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Bad news for chordal partitions

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Abstract

Reed and Seymour [1998] asked whether every graph has a partition into induced connected nonempty bipartite subgraphs such that the quotient graph is chordal. If true, this would have significant ramifications for Hadwiger's Conjecture. We prove that the answer is "no." In fact, we show that the answer is still "no" for several relaxations of the question.

KEYWORDS

chordal partition, graph, Hadwiger's conjecture, minor, partition

1 | INTRODUCTION

Hadwiger's Conjecture [9] says that for all $t \geq 0$ every graph with no K_{t+1}-minor is t-colorable. This conjecture is easy for $t \leq 3$, is equivalent to the 4-color theorem for $t = 4$, is true for $t = 5$ [18], and is open for $t \geq 6$. The best known upper bound on the chromatic number is $O(t\sqrt{\log t})$, independently due to Kostochka [14,15] and Thomason [21,22]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [20] for a survey.

Throughout this article, we employ standard graph-theoretic definitions (see [4]), with one important exception: we say that a graph G *contains* a graph H if H is isomorphic to an induced subgraph of G.

Motivated by Hadwiger's Conjecture, Reed and Seymour [17] introduced the following definitions[1]. A *vertex-partition*, or simply *partition*, of a graph G is a set $\mathcal{P}$ of nonempty induced subgraphs of G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $P, Q \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in P is adjacent in G to some vertex in Q. A partition of G is *connected* if each part is connected. We (almost) only consider connected partitions. In this case, the quotient is the minor of G obtained by contracting each part into a single vertex. A partition is *chordal* if it is connected and the quotient is chordal (i.e. contains no induced cycle of length at least four). Every graph has a chordal partition (with a 1-vertex quotient). Chordal partitions are a useful tool when studying graphs G with no K_{t+1} minor. Then for every connected partition $\mathcal{P}$ of G, the quotient $G/\mathcal{P}$ contains no K_{t+1}, so if in addition $\mathcal{P}$ is chordal, then $G/\mathcal{P}$ is t-colorable (since chordal graphs are perfect). Reed and Seymour [17] asked the following question (repeated in [13,20]).

Question 1. Does every graph have a chordal partition such that each part is bipartite?

If true, this would imply that every graph with no K_{t+1}-minor is $2t$-colorable, by taking the product of the t-coloring of the quotient with the 2-coloring of each part. This would be a major breakthrough for Hadwiger's Conjecture. The purpose of this note is to answer Reed and Seymour's question in the negative. In fact, we show the following stronger result.

Theorem 2. *For every integer $k \geqslant 1$ there is a graph G, such that for every chordal partition $\mathcal{P}$ of G, some part of $\mathcal{P}$ contains K_k. Moreover, for every integer $t \geqslant 4$ there is a graph G with tree-width at most $t-1$ (and thus with no K_{t+1}-minor) such that for every chordal partition $\mathcal{P}$ of $\mathcal{P}$ contains a complete graph on at least $\lfloor (3t-11)^{1/3} \rfloor$ vertices.*

Theorem 2 says that it is not possible to find a chordal partition in which each part has bounded chromatic number. What if we work with a larger class of partitions? The following natural class arises. A partition of a graph is *perfect* if it is connected and the quotient graph is perfect. If $\mathcal{P}$ is a perfect partition of a K_{t+1}-minor free graph G, then $G/\mathcal{P}$ contains no K_{t+1} and is therefore t-colorable. So if every part of $\mathcal{P}$ has small chromatic number, then we can control the chromatic number of G. We are led to the following relaxation of Question 1: does every graph have a perfect partition in which every part has bounded chromatic number? Unfortunately, this is not the case.

Theorem 3. *For every integer $k \geqslant 1$ there is a graph G, such that for every perfect partition $\mathcal{P}$ of G, some part of $\mathcal{P}$ contains K_k. Moreover, for every integer $t \geqslant 6$ there is a graph G with tree-width at most $t-1$ (and thus with no K_{t+1}-minor), such that for every perfect partition $\mathcal{P}$ of G, some part of $\mathcal{P}$ contains a complete graph on at least $\lfloor (\frac{3}{2}t-8)^{1/3} \rfloor$ vertices.*

Theorems 2 and 3 say that it is hopeless to improve on the $O(t\sqrt{\log t})$ bound for the chromatic number of K_t-minor-free graphs using chordal or perfect partitions directly. Indeed, the best possible upper bound on the chromatic number using the above approach would be $O(t^{4/3})$ (since the quotient is t-colorable, and the best possible upper bound on the chromatic number of the parts would be $O(t^{1/3})$.)

What about using an even larger class of partitions? Chordal graphs contain no 4-cycle, and perfect graphs contain no 5-cycle. These are the only properties of chordal and perfect graphs used in the proofs of Theorems 2 and 3. Thus the following result is a qualitative generalization of both Theorems 2 and 3. It says that there is no hereditary class of graphs for which the above coloring strategy works.

Theorem 4. *For every integer $k \geqslant 1$ and graph H, there is a graph G, such that for every connected partition $\mathcal{P}$ of G, either some part of $\mathcal{P}$ contains K_k or the quotient $G/\mathcal{P}$ contains H.*

Before presenting the proofs, we mention some applications of chordal partitions and related topics. Chordal partitions have proven to be a useful tool in the study of the following topics for K_{t+1}-minor-free graphs: cops and robbers pursuit games [1], fractional coloring [13,17], generalized coloring numbers [11], and defective and clustered coloring [12]. These articles show that every graph with no K_{t+1} minor has a chordal partition in which each part has desirable properties. For example, in [17], each part has a stable set on at least half the vertices, and in [12], each part has maximum degree $O(t)$ and is 2-colorable with monochromatic components of size $O(t)$.

Several articles [7,16,23] have shown that graphs with tree-width k have chordal partitions in which the quotient is a tree, and each part induces a subgraph with tree-width $k-1$, among other properties. Such partitions have been used for queue and track layouts [7] and nonrepetitive graph coloring [16]. A *tree partition* is a (not necessarily connected) partition of a graph whose quotient is a tree; these have also been widely studied [2,3,5,6,8,10,19,24]. Here the goal is to have few vertices in each part of the partition. For example, a referee of [5] proved that every graph with tree-width k and maximum degree Δ has a tree partition with $O(k\Delta)$ vertices in each part.

2 | CHORDAL PARTITIONS: PROOF OF THEOREM 2

Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a partition of a graph G, and let X be an induced subgraph of G. Then the *restriction* of $\mathcal{P}$ to X is the partition of X defined by

$$\mathcal{P}\langle X \rangle := \{G[V(P_i) \cap V(X)] : i \in \{1, \ldots, m\}, V(P_i) \cap V(X) \neq \emptyset\}.$$

Note that the restriction of a connected partition to a subgraph need not be connected. The following lemma gives a scenario where the restriction is connected.

Lemma 5. *Let X be an induced subgraph of a graph G, such that the neighborhood of each component of $G - V(X)$ is a clique (in X). Let $\mathcal{P}$ be a connected partition of G with quotient $G/\mathcal{P}$. Then $\mathcal{P}\langle X \rangle$ is a connected partition of X, and the quotient of $\mathcal{P}\langle X \rangle$ is the subgraph of $G/\mathcal{P}$ induced by those parts that intersect X.*

Proof. We first prove that for every connected subgraph G' of G, if $V(G') \cap V(X) \neq \emptyset$, then $G'[V(G') \cap V(X)]$ is connected. Consider nonempty sets A, B that partition $V(G') \cap V(X)$. Let P be a shortest path from A to B in G'. Then no internal vertex of P is in $V(X)$. If P has an internal vertex, then all its interior belongs to one component C of $G - V(X)$, implying the endpoints of P are in the neighborhood of C and are therefore adjacent, a contradiction. Thus P has no interior, and hence $G'[V(G') \cap V(X)]$ is connected.

Apply this observation with each part of $\mathcal{P}$ as G'. It follows that $\mathcal{P}\langle X \rangle$ is a connected partition of X. Moreover, if adjacent parts P and Q of $\mathcal{P}$ both intersect X, then by the above observation with $G' = G[V(P) \cup V(Q)]$, there is an edge between $V(P) \cap V(X)$ and $V(Q) \cap V(X)$. Conversely, if there is an edge between $V(P) \cap V(X)$ and $V(Q) \cap V(X)$ for some parts P and Q of $\mathcal{P}$, then PQ is an edge of $G/\mathcal{P}$. Thus the quotient of $\mathcal{P}\langle X \rangle$ is the subgraph of $G/\mathcal{P}$ induced by those parts that intersect X. ■

The next lemma with $r = 1$ implies Theorem 2. To obtain the second part of Theorem 2 apply Lemma 6 with $k = \lfloor (3t - 11)^{1/3} \rfloor$, in which case $s(k, 1) \leqslant t$.

Lemma 6. *For all integers $k \geqslant 1$ and $r \geqslant 1$, if*

$$s(k, r) := \tfrac{1}{3}(k^3 - k) + (r - 1)k + 4,$$

then there is a graph $G(k, r)$ with tree-width at most $s(k, r) - 1$ (and thus with no $K_{s(k,r)+1}$-minor), such that for every chordal partition $\mathcal{P}$ of G, either:

(1) *G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices, or*

(2) *some part of $\mathcal{P}$ contains K_{k+1}.*

Proof. Note that $s(k, r)$ is the upper bound on the size of the bags in the tree-decomposition of $G(k, r)$ that we construct. We proceed by induction on k and then r. When $k = r = 1$, the graph with one vertex satisfies (1) for every chordal partition and has a tree-decomposition with one bag of size $1 < s(1, 1)$.

First we prove that the $(k, 1)$ and (k, r) cases imply the $(k, r + 1)$ case. Let $A := G(k, 1)$ and $B := G(k, r)$. Let G be obtained from A as follows. For each k-clique C in A, add a copy B_C of B (disjoint from the current graph), where C is complete to B_C. We claim that G has the claimed properties of $G(k, r + 1)$.

By assumption, in every chordal partition of A some part contains K_k, A has a tree-decomposition with bags of size at most $s(k, 1)$, and for each k-clique C in A, there is a tree-decomposition of B_C with bags of size at most $s(k, r)$. For every tree-decomposition of a graph and for each clique C, there

is a bag containing C. Add an edge between the node corresponding to a bag containing C in the tree-decomposition of A and any node of the tree in the tree-decomposition of B_C, and add C to every bag of the tree-decomposition of B_C. We obtain a tree-decomposition of G with bags of size at most $\max\{s(k,1), s(k,r)+k\} = s(k,r)+k = s(k,r+1)$, as desired.

Consider a chordal partition $\mathcal{P}$ of G. By Lemma 5, $\mathcal{P}\langle A \rangle$ is a connected partition of A, and the quotient of $\mathcal{P}\langle A \rangle$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect A. Since $G/\mathcal{P}$ is chordal, the quotient of $\mathcal{P}\langle A \rangle$ is chordal. Since $A = G(k,1)$, by induction, $\mathcal{P}\langle A \rangle$ satisfies (1) with $r = 1$ or (2). If outcome (2) holds, then some part of $\mathcal{P}$ contains K_{k+1} and outcome (2) holds for G.

Now assume that $\mathcal{P}\langle A \rangle$ satisfies outcome (1) with $r = 1$; that is, some part P of $\mathcal{P}$ contains some k-clique C of A. If some vertex of B_C is in P, then P contains K_{k+1} and outcome (2) holds for G. Now assume that no vertex of B_C is in P. Since each part of $\mathcal{P}$ is connected, the parts of $\mathcal{P}$ that intersect B_C do not intersect $G - V(B_C)$. Thus, $\mathcal{P}\langle B_C \rangle$ is a connected partition of B_C, and the quotient of $\mathcal{P}\langle B_C \rangle$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect B_C, and is therefore chordal. Since $B = G(k,r)$, by induction, $\mathcal{P}\langle B_C \rangle$ satisfies (1) or (2). If outcome (2) holds, then the same outcome holds for G. Now assume that outcome (1) holds for B_C. Thus B_C contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices. None of these parts are P. Since C is complete to B_C, G contains a $K_{k(r+1)}$ subgraph intersecting each of $r+1$ distinct parts of $\mathcal{P}$ in k vertices, and outcome (1) holds for G. Hence G has the claimed properties of $G(k, r+1)$.

It remains to prove the $(k,1)$ case for $k \geqslant 2$. By induction, we may assume the $(k-1, k+1)$ case. Let $A := G(k-1, k+1)$. As illustrated in Figure 1, let G be obtained from A as follows: for each set $C = \{C_1, \ldots, C_{k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)(k+1)}$, add a K_{k+1} subgraph B_C (disjoint from the current graph), whose i-th vertex is adjacent to every vertex in C_i. We claim that G has the claimed properties of $G(k,1)$.

By assumption, A has a tree-decomposition with bags of size at most $s(k-1, k+1)$. For each set $C = \{C_1, \ldots, C_{k+1}\}$ of pairwise-disjoint $(k-1)$-cliques in A, whose union induces $K_{(k-1)(k+1)}$, choose a node x corresponding to a bag of the tree-decomposition of A containing $C_1 \cup \cdots \cup C_{k+1}$, and add a new node adjacent to x with corresponding bag $V(B_C) \cup C_1 \cup \cdots \cup C_{k+1}$. We obtain a tree-decomposition of G with bags of size at most $\max\{s(k-1, k+1), (k+1)k\} = s(k-1, k+1) = s(k,1)$, as desired.

Consider a chordal partition $\mathcal{P}$ of G. By Lemma 5, $\mathcal{P}\langle A \rangle$ is a connected partition of A and the quotient of $\mathcal{P}\langle A \rangle$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect A, and is therefore chordal. Since $A = G(k-1, k+1)$, by induction, $\mathcal{P}\langle A \rangle$ satisfies (1) or (2). If outcome (2) holds for $\mathcal{P}\langle A \rangle$, then some part of $\mathcal{P}$ contains K_k and outcome (1) holds for G (with $r = 1$). Now assume that
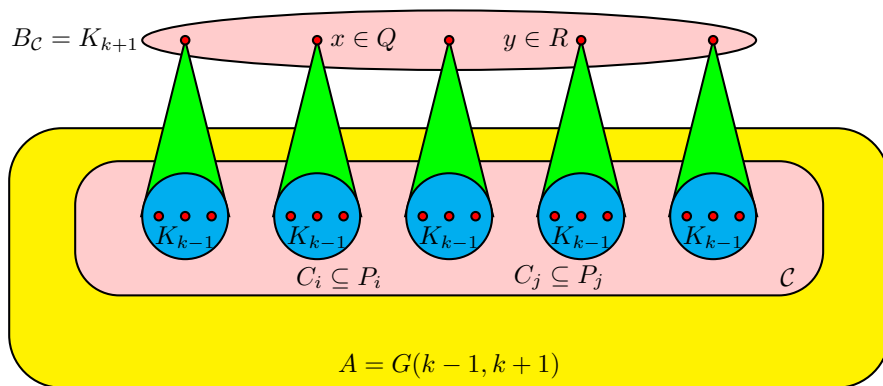

FIGURE 1 Construction of $G(k,1)$ in Lemma 6 [Color figure can be viewed at wileyonlinelibrary.com]

outcome (1) holds for $\mathcal{P}\langle A \rangle$. Thus A contains a $K_{(k-1)(k+1)}$ subgraph intersecting each of $k + 1$ distinct parts $P_1, \ldots, P_{k+1}$ of $\mathcal{P}$ in $k - 1$ vertices. Let C_i be the corresponding $(k - 1)$-clique in P_i. Let $C :=$ $\{C_1, \ldots, C_{k+1}\}$ and $\widehat{C} := C_1 \cup \cdots \cup C_{k+1}$.

If for some $i \in \{1, \ldots, k + 1\}$, the neighbor of C_i in B_C is in P_i, then P_i contains K_k and outcome (1) holds for G. Now assume that for each $i \in \{1, \ldots, k + 1\}$, the neighbor of C_i in B_C is not in P_i. Suppose that some vertex x in B_C is in P_i for some $i \in \{1, \ldots, k + 1\}$. Then since P_i is connected, there is a path in G between C_i and x avoiding the neighborhood of C_i in B_C. Every such path intersects $\widehat{C} \setminus C_i$, but none of these vertices are in P_i. Thus, no vertex in B_C is in $P_1 \cup \cdots \cup P_{k+1}$. If B_C is contained in one part, then outcome (2) holds. Now assume that there are vertices x and y of B_C in distinct parts Q and R of $\mathcal{P}$. Then x is adjacent to every vertex in C_i and y is adjacent to every vertex in C_j, for some distinct $i, j \in \{1, \ldots, k + 1\}$. Observe that (Q, R, P_j, P_i) is a 4-cycle in $G/\mathcal{P}$. Moreover, there is no QP_j edge in $G/\mathcal{P}$ because $(\widehat{C} \setminus C_j) \cup \{y\}$ separates $x \in Q$ from $C_j \subseteq P_j$, and none of these vertices are in $Q \cup P_j$. Similarly, there is no RP_i edge in $G/\mathcal{P}$. Hence (Q, R, P_j, P_i) is an induced 4-cycle in $G/\mathcal{P}$, which contradicts the assumption that $\mathcal{P}$ is a chordal partition. Therefore G has the claimed properties of $G(k, 1)$. ∎

3 | PERFECT PARTITIONS: PROOF OF THEOREM 3

The following lemma with $r = 1$ implies Theorem 3. To obtain the second part of Theorem 3 apply Lemma 6 with $k = \lfloor (\frac{3}{2}t - 8)^{1/3} \rfloor$, in which case $t(k, 1) \leqslant t$. The proof is very similar to Lemma 6 except that we force C_5 in the quotient instead of C_4.

Lemma 7. *For all integers $k \geqslant 1$ and $r \geqslant 1$, if*

$$t(k, r) := \tfrac{2}{3}(k^3 - k) + (r - 1)k + 6,$$

then there is a graph $G(k, r)$ with tree-width at most $t(k, r) - 1$ (and thus with no $K_{t(k,r)+1}$-minor), such that for every perfect partition $\mathcal{P}$ of G, either:

(1) *G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices, or*

(2) *some part of $\mathcal{P}$ contains K_{k+1}.*

Proof. Note that $t(k, r)$ is the upper bound on the size of the bags in the tree-decomposition of $G(k, r)$ that we construct. We proceed by induction on k and then r. For the base case, the graph with one vertex satisfies (1) for $k = r = 1$ and has a tree-decomposition with one bag of size $1 < t(1, 1)$. The proof that the $(k, 1)$ and (k, r) cases imply the $(k, r + 1)$ case is identical to the analogous step in the proof in Lemma 6, so we omit it.

It remains to prove the $(k, 1)$ case for $k \geqslant 2$. By induction, we may assume the $(k - 1, 2k + 1)$ case. Let $A := G(k - 1, 2k + 1)$. Let B be the graph consisting of two copies of K_{k+1} with one vertex in common. Note that $|V(B)| = 2k + 1$. As illustrated in Figure 2, let G be obtained from A as follows: for each set $C = \{C_1, \ldots, C_{2k+1}\}$ of pairwise-disjoint $(k - 1)$-cliques in A, whose union induces $K_{(k-1)(2k+1)}$, add a subgraph B_C isomorphic to B (disjoint from the current graph), whose i-th vertex is adjacent to every vertex in C_i. We claim that G has the claimed properties of $G(k, 1)$.

By assumption, A has a tree-decomposition with bags of size at most $t(k - 1, 2k + 1)$. For each set $C = \{C_1, \ldots, C_{2k+1}\}$ of pairwise-disjoint $(k - 1)$-cliques in A, whose union induces $K_{(k-1)(2k+1)}$, choose a node x corresponding to a bag containing $C_1 \cup \cdots \cup C_{2k+1}$ in the tree-decomposition of A, and add a new node adjacent to x with corresponding bag $V(B_C) \cup C_1 \cup \cdots \cup C_{2k+1}$. We obtain a
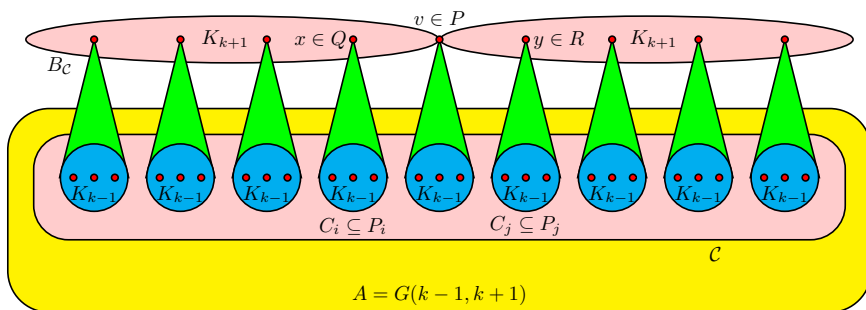
FIGURE 2 Construction of $G(k, 1)$ in Lemma 7 [Color figure can be viewed at wileyonlinelibrary.com]

tree-decomposition of G with bags of size at most $\max\{t(k-1, 2k+1), (2k+1)k\} = t(k-1, 2k+1) = t(k, 1)$, as desired.

Consider a perfect partition $\mathcal{P}$ of G. By Lemma 5, $\mathcal{P}\langle A \rangle$ is a connected partition of A and the quotient of $\mathcal{P}\langle A \rangle$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect A, and is therefore perfect. Recall that $A = G(k-1, 2k+1)$. If outcome (2) holds for $\mathcal{P}\langle A \rangle$, then some part of $\mathcal{P}$ contains K_k and outcome (1) holds for G (with $r = 1$). Now assume that outcome (1) holds for $\mathcal{P}\langle A \rangle$. Thus A contains a $K_{(k-1)(2k+1)}$ subgraph intersecting each of $2k+1$ distinct parts $P_1, \ldots, P_{2k+1}$ of $\mathcal{P}$ in $k-1$ vertices. Let C_i be the corresponding $(k-1)$-clique in P_i. Let $C := \{C_1, \ldots, C_{2k+1}\}$ and $\hat{C} := C_1 \cup \cdots \cup C_{2k+1}$.

If for some $i \in \{1, \ldots, 2k+1\}$, the neighbor of C_i in B_C is in P_i, then P_i contains a K_k subgraph and outcome (1) holds for G. Now assume that for each $i \in \{1, \ldots, 2k+1\}$, the neighbor of C_i in B_C is not in P_i. Suppose that some vertex x in B_C is in P_i for some $i \in \{1, \ldots, k+1\}$. Then since P_i is connected, there is a path in G between C_i and x avoiding the neighborhood of C_i in B_C. Every such path intersects $\hat{C} \setminus C_i$, but none of these vertices are in P_i. Thus, no vertex in B_C is in $P_1 \cup \cdots \cup P_{2k+1}$.

By construction, B_C consists of two $(k+1)$-cliques B^1 and B^2, intersecting in one vertex v. Say v is in part P of $\mathcal{P}$. If $B^1 \subseteq V(P)$, then outcome (2) holds. Now assume that there is a vertex x of B^1 in some part Q distinct from P. Similarly, assume that there is a vertex y of B^2 in some part R distinct from P. Now, $Q \neq R$, since $\hat{C} \cup \{v\}$ separates x and y, and none of these vertices are in $Q \cup R$. By construction, x is adjacent to every vertex in C_i and y is adjacent to every vertex in C_j, for some distinct $i, j \in \{1, \ldots, 2k+1\}$. Observe that (Q, P, R, P_j, P_i) is a 5-cycle in $G/\mathcal{P}$. Moreover, there is no QP_j edge in $G/\mathcal{P}$ because $(\hat{C} \setminus C_j) \cup \{y\}$ separates $x \in Q$ from $C_j \subseteq P_j$, and none of these vertices are in $Q \cup P_j$. Similarly, there is no RP_i edge in $G/\mathcal{P}$. There is no PP_j edge in $G/\mathcal{P}$ because $(\hat{C} \setminus C_j) \cup \{y\}$ separates $v \in P$ from $C_j \subseteq P_j$, and none of these vertices are in $P \cup P_j$. Similarly, there is no PP_i edge in $G/\mathcal{P}$. Hence (Q, P, R, P_j, P_i) is an induced 5-cycle in $G/\mathcal{P}$, which contradicts the assumption that $\mathcal{P}$ is a perfect partition. Therefore G has the claimed properties of $G(k, 1)$. ∎

4 | GENERAL PARTITIONS: PROOF OF THEOREM 4

To prove Theorem 4, we show the following stronger result, in which G only depends on $|V(H)|$.

Lemma 8. *For all integers $k, t, r \geqslant 1$, there is a graph $G = G(k, t, r)$, such that for every connected partition $\mathcal{P}$ of G either:*

(1) *G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathcal{P}$ in k vertices, or*

(2) *$G/\mathcal{P}$ contains every t-vertex graph, or*

(3) *some part of $\mathcal{P}$ contains K_{k+1}.*

Proof. We proceed by induction on $k + t$ and then r. We first deal with two base cases. First suppose that $t = 1$. Let $G := G(k, 1, r) := K_1$. Then for every partition $\mathcal{P}$ of G, the quotient $G/\mathcal{P}$ has at least one vertex, and (2) holds. Now assume that $t \geqslant 2$. Now suppose that $k = 1$. Let $G := G(1, t, r) := K_r$. Then for every connected partition $\mathcal{P}$ of G, if some part of $\mathcal{P}$ contains an edge, then (3) holds; otherwise each part is a single vertex, and (1) holds. Now assume that $k \geqslant 2$.

The proof that the $(k, t, 1)$ and (k, t, r) cases imply the $(k, t, r + 1)$ case is identical to the analogous step in the proof in Lemma 6, so we omit it.

It remains to prove the $(k, t, 1)$ case for $k \geqslant 2$ and $t \geqslant 2$. By induction, we may assume the $(k, t - 1, 1)$ case and the $(k - 1, t, r)$ case for all r. Let $B := G(k, t - 1, 1)$ and $n := |V(B)|$. Let $S^1, \ldots, S^{2^n}$ be the distinct subsets of $V(B)$. Let $A := G(k - 1, t, 2^n)$. Let G be obtained from A as follows: for each set $C = \{C_1, \ldots, C_{2^n}\}$ of pairwise-disjoint $(k - 1)$-cliques in A, whose union induces $K_{(k-1)2^n}$, add a copy B_C of B (disjoint from the current graph), where C_i is complete to S_C^i for all $i \in \{1, \ldots, 2^n\}$, where we write S_C^i for the subset of $V(B_C)$ corresponding to S^i. We claim that G has the claimed properties of $G(k, t, 1)$.

Consider a connected partition $\mathcal{P}$ of G. By Lemma 5, $\mathcal{P}\langle A \rangle$ is a connected partition of A, and the quotient of $\mathcal{P}\langle A \rangle$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect A. Recall that $A = G(k - 1, t, 2^n)$. If $\mathcal{P}\langle A \rangle$ satisfies outcome (2), then the quotient of $\mathcal{P}\langle A \rangle$ contains every t-vertex graph and outcome (2) is satisfied for G. If outcome (3) holds for $\mathcal{P}\langle A \rangle$, then some part of $\mathcal{P}$ contains K_k and outcome (1) holds for G (with $r = 1$). Now assume that outcome (1) holds for $\mathcal{P}\langle A \rangle$. Thus A contains a $K_{(k-1)2^n}$ subgraph intersecting each of 2^n distinct parts $P_1, \ldots, P_{2^n}$ of $\mathcal{P}$ in $k - 1$ vertices. Let C_i be the corresponding $(k - 1)$-clique in P_i. Let $C := \{C_1, \ldots, C_{2^n}\}$.

If for some $i \in \{1, \ldots, 2^n\}$, some neighbor of C_i in B_C is in P_i, then P_i contains K_k and outcome (1) holds for G. Now assume that for each $i \in \{1, \ldots, 2^n\}$, no neighbor of C_i in B_C is in P_i. Suppose that some vertex x in B_C is in P_i for some $i \in \{1, \ldots, 2^n\}$. Then since P_i is connected, G contains a path between C_i and x avoiding the neighborhood of C_i in B_C. Every such path intersects $C_1 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup \cdots \cup C_{2^n}$, but none of these vertices are in P_i. Thus, no vertex in B_C is in $P_1 \cup \cdots \cup P_{2^n}$. Hence, no part of $\mathcal{P}$ contains vertices in both B_C and in the remainder of G. Therefore, $\mathcal{P}\langle B_C \rangle$ is a connected partition of B_C, and the quotient of $\mathcal{P}\langle B_C \rangle$ equals the subgraph of $G/\mathcal{P}$ induced by those parts that intersect B_C. Since $B = G(k, t - 1, 1)$, by induction, $\mathcal{P}\langle B_C \rangle$ satisfies (1), (2), or (3). If outcome (1) or (3) holds for $\mathcal{P}\langle B_C \rangle$, then the same outcome holds for G. Now assume that outcome (2) holds for $\mathcal{P}\langle B_C \rangle$.

We now show that outcome (2) holds for G. Let H be a t-vertex graph, let v be a vertex of H, and let $N_H(v) = \{w_1, \ldots, w_d\}$. Since outcome (2) holds for $\mathcal{P}\langle B_C \rangle$, the quotient of $\mathcal{P}\langle B_C \rangle$ contains $H - v$. Let $Q_1, \ldots, Q_d$ be the parts corresponding to $w_1, \ldots, w_d$. Then $S_C^i = V(Q_1 \cup \cdots \cup Q_d)$ for some $i \in \{1, \ldots, 2^n\}$. In $G/\mathcal{P}$, the vertex corresponding to P_i is adjacent to $Q_1, \ldots, Q_d$ and to no other vertices corresponding to parts contained in B_C. Thus, including P_i, $G/\mathcal{P}$ contains H and outcome (2) holds for $\mathcal{P}$. Hence G has the claimed properties of $G(k, t, 1)$. ∎

ENDNOTE

[1] Reed and Seymour [17] used different terminology: "chordal decomposition" instead of chordal partition, and "touching pattern" instead of quotient.

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