their y-coordinates are preserved. If $z \notin R$, then assign x- and y-coordinates to z, and connect z to its neighbours in $\mathcal{H}$, such that the resulting geometric graph H is crossing-free and all the internal faces of H are star-shaped. H is an untangling of $G[V(\mathcal{H}) \cup \{z\}]$.

Now we place the remaining free vertices (the vertices in $V(G) \setminus V(H)$). Partition $V(G) \setminus V(H)$ into sets I_j ($j \in \{1, 2, \ldots, |E(H)| - |V(H)| + 1\}$), where each vertex in I_j is inside the cycle in $\mathcal{E}$ determined by the internal face f_j of H. For each internal face f_j of H, let G^j be the subgraph of $\mathcal{E}$ with vertex set $V(f_j) \cup I_j$, and comprised of the edges of the cycle f_j, the edges in $\mathcal{E}[I_j]$, and the edges between $V(f_j)$ and I_j. Each f_j is star-shaped in H, by Lemma 4.2. The proof of that G^j is 3-connected is simple and we omit it due to space limitations. Applying Theorem 4.3 to embed each subgraph G^j yields an untangling of G in which the vertices in R are on the y-axis and their y-coordinates are preserved. Applying Lemma 4.1 completes the proof. □

Proof. [of Lemma 4.2] We start by proving some properties of the roofs of vertices in R. Consider two incomparable vertices u and v in R, where $u <_\sigma v$. Let x' be a vertex of $\mathcal{F}$ such that $x' \in \mathsf{Lroof}(u)$ and $x' \in \mathsf{Lroof}(v)$, and the vertex following x' in $\mathsf{Lroof}(u)$ is not the same as the vertex following x' in $\mathsf{Lroof}(v)$. Similarly, let y' be a vertex of $\mathcal{F}$ such that $y' \in \mathsf{Rroof}(u)$ and $y' \in \mathsf{Rroof}(v)$, and the vertex before y' in $\mathsf{Rroof}(u)$ is not the same as the vertex before y' in $\mathsf{Rroof}(v)$. Such vertices, x' and y', exist since u and v are incomparable in $\mathcal{F}$. Then the paths between x and x' both equal $\mathsf{Lroof}(x')$. Similarly, the paths between y' and y both equal $\mathsf{Rroof}(y')$. The path between x' and y' in $\mathsf{roof}(u)$ contains u, the path between x' and y' in $\mathsf{roof}(v)$ contains v, and the two paths have only x' and y' in common. Finally, u is inside the cycle determined by $\mathsf{roof}(v)$ and the edge xy in $\mathcal{F}$.

We proceed by induction on the number of vertices in R, but require a somewhat stronger inductive hypothesis. A simple strictly x-monotone polygonal chain C is *ε-ray-monotone* from a point $p = (x_p, y_p)$ if for every point $r = (x_p, y_p+t)$ with $t \geq \varepsilon$, and every point $q \in C$, $\overleftrightarrow{rq} \cap C = \emptyset$, where $\overleftrightarrow{rq}$ is the *open* line-segment with endpoints r and q. Under this definition, if C is ε-ray-monotone from p then C is ε-ray-monotone from every point $q = (x_p, y_p + t)$, $t > 0$, above p. Furthermore, there exists a value $\delta = \delta(p, C, \varepsilon)$ such that C is ε-ray-monotone from every point p' at distance at most δ from p.

Let ε' be the minimum difference between the y-coordinates of some pair of vertices in R. Below we construct an untangling $\overline{\mathcal{H}}$ of $G[V(\mathcal{H})]$ that satisfies the following property (in addition to the conditions of the lemma): If $|R| > 0$ then the outer face of $\overline{\mathcal{H}}$ is bounded by the edge xy and a path C from x to y such that $C \cap R = \{v\}$, for some vertex $v \in R$, and C is ε-ray-monotone from v for some $\varepsilon < \varepsilon'$.

For the base case, with $|R| = 0$, $\mathcal{H}$ consists of the single edge xy, which can be untangled by placing x at $(-1, t)$ and y at $(1, t)$, where t is less than every y-coordinate in G. Now assume that $|R| \geq 1$. Let v be the vertex in R, right-most in the total order σ. If $|R| = 1$ then let $\mathcal{H}'$ be the subgraph of $\mathcal{H}$ induced by $\{x, y\}$. Otherwise $|R| > 1$ and let $\mathcal{H}'$ be the subgraph of $\mathcal{H}$ induced by $\cup\{\mathsf{roof}(u) : u \in R \setminus v\}$. By induction, there is an untangling $\overline{\mathcal{H}'}$ of $G[V(\mathcal{H}')]$ that satisfies the inductive hypothesis. It remains to place v and the vertices of $\mathsf{roof}(v)$ that are not yet placed. These vertices form a path P from some vertex x' of $\mathcal{H}'$ to v to some vertex y' of $\mathcal{H}'$.

The conditions of the lemma specify the location of v. In particular, v is on the y-axis, with its y-coordinate equal to its y-coordinate in G. The inductive hypothesis guarantees that the vertex v and every point sufficiently close to v can see every vertex on the outer face of $\overline{\mathcal{H}'}$. Note that if $|R| > 1$, then directly below v, on the y-axis, is a vertex $u \in R$. Since u is on the y-axis and the outer face of $\overline{\mathcal{H}'}$ is strictly x-monotone, the x-coordinate of x' is less than 0 and that the x-coordinate of y' is greater than 0.

To obtain the crossing-free geometric graph $\overline{\mathcal{H}}$, draw a unit circle c containing v, whose center is on the y-axis and below v, and place the interior vertices of P on c sufficiently close to v so that: (1) the path on the outer face of $\overline{\mathcal{H}}$ from x to y not containing xy is strictly x-monotone, (2) all interior vertices of P see all other vertices of P in $\overline{\mathcal{H}}$, (3) all interior vertices of P see all vertices on the outer face of $\overline{\mathcal{H}'}$ between x' and y', and (4) the path on the outer face of $\overline{\mathcal{H}}$ from x to y not containing xy is ε-ray-monotone from v for some $\varepsilon < \varepsilon'$. It is simple to verify that all four conditions can be achieved.

Consider the path in $\overline{\mathcal{H}'}$ from x to y not containing xy along the outer face of $\overline{\mathcal{H}'}$. This path is comprised of the same vertices and edges as a directed path from x to y in $\mathcal{F}$. Thus, as in the proof of Lemma 2.1, the outer face of $\overline{\mathcal{H}'}$ has no outer chords in $\overline{\mathcal{H}}$. Therefore, an edge of $\overline{\mathcal{H}}$ that is not an edge of $\overline{\mathcal{H}'}$ is either an edge on P, or it is an edge accounted for in Conditions (2) or (3). Thus $\overline{\mathcal{H}}$ is crossing-free. The vertices in R are on the y-axis and have the same y-coordinates in G as in $\overline{\mathcal{H}}$. Conditions (1) and (4) imply that the path between x and y on the outer face of $\overline{\mathcal{H}}$ is strictly x-monotone. The internal faces of $\overline{\mathcal{H}}$ are star-shaped since the only faces in $\overline{\mathcal{H}}$ not present in $\overline{\mathcal{H}'}$ have interior vertices of P on their boundary, and conditions (2) and (3) imply that each such face is star-shaped from some interior vertex of P. □

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