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No-Three-in-Line-in-3D[1]

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Abstract. The *no-three-in-line* problem, introduced by Dudeney in 1917, asks for the maximum number of points in the $n \times n$ grid with no three points collinear. Erdős proved that the answer is $\Theta(n)$. We consider the analogous problem in three dimensions, and prove that the maximum number of points in the $n \times n \times n$ grid with no three collinear is $\Theta(n^2)$. This result is generalised by the notion of a 3*D drawing* of a graph. Here each vertex is represented by a distinct gridpoint in $\mathbb{Z}^3$, such that the line-segment representing each edge does not intersect any vertex, except for its own endpoints. Note that edges may cross. A 3D drawing of a complete graph K_n is nothing more than a set of n gridpoints with no three collinear. A slight generalisation of our first result is that the minimum volume for a 3D drawing of K_n is $\Theta(n^{3/2})$. This compares favourably with $\Theta(n^3)$ when edges are not allowed to cross. Generalising the construction for K_n, we prove that every k-colourable graph on n vertices has a 3D drawing with $\mathcal{O}(n\sqrt{k})$ volume, which is optimal for the k-partite Turán graph.

Key Words. Graph drawing, No-three-in-line problem, Three-dimensional graph drawing.

1. Introduction. In 1917 Dudeney [7] asked for the maximum number of points in the $n \times n$ grid with no three points collinear? This question, dubbed the *no-three-in-line* problem, has since been widely studied [1], [2], [5], [10]–[17]. A breakthrough came in 1951, when Erdős [10] proved that for every prime p, the set $\{(x, x^2 \bmod p)\colon 0 \le x \le p-1\}$ contains no three collinear points. It follows that the $n \times n$ grid contains $n/2$ points with no three collinear, and for all $\varepsilon > 0$ and $n > n(\varepsilon)$, there are $(1-\varepsilon)n$ points with no three collinear. The result has been improved to $(3/2 - \varepsilon)n$ by Hall et al. [15] using a different construction. Ignoring constant factors, these bounds are optimal since each gridline contains at most two points, and thus the number of points is at most $2n$. Guy and Kelley [14] conjectured that for large n the maximum number of points in the $n \times n$ grid with no three collinear tends to $(2\pi^2/3)^{1/3}n$, which was recently revised to $\pi n/\sqrt{3}$; see [19].

We consider the *no-three-in-line-in-*3*D* problem: what is the maximum number of points in the $n \times n \times n$ grid with no three points collinear? The following is our primary result.

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THEOREM 1. *The maximum number of points in the $n \times n \times n$ grid with no three collinear is $\Theta(n^2)$.*

Cohen et al. [4] considered a similar three-dimensional generalisation of the no-three-in-line problem. They proved that for every prime p, no four points in the set $\{(x, x^2 \bmod p, x^3 \bmod p)\colon 0 \le x \le p-1\}$ are coplanar. It follows that the $n \times n \times n$ grid contains at least $n/2$ and $(1-\varepsilon)n$ points with no four coplanar. Note that $3n$ is an upper bound since each gridplane contains at most three points.

Cohen et al. [4] were motivated by three-dimensional graph visualisation. Let G be an (undirected, finite, simple) graph with vertex set $V(G)$ and edge set $E(G)$. A 3*D drawing* of G represents each vertex by a distinct point in $\mathbb{Z}^3$ (a *gridpoint*), such that with each edge represented by the line segment between its endpoints, the only vertices that an edge intersects are its own endpoints. That is, an edge does not "pass through" a vertex. The *bounding box* of a 3D drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. That is, the volume of a 3D drawing is the number of gridpoints in the bounding box. This definition is formulated so that 2D drawings have positive volume.

Distinct edges in a 3D drawing *cross* if they intersect at a point other than a common endpoint. Based on the observation that the endpoints of a pair of crossing edges are coplanar, Cohen et al. [4] proved that the minimum volume for a crossing-free 3D drawing of K_n is $\Theta(n^3)$. The lower bound follows from the observation that no axis-perpendicular gridplane can contain five vertices, as otherwise there is a planar K_5. Note that it is possible for four vertices to be in a single gridplane, provided that they are not in convex position. Subsequent to the work of Cohen et al. [4], crossing-free 3D drawings have been widely studied; see [4], [6], [8], and [18] for example. This paper initiates the study of volume bounds for 3D drawings of graphs in which crossings are allowed. The following simple observation is immediate:

OBSERVATION 1. *A set V of n gridpoints in $\mathbb{Z}^3$ determines a* 3*D drawing of K_n if and only if no three points in V are collinear.*

Thus the following result is a slight strengthening of Theorem 1:

THEOREM 2. *The minimum volume for a* 3*D drawing of K_n is $\Theta(n^{3/2})$.*

A *k-colouring* of a graph G is an assignment of one of k colours to each vertex, so that adjacent vertices receive distinct colours. The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colourable. The Turán graph $T(n,k)$ is the n-vertex complete k-partite graph with $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ vertices in each colour class. Theorem 2 generalises as follows:

THEOREM 3. *Every k-colourable graph on n vertices has a* 3*D drawing with $\mathcal{O}(n\sqrt{k})$ volume. Moreover, every* 3*D drawing of the Turán graph $T(n,k)$ has $\Omega(n\sqrt{k})$ volume.*

Note that 2D drawings of k-colourable graphs were studied by Wood [21], who proved an $\mathcal{O}(kn)$ area bound, which is best possible for the Turán graph.

The remainder of this paper is organised as follows. In Section 2 we prove the lower bounds in Theorems 2 and 3, which imply the upper bound in Theorem 1. In Section 3 we prove the upper bounds in Theorems 2 and 3, which imply the lower bound in Theorem 1.

2. Lower Bounds. An axis-parallel line through a gridpoint is a *gridline*. A gridline that is parallel to the X-axis (respectively, Y-axis and Z-axis) is an *X-line* (*Y-line* and *Z-line*). An axis-perpendicular plane through a gridpoint is a *gridplane*. A gridplane that is perpendicular to the X-axis (respectively, Y-axis and Z-axis) is an *X-plane* (*Y-plane* and *Z-plane*). Let $[n] := \{1, 2, \dots, n\}$.

LEMMA 1. *There are at most $2n^2$ points in the $n \times n \times n$ grid with no three collinear.*

PROOF. Every X-line contains at most two points, and there are n^2 X-lines. □

Lemma 1 can be generalised to give a universal lower bound on the volume of a 3D drawing.

LEMMA 2. *Every 3D drawing of a graph G has at least $\chi(G)^{3/2}/\sqrt{8}$ volume.*

PROOF. Say G has an $A \times B \times C$ drawing. The vertices on a single Z-line induce a set of paths, as otherwise an edge passes through a vertex. The paths are 2-colourable. Using a distinct pair of colours for each Z-line, we obtain a $2AB$-colouring of G. Thus $\chi(G) \leq 2AB$. Similarly, $\chi(G) \leq 2AC$ and $\chi(G) \leq 2BC$. Thus $8(ABC)^2 \geq \chi(G)^3$, and the volume $ABC \geq \sqrt{\chi(G)^3/8}$. □

LEMMA 3. *Let S be a nonempty set of gridpoints. Let x (respectively, y and z) be the number of X-lines (Y-lines and Z-lines) that contain a point in S. Then $xyz \geq |S|^2$.*

PROOF. Number the Z-planes that contain a point in S by $1, 2, \dots, \ell$. For each $i \in [\ell]$, let z_i be the number of points in S that are in the ith Z-plane, and let x_i (respectively, y_i) be the number of X-lines (Y-lines) in the ith Z-plane that contain a point in S. Thus $x = \sum_i x_i$ and $y = \sum_i y_i$. Observe that $z_i \leq x_i y_i$. Let $z_* := \max\{z_1, \dots, z_\ell\}$. By Lemma 10 in the Appendix, $xyz_* \geq |S|^2$. Since each point in a fixed Z-plane defines a distinct Z-line, $z \geq z_*$. Thus $xyz \geq |S|^2$. □

Note that the bound in Lemma 3 is tight when S is contained in a single gridline.

The following lemma proves the lower bound in Theorem 3.

LEMMA 4. *For all $n \equiv 0 \pmod{k}$, every 3D drawing of $T(n,k)$ has at least $n\sqrt{k/8}$ volume.*

PROOF. Consider an $A \times B \times C$ drawing of $T(n,k)$. Let α_i (respectively, β_i and γ_i) be the number of X-lines (respectively, Y-lines and Z-lines) that contain a vertex coloured i. There are at most two distinct colours represented in each gridline, as otherwise an edge passes through a vertex. There are BC distinct X-lines, and at most α_i X-lines that contain a vertex coloured i. Thus $\sum_i \alpha_i \leq 2BC$. Similarly $\sum_i \beta_i \leq 2AC$ and $\sum_i \gamma_i \leq 2AB$. There are n/k vertices coloured i. Thus $\alpha_i\beta_i\gamma_i \geq n^2/k^2$ by Lemma 3. By Lemma 11 in the Appendix,

$$\left(k\left(\frac{n^2}{k^2}\right)^{1/3}\right)^3 \leq \left(\sum_i (\alpha_i\beta_i\gamma_i)^{1/3}\right)^3 \leq \left(\sum_i \alpha_i\right)\left(\sum_i \beta_i\right)\left(\sum_i \gamma_i\right)$$
$$\leq (2BC)(2AC)(2AB).$$

That is, $n^2 k \leq 8(ABC)^2$, which implies that the volume $ABC \geq n\sqrt{k/8}$. □

Since $\chi(K_n) = n$ and $K_n = T(n, n)$, Lemmata 2 and 4 both prove the lower bound in Theorem 2: every 3D drawing of K_n has volume at least $n^{3/2}/\sqrt{8}$.

3. Upper Bounds. The next lemma is the key idea in our upper bounds. For every prime p, define

$$V_p := \{(x,\ y,\ (x^2 + y^2) \bmod p) \colon\ 0 \leq x, y \leq p - 1\}.$$

LEMMA 5. *The set V_p (p prime) contains three collinear points if and only if $p \equiv 1 \pmod 4$.*

PROOF. The result is trivial for $p = 2$. Now assume that p is odd. Suppose that V_p contains three collinear points a, b and c. Then there exists a vector $\vec{v} = (v_x, v_y, v_z)$ such that $b = k\vec{v} + a$ and $c = \ell\vec{v} + a$, for distinct nonzero integers k and ℓ. (Precisely, $v_x = \gcd(b_x - a_x, c_x - a_x)$, $v_y = \gcd(b_y - a_y, c_y - a_y)$ and $v_z = \gcd(b_z - a_z, c_z - a_z)$.) Since $b \in V_p$,

$$(kv_x + a_x)^2 + (kv_y + a_y)^2 \equiv kv_z + a_z \pmod p.$$

That is, $k^2(v_x^2 + v_y^2) + a_x^2 + a_y^2 \equiv kv_z + a_z - 2k(v_x a_x + v_y a_y) \pmod p$. Since $a \in V_p$, we have $a_x^2 + a_y^2 \equiv a_z \pmod p$. Since p is a prime and $k \neq 0$,

$$k(v_x^2 + v_y^2) \equiv v_z - 2(v_x a_x + v_y a_y) \pmod p.$$

By symmetry, $\ell(v_x^2 + v_y^2) \equiv v_z - 2(v_x a_x + v_y a_y) \pmod p$. Thus,

$$k(v_x^2 + v_y^2) \equiv \ell(v_x^2 + v_y^2) \pmod p.$$

That is, $(k - \ell)(v_x^2 + v_y^2) \equiv 0 \pmod p$. Since $k \neq \ell$ and p is a prime, $v_x^2 + v_y^2 \equiv 0 \pmod p$. Now v_x and v_y are both not zero, as otherwise a, b and c would be in a single Z-line. Without loss of generality, $v_x \neq 0$. Thus v_x has a multiplicative inverse modulo p, and $(v_y v_x^{-1})^2 \equiv -1 \pmod p$. That is, -1 is a quadratic residue. A classical result states that -1 is a quadratic residue modulo an odd prime p if and only if $p \equiv 1 \pmod 4$.

Now we prove the converse. Suppose that $p \equiv 1 \pmod 4$. By the above-mentioned result there is an integer t such that $1 + t^2 \equiv 0 \pmod p$. We can assume that $0 \le t \le (p-1)/2$ as otherwise $p - t$ would do. Thus $(1, t, 0) \in V_p$ and $(2, 2t, 0) \in V_p$, and the three points $\{(0, 0, 0), (1, t, 0), (2, 2t, 0)\}$ are collinear. $\square$

To apply Lemma 5 we need primes $p \not\equiv 1 \pmod 4$.

LEMMA 6 [3], [9].

(a) *For all* $t \in \mathbb{N}$, *there is a prime* $p \not\equiv 1 \pmod 4$ *with* $t \le p \le 2t$.
(b) *For all* $\varepsilon > 0$ *and* $t > t(\varepsilon)$, *there is a prime* $p \equiv 3 \pmod 4$ *with* $t \le p \le (1+\varepsilon)t$.

PROOF. Part (a) is a strengthening of Bertrand's Postulate due to Erdős [9]. Baker et al. [3] proved that for all sufficiently large t, the interval $[t, t + t^{0.525}]$ contains a prime. The proof can be modified to give primes $\equiv 3 \pmod 4$ in the same interval [Glyn Harman, personal communication, 2004]. Clearly this implies (b). $\square$

We can now prove the upper bound in Theorem 2.

LEMMA 7. *Every complete graph* K_n *has a* 3D *drawing with at most* $(2 + o(1))n^{3/2}$ *volume*, *and for all* $\varepsilon > 0$ *and* $n > n(\varepsilon)$, K_n *has a* 3D *drawing with at most* $(1+\varepsilon)n^{3/2}$ *volume*.

PROOF. By Lemma 6 with $t = \lceil \sqrt{n} \rceil$, there is a prime $p \not\equiv 1 \pmod 4$ with $\lceil \sqrt{n} \rceil \le p \le 2\lceil \sqrt{n} \rceil$ and $p \le (1+\varepsilon)\lceil \sqrt{n} \rceil$. By Observation 1 and Lemma 5, the set V_p defines a $p \times p \times p$ drawing of K_{p^2}. By choosing the appropriate vertices, we obtain a $\lceil n/p \rceil \times p \times p$ drawing of K_n. The volume is at most $(2 + o(1))n^{3/2}$ and $(1+\varepsilon)n^{3/2}$. $\square$

The same proof gives the lower bound in Theorem 1.

LEMMA 8. *The* $n \times n \times n$ *grid contains at least* $n^2/4$ *points with no three collinear. For all* $\varepsilon > 0$ *and* $n > n(\varepsilon)$, *the* $n \times n \times n$ *grid contains at least* $(1-\varepsilon)n^2$ *points with no three collinear.*

Lemma 7 generalises to give the following construction of a 3D drawing of $T(n, k)$.

LEMMA 9. *Every Turán graph* $T(n, k)$ *has a* 3D *drawing with at most* $(2 + o(1))n\sqrt{k}$ *volume. For all* $\varepsilon > 0$ *and* $k > k(\varepsilon)$, $T(n, k)$ *has a* 3D *drawing with at most* $(1+\varepsilon)n\sqrt{k}$ *volume*.

PROOF. Index the colour classes $\{(x, y) \colon 0 \le x, y \le \lceil \sqrt{k} \rceil - 1\}$. By Lemma 6, there is a prime $p \not\equiv 1 \pmod 4$ with $\lceil \sqrt{k} \rceil \le p \le 2\lceil \sqrt{k} \rceil$ and $p \le (1+\varepsilon)\lceil \sqrt{k} \rceil$. For each $i \in [\lceil n/k \rceil]$, position the ith vertex in colour class (x, y) at $(x,\ y,\ ip + (x^2+y^2) \bmod p)$.

Each colour class occupies its own Z-line. Thus, if an edge passes through a vertex, then three vertices from distinct colour classes are collinear. Observe that for every vertex at (a_x, a_y, a_z), we have $a_x^2 + a_y^2 \equiv a_z \pmod p$. Thus the same argument from Lemma 5 applies here, and no three vertices from distinct colour classes are collinear. Thus no edge passes through a vertex, and we obtain a 3D drawing of $T(n, k)$. The bounding box is $\lceil\sqrt{k}\rceil \times \lceil\sqrt{k}\rceil \times p\lceil n/k\rceil$. The volume is $(1+o(1))np$, which is at most $(2+o(1))n\sqrt{k}$ and $(1+\varepsilon)n\sqrt{k}$. $\square$

Pach et al. [18] proved that every k-colourable graph on n vertices is a subgraph of $T(2n+2k, 2k-1)$. Thus Lemma 9 implies the upper bound in Theorem 3.

COROLLARY 1. *Every k-colourable graph on n vertices has a 3D drawing with $(4\sqrt{2}+o(1))n\sqrt{k}$ volume. For all $\varepsilon > 0$ and $k > k(\varepsilon)$, every k-colourable graph on n vertices has a 3D drawing with $(2\sqrt{2}+\varepsilon)n\sqrt{k}$ volume.*

4. Open Problems

OPEN PROBLEM 1. Does every k-colourable graph have a crossing-free 3D drawing with $\mathcal{O}(kn^2)$ volume? The best known upper bound is $\mathcal{O}(k^2n^2)$ due to Pach et al. [18]. A $\mathcal{O}(kn^2)$ bound would match the $\Theta(n^3)$ bound for the minimum volume of a crossing-free 3D drawing of K_n.

OPEN PROBLEM 2. What is $\mathrm{vol}(n, d, \ell)$? For $\ell \in [d-1]$, let $\mathrm{vol}(n, d, \ell)$ be the minimum bounding box volume for n points in $\mathbb{Z}^d$, such that no $\ell+2$ points are in any ℓ-dimensional subspace. The box can be partitioned into $\mathrm{vol}(n, d, \ell)^{(d-\ell)/d}$ subspaces of dimension ℓ (each with at most $\ell+1$ vertices). Thus $n \leq (\ell+1)\mathrm{vol}(n, d, \ell)^{(d-\ell)/d}$ and

$$\mathrm{vol}(n, d, \ell) \geq \left(\frac{n}{\ell+1}\right)^{d/(d-\ell)}. \tag{1}$$

Consider the case of $\mathrm{vol}(n, d, d-1)$. Erdős [10] and Cohen et al. [4] proved that $\mathrm{vol}(n, 2, 1) \in \Theta(n^2)$ and $\mathrm{vol}(n, 3, 2) \in \Theta(n^3)$, respectively. Let $V = \{(x, x^2 \bmod p, \ldots, x^d \bmod p)\colon 0 \leq x \leq n-1\}$, where p is a prime with $n-1 \leq p \leq 2n$. The proofs of Erdős [10] and Cohen et al. [4] generalise to show that V contains no $d+1$ points in any $(d-1)$-dimensional subspace. Thus $\mathrm{vol}(n, d, d-1) \leq 2^{d-1}n^d$. By (1), $\mathrm{vol}(n, d, d-1) \in \Theta(n^d)$ for constant d.

OPEN PROBLEM 3. What is $\mathrm{vol}(n, d, 1)$? Erdős [10] proved that $\mathrm{vol}(n, 2, 1) \in \Theta(n^2)$. Theorem 2 proves that $\mathrm{vol}(n, 3, 1) \in \Theta(n^{3/2})$. This problem is unsolved for all constant $d \geq 4$. If $d \geq \log_2 n$ then trivially $\mathrm{vol}(n, d, 1) \in \Theta(n)$: just place the vertices at $\{(x_1, \ldots, x_d)\colon x_i \in \{0, 1\}\}$.

OPEN PROBLEM 4. What is $\mathrm{vol}(n, d, 2)$? This case is interesting as it relates to crossing-free drawings. Cohen et al. [4] proved $\mathrm{vol}(n, 3, 2) \in \Theta(n^3)$. Wood [20] proved that K_n has a $2 \times 2 \times \cdots \times 2$ crossing-free d-dimensional drawing for $d = 2\log n + \mathcal{O}(1)$; thus $\mathrm{vol}(n, d, 2) \in \mathcal{O}(n^2)$. What is the minimum volume of a crossing-free drawing of K_n irrespective of dimension?

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Appendix. Useful Inequalities

LEMMA 10. *Let $\{x_i, y_i, z_i : i \in [\ell]\}$ be a set of positive real numbers such that $x_i y_i \geq z_i$ for all $i \in [\ell]$. Let $z_* := \max\{z_1, \dots, z_\ell\}$. Then*

$$\left(\sum_i x_i\right)\left(\sum_i y_i\right) \geq \left(\sum_i z_i\right)^2 / z_*.$$

PROOF. By decreasing x_i or y_i we can assume that $x_i y_i = z_i$ for all i. Thus $(x_i y_j)(x_j y_i) = z_i z_j$ for all i, j. Hence $x_i y_j + x_j y_i \geq 2\sqrt{z_i z_j}$. Therefore

$$\left(\sum_i x_i\right)\left(\sum_i y_i\right) = \left(\sum_i x_i y_i + \sum_{i<j} \left(x_i y_j + x_j y_i\right)\right) \geq \left(\sum_i z_i + \sum_{i<j} 2\sqrt{z_i z_j}\right).$$

Now $z_* \geq z_i$ for all i. Thus

$$\left(\sum_i x_i\right)\left(\sum_i y_i\right) z_* \geq \sum_i z_i z_* + \sum_{i<j} 2\sqrt{z_i z_j}\, z_* \geq \sum_i z_i^2 + \sum_{i<j} 2 z_i z_j = \left(\sum_i z_i\right)^2,$$

as claimed. □

LEMMA 11. *For all positive real numbers $\alpha_i, \beta_i, \gamma_i$,*

$$\left(\sum_i (\alpha_i \beta_i \gamma_i)^{1/3}\right)^3 \leq \left(\sum_i \alpha_i\right)\left(\sum_i \beta_i\right)\left(\sum_i \gamma_i\right).$$

PROOF. Hölder's inequality states that if $p > 1$ and $1/p + 1/q = 1$, then

$$\sum_i x_i y_i \leq \left(\sum_i x_i^p\right)^{1/p} \left(\sum_i y_i^q\right)^{1/q}. \tag{2}$$

Apply (2), first with $p = \frac{3}{2}$ and $q = 3$, then with $p = q = 2$. We have

$$\begin{aligned}\sum_i (x_i y_i) z_i &\leq \left(\sum_i (x_i y_i)^{3/2}\right)^{2/3} \left(\sum_i z_i^3\right)^{1/3} = \left(\sum_i x_i^{3/2} y_i^{3/2}\right)^{2/3} \left(\sum_i z_i^3\right)^{1/3} \\ &\leq \left(\sum_i x_i^3\right)^{1/3} \left(\sum_i y_i^3\right)^{1/3} \left(\sum_i z_i^3\right)^{1/3}.\end{aligned}$$

The result follows by substituting $\alpha_i = x_i^3$, $\beta_i = y_i^3$ and $\gamma_i = z_i^3$. □

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