Circumference and Pathwidth of Highly Connected Graphs

Emily A. Marshall[1] **and David R. Wood**[2]

[1]DEPARTMENT OF MATHEMATICS,
VANDERBILT UNIVERSITY,
NASHVILLE, TENNESSEE
E-mail: emily.a.marshall@vanderbilt.edu

[2]SCHOOL OF MATHEMATICAL SCIENCES,
MONASH UNIVERSITY,
MELBOURNE, AUSTRALIA
E-mail: david.wood@monash.edu

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Abstract: Birmele [*J Graph Theory* 2003] proved that every graph with circumference t has treewidth at most $t-1$. Under the additional assumption of 2-connectivity, such graphs have bounded pathwidth, which is a qualitatively stronger conclusion. Birmele's theorem was extended by Birmele et al. [*Combinatorica* 2007] who showed that every graph without k disjoint cycles of length at least t has treewidth $\mathcal{O}(tk^2)$. Our main result states that, under the additional assumption of $(k+1)$-connectivity, such graphs have bounded pathwidth. In fact, they have pathwidth $\mathcal{O}(t^3+tk^2)$. Moreover, examples show that $(k+1)$-connectivity is required for bounded pathwidth to hold. These results suggest the following general question: for which

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values of k and graphs H does every k-connected H-minor-free graph have bounded pathwidth? We discuss this question and provide a few observations. J. Graph Theory 00: 1–11, 2014

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1. INTRODUCTION

Birmele [7] proved that every graph with circumference t has treewidth at most $t-1$, and this bound is tight for the complete graph K_t. Nešetřil and Ossona de Mendez [16, p. 118] showed that under the additional assumption of 2-connectivity, such graphs have treedepth at most $1+(t-2)^2$. Since pathwidth is at most treedepth minus 1, every 2-connected graph with circumference t has pathwidth at most $(t-2)^2$. Our first result strengthens this bound.

Theorem 1. *Every 2-connected graph with circumference t has pathwidth at most* $\lfloor \frac{t}{2} \rfloor (t-1)$.

The 2-connectivity assumption is needed in Theorem 1 since complete binary trees have unbounded pathwidth. In particular, the complete binary tree of height h has pathwidth $\lceil \frac{h}{2} \rceil$.

Birmele's theorem was extended by Birmele et al. [6], who showed that graphs without k disjoint cycles of length at least t have treewidth $\mathcal{O}(tk^2)$. Under the additional assumption of $(k+1)$-connectivity, we prove that such graphs have bounded pathwidth.

Theorem 2. *Every* $(k+1)$*-connected graph without k disjoint cycles of length at least t has pathwidth at most* $\mathcal{O}(t^3+tk^2)$.

We now show that the assumption of $(k+1)$-connectivity is needed in Theorem 2. Suppose on the contrary that every k-connected graph without k disjoint cycles of length at least t has pathwidth at most $f(k,t)$ for some function f. Let G be the graph obtained from the complete binary tree of height h by adding $k-1$ dominant vertices. Observe that G is k-connected. Since every cycle in G uses at least one of the dominant vertices, G contains no k disjoint cycles. Thus G has pathwidth at most $f(k,t)$ for all $t \geq 3$. On the other hand, the pathwidth of G equals $\lceil \frac{h}{2} \rceil + k - 1$. We obtain a contradiction by choosing $h > 2 \cdot f(k,t)$.

The proofs of Theorems 1 and 2 are given in Sections 3 and 4, respectively. We conclude in Section 5 by reinterpreting these results in terms of excluded minors. In general, we observe that highly connected H-minor-free graphs have bounded pathwidth. Determining the minimum connectivity required for this behavior to occur is an interesting line of future research.

2. DEFINITIONS

Let G be an (undirected, simple, finite) graph. The *circumference* of G is the length of the longest cycle in G, or is 0 if G is acyclic. A *tree decomposition* $(T, \{B_x \subseteq V(G) : x \in V(T)\})$ of G consists of a tree T and a set $\{B_x \subseteq V(G) : x \in V(T)\}$ of sets of vertices of G indexed by the nodes of T, such that:

- for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a nonempty (connected) subtree of T, and
- for each edge $uv \in E(G)$, there is some $x \in V(T)$ such that $u, v \in B_x$.

We refer to the sets B_x in the decomposition as *bags*. The *width* of a decomposition is the maximum size of a bag minus 1. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width over all tree decompositions of G. A *path decomposition* of G is a tree decomposition whose underlying tree is a path. The *pathwidth* of a graph G, denoted by $\mathrm{pw}(G)$, is the minimum width over all path decompositions of G. For simplicity, we describe a path decomposition by $(B_1, B_2, \ldots, B_n)$, where B_i is the bag associated with the i-th vertex in the path. In such a decomposition, for each vertex v of G, let $L(v)$ be the bag B_i containing v with i minimum. If $L(v) = L(w) = B_i$ for distinct $v, w \in V(G)$, then replace B_i by the two bags $B_i \setminus \{v\}$ and B_i. Now $L(v) \neq L(w)$. Repeat this step until $L(v) \neq L(w)$ for all distinct $v, w \in V(G)$. Such a path decomposition is said to be *normalized*. Hence, every graph has a normalized path decomposition with width $\mathrm{pw}(G)$.

A graph H is a *minor* of a graph G if H is isomorphic to a graph formed from a subgraph of G by contracting edges. When H is a minor of G, for each vertex $v \in V(H)$ there is a connected subgraph C of G that contracts to form v in the minor. We call C the *branch set* of v.

In a rooted forest F, the *height* of a vertex v in F is the distance between v and the root of the component of F that contains v. The *height* of F is the maximum height over all vertices of F. The *closure* of F, denoted $\mathrm{clos}(F)$, is the graph with vertex set $V(F)$ and edge set $\{xy : x \text{ is an ancestor of } y, x \neq y\}$. The *treedepth* of a graph G, denoted $\mathrm{td}(G)$, is the minimum height plus 1 of a forest F such that $G \subseteq \mathrm{clos}(F)$. Treedepth is equivalent to several other notions including minimal elimination tree height and is closely related to a number of graph invariants including pathwidth and treewidth; see [3], [16].

3. PROOF OF THEOREM 1

Lemma 3. *Every 2-connected graph G with circumference t has treedepth at most* $\lfloor \frac{t}{2} \rfloor (t-1) + 1$.

Proof. Let T be a *depth-first* spanning subtree T of G rooted at some vertex r. Thus $G \subseteq \mathrm{clos}(F)$. Say an edge vw of T has *span* $|i - j|$, where v and w are respectively at height i and j in T. For each edge vw of span $s \geq 2$, the vw-path in T plus vw forms a cycle of length $s + 1$. Thus $s \leq t - 1$. Consider a vertex v in G. By Menger's Theorem, there are two internally disjoint vr-paths in G. Their union is a cycle of length at most t. Thus there is a vr-path P in G of length at most $\lfloor \frac{t}{2} \rfloor$. Since each edge in P has span at most $t - 1$, the height of v is at most $\lfloor \frac{t}{2} \rfloor (t - 1)$. Hence the height of T is at most $\lfloor \frac{t}{2} \rfloor (t - 1)$. The result follows. ■

Theorem 1 follows directly from Lemma 3 since $\mathrm{pw}(G) \leq \mathrm{td}(G) - 1$ (see [16]).

4. PROOF OF THEOREM 2

A *block* in a graph G is a maximal 2-connected subgraph of G, or the subgraph of G induced by a bridge edge or an isolated vertex. It is well known that the blocks of G form

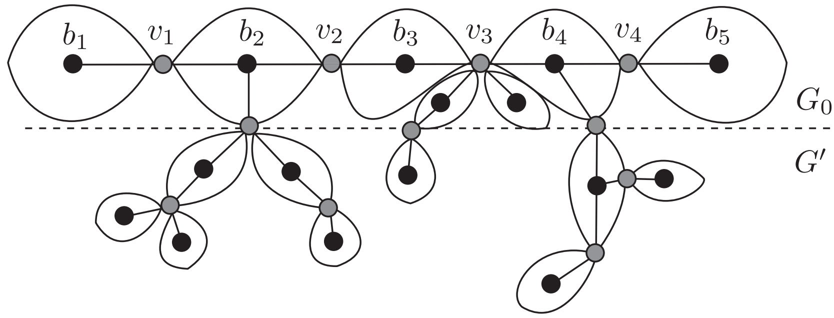

FIGURE 1. Example of a block-cut tree: P is the path $(b_1; v_1; b_2; v_2; b_3; v_3; b_4; v_4; b_5)$. The subgraphs G_0 and G' respectively consist of the blocks above and below the dotted line.

a proper partition of $E(G)$. The *block-cut-forest* T of a graph G is defined as follows: $V(T)$ is the set of cut-vertices and blocks of G, where a cut-vertex v is adjacent to a block B whenever $v \in B$. It is well known that T is a forest, and if G is connected, then T is a tree called the *block-cut-tree*.

Lemma 4. *Let T be the block-cut-forest of a graph G. Assume that* $\text{pw}(T) \leq n$ *and* $\text{pw}(B) \leq m$ *for each block B of G. Then* $\text{pw}(G) \leq (m+3)(n+1) - 3$.

Proof. We proceed by induction on $\text{pw}(T)$. For the base case, say $\text{pw}(T) = 0$. Then T has no edges, and each component of G is 2-connected. Clearly, the pathwidth of G equals the maximum pathwidth of the components of G. Thus $\text{pw}(G) \leq m = (m+3)(n+1) - 3$.

Now assume that $\text{pw}(T) \geq 1$. Since the pathwidth of G equals the maximum pathwidth of the components of G, we may assume that G is connected. Thus T is connected. Let $(X_1, X_2, \dots, X_s)$ be a path decomposition of T with width at most n. Choose vertices $x \in X_1$ and $y \in X_s$. Let P be a maximal path in T that contains an xy-path. Then $V(P) \cap X_i \neq \emptyset$ for all i. Let $X'_i = X_i - V(P)$; then $|X'_i| \leq |X_i| - 1$. Now $(X'_1, X'_2, \dots, X'_s)$ is a path decomposition of $T - V(P)$ with width at most $n - 1$. By the maximality of P, the endpoints of P are leaf vertices of T. No cut-vertex of G is a leaf of T. Thus the endpoints of P correspond to blocks. Say $P = b_1 v_1 b_2 v_2 \dots b_{s-1} v_{s-1} b_s$, where b_i represents the block B_i in G, and v_i is a cut-vertex in G. For each v_i, let $C_{i,1}, C_{i,2}, \dots, C_{i,t_i}$ be the blocks in G corresponding to neighbors of v_i in $T - V(P)$. Let $G_0 := \bigcup\{B_i : 1 \leq i \leq s\} \bigcup\{C_{i,j} : 1 \leq i \leq s-1, 1 \leq j \leq t_i\}$. Let G' be the union of the blocks not in G_0. Then $G = G_0 \bigcup G'$, as illustrated in Figure 1.

Let T' be the forest obtained from $T - V(P)$ by removing the leaf vertices that correspond to cut-vertices in G. This step removes all cut-vertices in G that are not cut-vertices in G', and the blocks that remain are blocks in G'. Thus T' is the block-cut-forest of G'. Since T' is a subgraph of $T - V(P)$, we have $\text{pw}(T') \leq \text{pw}(T - V(P)) \leq n - 1$. Furthermore, since each block B of G' is also a block of G, $\text{pw}(B) \leq m$. Let $G'_1, G'_2, \dots, G'_r$ be the components of G'. By induction, $\text{pw}(G'_j) \leq (m+3)n - 3$ for $1 \leq j \leq r$. Let $(H_{j,1}, H_{j,2}, \dots, H_{j,k_j})$ be a path decomposition of G'_j with $|H_{j,\ell}| \leq (m+3)n - 2$.

We now construct a path decomposition of G_0. For $1 \leq i \leq s$, let $(X_{i,1}, X_{i,2}, \dots, X_{i,k_i})$ be a path decomposition of B_i with $|X_{i,j}| \leq m + 1$ for $1 \leq j \leq k_i$. Define $X^+_{1,j} := X_{1,j} \cup \{v_1\}$ and $X^+_{s,j} := X_{s,j} \cup \{v_{s-1}\}$ and $X^+_{i,j} := X_{i,j} \cup \{v_{i-1}, v_i\}$ for $1 < i < s$. For each $C_{i,j}$, let

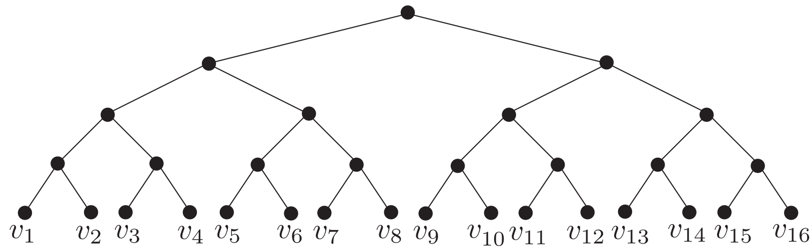

FIGURE 2. Left-to-right labeling of the leaves of the complete binary tree.

$(S_{i,j,1}, S_{i,j,2}, \ldots, S_{i,j,k_{i,j}})$ be a path decomposition with $|S_{i,j,\ell}| \leq m+1$. Define $S^+_{i,j,\ell} := S_{i,j,\ell} \cup \{v_i\}$. Denote by $T_{i,j}$ the sequence of bags $(S^+_{i,j,1}, S^+_{i,j,2}, \ldots, S^+_{i,j,k_{i,j}})$. It is easily proved that

$$\big(X^+_{1,1}, \ldots, X^+_{1,k_1}, T_{1,1}, \ldots, T_{1,t_1}, X^+_{2,1}, \ldots, X^+_{2,k_2}, T_{2,1}, \ldots, T_{2,t_2}, \ldots, T_{s-1,1}, \ldots,$$
$$T_{s-1,t_{s-1}}, X^+_{s,1}, \ldots, X^+_{s,k_s}\big)$$

is a path decomposition of G_0. The maximum bag size is at most $m+3$. Let $(Y_1, Y_2, \ldots, Y_p)$ be a normalized path decomposition of G_0 with $|Y_i| \leq m+3$ for $1 \leq i \leq p$. Then $L(v) \neq L(w)$ for $v \neq w$.

We now construct a path decomposition of G. For each component G'_j of G', let w_j be the cut-vertex in $G_0 \cap G'_j$. Note that w_j is distinct for each G'_j. Replace the bag $L(w_j)$ with the bags

$$(L(w_j) \cup H_{j,1}, L(w_j) \cup H_{j,2}, \ldots, L(w_j) \cup H_{j,k_j}).$$

The bag size is at most $m+3+(m+3)n-2 = (m+3)(n+1)-2$. For simplicity, rename the decomposition $(Z_1, \ldots, Z_q)$. It remains to show that $(Z_1, \ldots, Z_q)$ is a path decomposition of G. For each edge xy in G, we have $x, y \in Z_i$ for some i. Suppose $v \in Z_i \cap Z_j$ for $j > i+1$. Furthermore, assume $v \in V(G' - G_0)$ and without loss of generality, $v \in V(G'_1)$. Then by construction, $H_{1,r} \subset Z_i, H_{1,r+1} \subset Z_{i+1}, \ldots, H_{1,r+j-i} \subset Z_j$ for some r. $v \in H_{1,t}$ for $r \leq t \leq r+j-i$ so $v \in Z_s$ for $i \leq s \leq j$. Now instead assume $v \in V(G_0)$. Then by construction, $Y_r \subset Z_i$ and $Y_s \subset Z_j$ for some r, s with $s \geq r$. $v \in Y_t$ for $r \leq t \leq s$ so $v \in Z_\ell$ for $i \leq \ell \leq j$.

We conclude that $(Z_1, \ldots, Z_q)$ is a valid path decomposition. Since $|Z_i| \leq (m+3)(n+1)-2$, we have $\mathrm{pw}(G) \leq (m+3)(n+1)-3$. ■

Let T be a complete binary tree embedded in the plane as illustrated in Figure 2. Vertices at the same distance from the root are at the same level. Number the leaf vertices from left to right; let v_i be the leaf labeled i as shown.

Lemma 5. *Let T be a complete binary tree with leaf vertices numbered as in Figure 2. Then the path in T between v_a and v_b has length at least $2\log_2(b-a+1)$ where $a \leq b$.*

Proof. Let V_0 be the set of all leaf vertices of T. Let V_i be the set of all vertices u of T such that the shortest path from u to a vertex in V_0 has length i. Since T is a complete binary tree, each $u \in V_i$ has exactly 2^i descendants in V_0; furthermore, the descendants are $v_j, v_{j+1}, \ldots, v_{j+2^i-1}$ for some number j. Consider the vertex v_a and suppose $u \in V_i$ is an ancestor of v_a. Then if $v_j \in V_0$ also has u as an ancestor, then $j \in [a - (2^i - 1), a + (2^i - 1)]$.

For all $b \geq a$, there exists k such that $2^k \leq (b - a + 1) < 2^{k+1}$. Then, for $i < k$, $b \notin [a - (2^i - 1), a + (2^i - 1)]$, so v_a and v_b do not have a common ancestor in V_i. However, $b \in [a - (2^j - 1), a + (2^j - 1)]$ for all $j \geq k$ and there exists some $j \geq k$ such that v_a and v_b have a common ancestor u in V_j. Then by the definition of V_j, the path P_1 from v_a to u has length j and the path P_2 from u to v_b has length j. Thus P_1P_2 is a path of length $2j$ from v_a to v_b. Since $2^k \leq b - a + 1 < 2^{k+1}$ and $j \geq k$, we have $2j \geq 2\log_2(b - a + 1)$. ∎

Lemma 6. *Let T be a forest with $\mathrm{pw}(T) \geq t \geq 1$. Then T contains a complete binary tree of height $t - 1$ as a minor. Moreover, for any vertex $v \in V(T)$, there is such a minor in T with the property that v is in the branch set of the root of the binary tree.*

Proof. Since the pathwidth of a graph equals the maximum pathwidth of its components, we may assume that T is a tree. For a vertex v of T, define the *rooted pathwidth* of T at v, denoted $\mathrm{rpw}(T, v)$, as the minimum width of a path decomposition of T such that v is in the last bag of the decomposition. We say such a decomposition is *rooted* at v.

We prove, by induction on $|V(T)|$, that if $\mathrm{rpw}(T, v) \geq t$ for some vertex v of a tree T, then T contains a complete binary tree of height $t - 1$ as a minor with v in the branch set of the root. Since $\mathrm{rpw}(T, v) \geq \mathrm{pw}(T)$, the result follows when $\mathrm{pw}(T) \geq t$.

In the base case with $|V(T)| = 2$, the rooted pathwidth at a given vertex is 1 and the tree trivially contains a complete binary tree of height 0 rooted at the given vertex.

Now suppose $|V(T)| \geq 3$ and let v be such that $\mathrm{rpw}(T, v) \geq t$. Let $w_1, w_2, \ldots, w_d$ be the neighbors of v and let T_i be the component of $T - v$ rooted at w_i for $1 \leq i \leq d$. Let $r_i = \mathrm{rpw}(T_i, w_i)$. Without loss of generality, $r_1 \geq r_2 \geq \cdots \geq r_d$.

Let $(X_{i,1}, X_{i,2}, \ldots, X_{i,k_i})$ be a path decomposition of T_i rooted at w_i with width r_i. For $2 \leq i \leq d$, let $X^+_{i,j} = X_{i,j} \cup \{v\}$. Then

$$\big(X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}, \{w_1, v\}, X^+_{2,1}, X^+_{2,2}, \ldots, X^+_{2,k_2}, X^+_{3,1}, X^+_{3,2}, \ldots, X^+_{3,k_3}, \ldots, X^+_{d,1}, X^+_{d,2}, \ldots, X^+_{d,k_d}\big)$$

is a path decomposition of T rooted at v with width $\max\{r_1, r_2 + 1\}$. Here we use the fact that $w_1 \in X_{1,k_1}$. Thus $\mathrm{rpw}(T, v) \leq \max\{r_1, r_2 + 1\}$.

First suppose that $r_1 \geq r_2 + 1$. Then $\mathrm{rpw}(T_1, w_1) = r_1 \geq \mathrm{rpw}(T, v) \geq t$. By induction, T_1 contains a complete binary tree of height $t - 1$ rooted at w_1 as a minor. Extend the branch set containing w_1 to include v. We obtain a complete binary tree of height $t - 1$ rooted at v as a minor in T.

Now suppose that $r_2 + 1 > r_1$. Then $r_1 = r_2 \geq \mathrm{rpw}(T, v) - 1 \geq t - 1$. By induction, T_1 and T_2 each contain a complete binary tree of height $t - 2$ as a minor rooted at w_1 and w_2, respectively. Thus T contains a complete binary tree of height $t - 1$ rooted at v as a minor. ∎

To prove Theorem 2, we need the following. Let $\mathcal{F}$ be a family of graphs. For a graph G, a *hitting set* H of $\mathcal{F}$ is a set of vertices of G such that $G - H$ contains no member of $\mathcal{F}$. The family $\mathcal{F}$ is said to satisfy the *Erdős-Pósa property* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all graphs G, either G contains k vertex-disjoint members of $\mathcal{F}$ or G contains a hitting set H of size at most $f(k)$. Birmele et al. [6] proved that if $\mathcal{F}_t$ is the family of cycles of length at least t, then $\mathcal{F}_t$ satisfies the Erdős-Pósa property with $f(k) = 13t(k - 1)(k - 2) + (2t + 3)(k - 1)$.

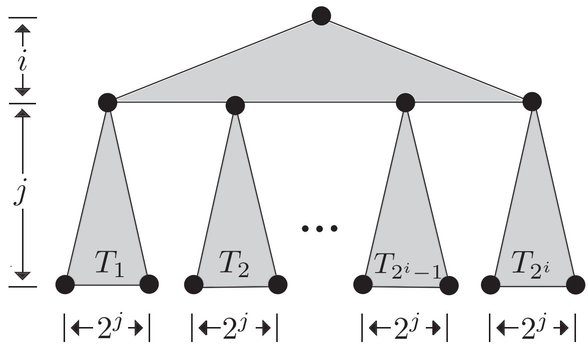

FIGURE 3. Complete binary tree of height $i + j$ with 2^i disjoint subtrees of height j.

Proof of Theorem 2. Since G contains no k vertex-disjoint cycles of length at least t, by the above-mentioned result of Birmele et al. [6], there is a hitting set $H \subseteq V(G)$ such that $|H| \le h := 13t(k-1)(k-2) + (2t+3)(k-1)$. Let T be the block-cut-forest of $G - H$. Define

$$i := \lfloor \log_2 (k-1)(2h-2k+1) \rfloor + 1 \quad \text{and} \quad j := \left\lceil \frac{t}{2} + \log_2(h-k+1) \right\rceil .$$

Since G is $(k+1)$-connected, $|H| \ge k$. Hence $h \ge |H| \ge k$, and i and j are well defined.

First suppose that $\mathrm{pw}(T) \le i + j$. Since H is a hitting set, $G - H$ has circumference at most $t - 1$. Thus the 2-connected blocks of $G - H$ have pathwidth at most $\lfloor \frac{t-1}{2} \rfloor (t-2)$ by Theorem 1. The blocks that are not 2-connected consist of bridges or isolated vertices, which have pathwidth at most 1. By Lemma 4 with $m = \lfloor \frac{t-1}{2} \rfloor (t-2)$ and $n = i + j$, we have $\mathrm{pw}(G - H) \le (\lfloor \frac{t-1}{2} \rfloor (t-2) + 3)(i + j + 1) - 3$. Add H to each bag of an optimal path decomposition of $G - H$ to obtain a path decomposition of G with width at most $(\lfloor \frac{t-1}{2} \rfloor (t-2) + 3)(i + j + 1) - 3 + h \in \mathcal{O}(t^3 + tk^2)$.

It remains to handle the case when $\mathrm{pw}(T) > i + j$. We claim, however, that this case does not occur. Suppose it does and assume $\mathrm{pw}(T) > i + j$. By Lemma 6, T contains a complete binary tree T' of height $i + j$ as a minor. It is well known and easily proved that if a graph A contains a graph B with maximum degree 3 as a minor, then A contains a subdivision of B as a subgraph. Thus, T contains a subdivision S of T' as a subgraph. By taking S maximal, each leaf of S is a leaf of T.

For each $v \in H$, let $d(v)$ be the number of leaves u of S such that v is adjacent in G to some vertex in the block corresponding to u (in which case we say that v is *adjacent* to u). Since G is $(k+1)$-connected, each leaf of S has at least k neighbors in H. Since S contains 2^{i+j} leaves, $\sum_{v \in H} d(v) \ge k\, 2^{i+j}$. Let $H = \{v_1, v_2, \ldots, v_h\}$ and $d_m := d(v_m)$. Without loss of generality, $d_1 \ge d_2 \ge \cdots \ge d_h$. Since $d_m \le 2^{i+j}$ for $1 \le m \le h$,

$$d_k + d_{k+1} + \cdots + d_h \ge k\, 2^{i+j} - (d_1 + d_2 + \cdots + d_{k-1}) \ge k\, 2^{i+j} - (k-1)2^{i+j} = 2^{i+j}.$$

Hence $d_1 \ge d_2 \ge \cdots \ge d_k \ge 2^{i+j}/(h-k+1)$. Let $X := \{v_1, v_2, \ldots, v_k\}$.

Since T' has height $i + j$, there are 2^i pairwise disjoint subtrees $T_1, T_2, \ldots, T_{2^i}$ in S, each a subdivision of a complete binary tree of height j, such that for $1 \le m \le 2^i$, the leaves of T_m are leaves of S and the root of T_m is at height i in T', as illustrated in Figure 3.

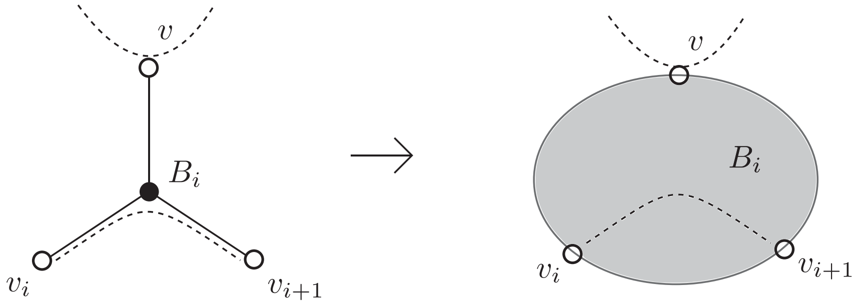

FIGURE 4. Routing the cycles in G.

For each $v \in X$, we say the pair (v, T_m) is *good* if v is adjacent to at least $2^{j-1}/(h-k+1)$ leaves of T_m. We claim that each $v \in X$ is in at least k good pairs. Suppose for the sake of contradiction that some $v \in X$ is in at most $k-1$ good pairs. Then

$$\frac{2^{i+j}}{h-k+1} \leq d(v) \leq (k-1)2^j + (2^i - k + 1)\frac{2^j}{2(h-k+1)}.$$

Thus $2^i \leq (k-1)(2h-2k+1)$, which contradicts the definition of i. Thus each $v \in X$ is in at least k good pairs. Since $|X| = k$, there is a distinct T_m for each $v \in X$ such that (v, T_m) is a good pair.

For each such pair (v, T_m), label the leaf vertices of T_m as in Figure 2. Since v is adjacent to at least $2^{j-1}/(h-k+1)$ leaf vertices, there are two leaves x and y labeled a and b respectively such that $b - a + 1 \geq \frac{2^{j-1}}{h-k+1}$. Then by Lemma 5, there is a path P of length at least $2\log_2(\frac{2^{j-1}}{h-k+1}) = 2j - 2 - 2\log_2(h-k+1)$ in T_m between x and y. Thus vPv is a cycle of length $2j - 2\log_2(h-k+1) \geq t$ in $T_m \cup \{v\}$. Since the T_m are pairwise disjoint, we have k pairwise disjoint cycles $C_1, C_2, \ldots, C_k$ of length at least t in $T \cup H$.

We now construct pairwise disjoint cycles $C'_1, C'_2, \ldots, C'_k$ in G. Say $C_1 = v_1 B_1 v_2 B_2 \ldots B_r v_1$, where $v_1 \in H$, v_i is a cut-vertex in $G - H$ for $2 \leq i \leq r-1$, and B_i is a block in $G - H$. The vertex v_1 is adjacent to a vertex x in B_1. Let P_1 be a path from x to v_2 in B_1. Next, for $2 \leq i \leq r-1$, let P_i be a path from v_i to v_{i+1} in B_i, such that if there is a vertex v in $B_i \cap V(C_j)$ for some $j \neq 1$, then choose P_i such that $v \notin V(P_i)$, as illustrated in Figure 4. Since each vertex in S has degree at most 3, there is at most one such vertex v to be avoided. Therefore, since B_i is 2-connected, such a P_i exists. For B_r, let P_r be a path from v_r to y in B_r, where y is a neighbor of v_1. Let $C'_1 = v_1 x P_1 v_2 P_2 v_3 \ldots P_r y v_1$. From each C_i, construct C'_i in G in this same manner. The cycles $C'_1, C'_2, \ldots, C'_k$ by construction are pairwise disjoint with length at least t in G, which is a contradiction. ■

5. RELATIONSHIP TO FORBIDDEN MINORS

Another way to describe a graph G with circumference $t-1$ is to say G is C_t-minor-free where C_t is a cycle on t vertices. Our two main theorems can thus be restated in terms of minors:

Theorem 7. *Let G be a 2-connected C_t-minor-free graph. Then* $\mathrm{pw}(G) \leq \lfloor \frac{t-1}{2} \rfloor (t-2)$.

Let $C_{t,k}$ be the graph consisting of k disjoint cycles of length t.

Theorem 8. *Let G be a $(k+1)$-connected $C_{t,k}$-minor-free graph. Then* $\mathrm{pw}(G) \leq \mathcal{O}(t^3 + tk^2)$.

These results suggest the following definition. For a graph H, let $g(H)$ be the minimum integer for which there exists a number $c = c(H)$ such that every $g(H)$-connected H-minor-free graph has pathwidth at most c. Mader [12] exhibited a function ℓ such that every $\ell(H)$-connected graph contains H as a minor. (Kostochka [13,14] and Thomason [18] independently proved that if $t = |V(H)|$ then $\ell(H) \leq \ell(K_t) \in \Theta(t\sqrt{\log t})$.) Thus every H-minor-free $\ell(H)$-connected graph has bounded pathwidth (since there is no such graph). Hence $g(H)$ is well defined, and $g(H) \leq \ell(H)$. We conclude with some observations about $g(H)$.

For some graphs, $g(H) = \ell(H)$. For example, $g(K_5) = \ell(K_5) = 6$ (since every 6-connected graph contains K_5 as a minor, but 5-connected planar (and thus K_5-minor-free) graphs have unbounded pathwidth).

On the other hand, $g(H)$ and $\ell(H)$ can be far apart. For example, we showed that $g(C_t) = 2$ but $\ell(C_t) \geq t-1$ since K_{t-1} is $(t-2)$-connected and contains no C_t-minor.

Observe that if H_1 is a minor of H_2, then $g(H_1) \leq g(H_2)$. Thus, for each integer c, the class $\mathcal{H}_c := \{H : g(H) \leq c\}$ is minor closed. By Robertson and Seymour's graph minor theorem, for each c, there is a finite set of minimal excluded minors for $\mathcal{H}_c$.

Bienstock et al. [1] proved that for every forest F, every graph with pathwidth at least $|F| - 1$ contains F as a minor. Thus $g(F) = 0$. Moreover, since complete binary trees have unbounded pathwidth, $g(F) = 0$ if and only if F is a forest. And K_3 is the only minimal excluded minor for $\mathcal{H}_0$.

There is no graph H with $g(H) = 1$ since the pathwidth of a graph equals the maximum pathwidth of its connected components.

We showed that $g(C_t) = 2$ for all $t \geq 3$. It is an interesting open problem to characterize the graphs H with $g(H) = 2$. (An answer is conjectured below.)

The following example is important. Consider $G_0 := K_3$ embedded in the plane. For $i \geq 0$, construct G_{i+1} from G_i as follows: for each edge vw on the outerface of G_i, add one new vertex adjacent to v and w. Thus G_i is 2-connected and outerplanar. Hence G_i is K_4-minor-free and $K_{2,3}$-minor-free. Observe that the dual of G_i contains a complete binary tree of height i as a minor, which has pathwidth i. By a result of Bodlaender and Fomin [2], the class $\{G_i : i \geq 0\}$ has unbounded pathwidth. Hence $g(K_4) \geq 3$ and $g(K_{2,3}) \geq 3$.

Dirac [9] proved that every 3-connected graph has a K_4-minor. Thus $g(K_4) = \ell(K_4) = 3$.

An unfinished result of Ding [8] implies that, for some function f, every 3-connected $K_{2,t}$-minor-free graph has pathwidth at most $f(t)$, implying $g(K_{2,t}) \leq 3$. Thus $g(K_{2,t}) \geq g(K_{2,3})$ and $g(K_{2,t}) = 3$ for $t \geq 3$ (assuming Ding's result).

We proved that $g(C_{t,k}) = k+1$ for all $t \geq 3$, where the lower bound follows from the example given after the statement of Theorem 2. This leads to the following lower bound on $g(H)$: If H contains k disjoint cycles, then $C_{3,k}$ is a minor of H, and $g(H) \geq k+1$. This observation can be strengthened as follows. A *transversal* in a graph H is a set X of vertices such that $H - X$ is acyclic. Let $\tau(H)$ be the minimum size of a transversal in H. Note that if H is a minor of G, then $\tau(H) \leq \tau(G)$.

Proposition 9. $g(H) \geq \tau(H) + 1$ *for every graph H with $\tau(H) \geq 1$.*

Proof. Suppose on the contrary that $g(H) \leq \tau(H)$ for some graph H. Let G be the graph obtained from the complete binary tree of height h by adding $\tau(H) - 1$ dominant vertices. Then G is $\tau(H)$-connected, and $\tau(G) = \tau(H) - 1$, implying G is H-minor-free. By the definition of $g(H)$, for some $c = c(H)$, the pathwidth of G is at most c. This is a contradiction for $h > 2c$, since G has pathwidth $\lceil \frac{h}{2} \rceil + \tau(H) - 1$. Therefore $g(H) \geq \tau(H) + 1$. ∎

We have described three minor-minimal graphs H with $g(H) = 3$. Namely, K_4, $K_{2,3}$, and $K_3 \cup K_3$. (It is easily seen that these graphs are minor-minimal.) There is one more key example. Let Q be the octahedron graph $K_{2,2,2}$ minus the edges of a triangle. Observe that $\tau(Q) = 2$, and thus $g(Q) \geq 3$ by Proposition 9. Moreover, Q contains no K_4, $K_{2,3}$, or $K_3 \cup K_3$ minor.

Conjecture 10. *The minimal excluded minors for $\mathcal{H}_2$ are $\{K_4, K_{2,3}, K_3 \cup K_3, Q\}$.*

It is well known that H is outerplanar if and only if H contains no K_4 or $K_{2,3}$ minor, and it follows from a result of Lovász [15] that $\tau(H) \leq 1$ if and only if H contains no K_4, $K_3 \cup K_3$ or Q minor. Thus Conjecture 10 is equivalent to saying that $g(H) \leq 2$ if and only if H is outerplanar and $\tau(H) \leq 1$.

In the above examples H is planar. Planarity is significant for these types of questions since the class of H-minor-free graphs has bounded treewidth if and only if H is planar [17]. However, $g(H)$ is well defined for all graphs, and is interesting for certain nonplanar graphs. For example, Böhme et al. [5] proved that there is a function n such that every 7-connected graph with at least $n(k)$ vertices contains $K_{3,k}$ as a minor. That is, every 7-connected $K_{3,k}$-minor-free graph has less than $n(k)$ vertices, implying $g(K_{3,k}) \leq 7$. More generally, Böhme et al. [4] conjectured that for all a, k there is an integer $n(a, k)$ such that every $(2a + 1)$-connected graph on at least $n(a, k)$ vertices contains $K_{a,k}$ as a minor. This would imply that $g(K_{a,k}) \leq 2a + 1$.

In general, it would be interesting if some function of $\tau(H)$ was an upper bound on $g(H)$. Or is there a family of graphs H with bounded transversals, but with $g(H)$ unbounded?

NOTES ADDED IN PROOF

Fiorini and Herinckx [11] recently improved the above-mentioned result of Birmele et al. [6] by showing that cycles of length at least t satisfy the Erdős-Pósa property with $f(k) = \mathcal{O}(tk \log k)$ (which is optimal for fixed k or fixed t). It follows that the $\mathcal{O}(t^3 + tk^2)$ bound in Theorem 2 can be improved to $\mathcal{O}(t^3 + tk \log k)$.

In an early version of this paper, the graph Q was omitted from Conjecture 10. Proposition 9 and the importance of Q were jointly observed with János Barát and Gwenaël Joret.

Gwenaël Joret also pointed out the following alternative proof of a slightly weaker version of Theorem 1. Let G be a 2-connected graph with circumference t. Let p be the number of edges in the longest path in G. Dirac [10] proved that $t > \sqrt{2p}$. Thus $p < \lceil \frac{t^2}{2} \rceil$. That is, G contains no path on $\lceil \frac{t^2}{2} \rceil$ edges. Hence G contains no path on $\lceil \frac{t^2}{2} \rceil$ edges as a minor. Bienstock et al. [1] proved that every graph that excludes a fixed forest on k edges as a minor has pathwidth at most $k - 1$. Thus G has pathwidth at most $\lceil \frac{t^2}{2} \rceil - 1$.

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