Hadwiger's Conjecture for ℓ-Link Graphs

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Abstract: In this article, we define and study a new family of graphs that generalizes the notions of line graphs and path graphs. Let G be a graph with no loops but possibly with parallel edges. An *ℓ-link* of G is a walk of G of length $\ell \geqslant 0$ in which consecutive edges are different. The *ℓ-link graph* $\mathbb{L}_\ell(G)$ of G is the graph with vertices the ℓ-links of G, such that two vertices are joined by $\mu \geqslant 0$ edges in $\mathbb{L}_\ell(G)$ if they correspond to two subsequences of each of μ $(\ell+1)$-links of G. By revealing a recursive structure, we bound from above the chromatic number of ℓ-link graphs. As a corollary, for a given graph G and large enough ℓ, $\mathbb{L}_\ell(G)$ is 3-colorable. By investigating the shunting of ℓ-links in G, we show that the Hadwiger number of a nonempty $\mathbb{L}_\ell(G)$ is greater or equal to that of G. Hadwiger's conjecture states that the Hadwiger number of a graph is at least the chromatic number of that graph. The conjecture has been proved by Reed and

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Seymour (Eur J Combin 25(6) (2004), 873–876) for line graphs, and hence 1-link graphs. We prove the conjecture for a wide class of ℓ-link graphs.

Keywords: *ℓ-link graph; path graph; chromatic number; graph minor; Hadwiger's conjecture*

1. INTRODUCTION AND MAIN RESULTS

We introduce a new family of graphs, called *ℓ-link graphs*, which generalizes the notions of line graphs and path graphs. Such a graph is constructed from a certain kind of walk of length $\ell \geqslant 0$ in a given graph G. To ensure that the constructed graph is undirected, G is undirected. To avoid loops, G is loopless, and the consecutive edges in each walk are different. Such a walk is called an *ℓ-link*. For example, a 0-link is a vertex, a 1-link is an edge, and a 2-link consists of two distinct edges with an end vertex in common. An ℓ-path is an ℓ-link without repeated vertices. We use $\mathscr{L}_\ell(G)$ and $\mathscr{P}_\ell(G)$ to denote the sets of ℓ-links and ℓ-paths of G, respectively. There have been a number of families of graphs constructed from ℓ-links. For example, the *line graph* $\mathbb{L}(G)$, introduced by Whitney [23], is the simple graph with vertex set $E(G)$, in which two vertices are adjacent if their corresponding edges are incident to a common vertex. More generally, the *ℓ-path graph* $\mathbb{P}_\ell(G)$ is the simple graph with vertex set $\mathscr{P}_\ell(G)$, where two vertices are adjacent if the union of their corresponding ℓ-paths forms a path or a cycle of length $\ell + 1$. Note that an ℓ-path contains ℓ distinct edges and $\ell + 1$ distinct vertices. So $\mathbb{P}_\ell(G)$ is the $\mathbf{P}_{\ell+1}$-graph of G introduced by Broersma and Hoede [4]. Inspired by these graphs, we define the *ℓ-link graph* $\mathbb{L}_\ell(G)$ of G to be the graph with vertex set $\mathscr{L}_\ell(G)$, in which two vertices are joined by $\mu \geqslant 0$ edges in $\mathbb{L}_\ell(G)$ if they correspond to two subsequences of each of μ $(\ell + 1)$-links of G. More strict definitions can be found in Section 2, together with some other related graphs.

This article studies the structure, coloring, and minors of ℓ-link graphs including a proof of Hadwiger's conjecture for a wide class of ℓ-link graphs. By default $\ell \geqslant 0$ is an integer. And all graphs are finite, undirected, and loopless. Parallel edges are admitted unless we specify the graph to be *simple*.

1.1. Graph Coloring

Let $t \geqslant 0$ be an integer. A *t-coloring* of G is a map $\lambda : V(G) \to [t] := \{1, 2, \dots, t\}$ such that $\lambda(u) \neq \lambda(v)$ whenever $u, v \in V(G)$ are adjacent in G. A graph with a t-coloring is *t-colorable*. The *chromatic number* $\chi(G)$ is the minimum t such that G is t-colorable. Similarly, a *t-edge-coloring* of G is a map $\lambda : E(G) \to [t]$ such that $\lambda(e) \neq \lambda(f)$ whenever $e, f \in E(G)$ are incident to a common vertex in G. The *edge-chromatic number* $\chi'(G)$ of G is the minimum t such that G admits a t-edge-coloring. Let $\chi_\ell(G) := \chi(\mathbb{L}_\ell(G))$, and $\Delta(G)$ be the maximum degree of G. Brooks' theorem [5] states that, the chromatic number of a connected graph G equals $\Delta(G) + 1$ if G is an odd cycle or a complete graph with at least one vertex, and is at most $\Delta(G)$ otherwise. Shannon [18] proved that $\chi_1(G) = \chi'(G) \leqslant \frac{3}{2}\Delta(G)$. We prove a recursive structure for ℓ-link graphs, which leads to the following upper bounds for $\chi_\ell(G)$.

Theorem 1.1. *Let G be a graph, $\chi := \chi(G)$, $\chi' := \chi'(G)$, and $\Delta := \Delta(G)$.*

(1) *If $\ell \geqslant 0$ is even, then* $\chi_\ell(G) \leqslant \min\{\chi, \lfloor(\frac{2}{3})^{\ell/2}(\chi - 3)\rfloor + 3\}$.

(2) *If* $\ell \geqslant 1$ *is odd, then* $\chi_\ell(G) \leqslant \min\{\chi', \lfloor(\frac{2}{3})^{\frac{\ell-1}{2}}(\chi'-3)\rfloor+3\}$.
(3) *If* $\ell \neq 1$, *then* $\chi_\ell(G) \leqslant \Delta + 1$.
(4) *If* $\ell \geqslant 2$, *then* $\chi_\ell(G) \leqslant \chi_{\ell-2}(G)$.

Theorem 1.1 implies that $\mathbb{L}_\ell(G)$ is 3-colorable for large enough ℓ.

Corollary 1.2. *For each graph G,* $\mathbb{L}_\ell(G)$ *is* 3*-colorable in the following cases:*

(1) $\ell \geqslant 0$ *is even, and either* $\chi(G) \leqslant 3$ *or* $\ell > 2\log_{1.5}(\chi(G)-3)$.
(2) $\ell \geqslant 1$ *is odd, and either* $\chi'(G) \leqslant 3$ *or* $\ell > 2\log_{1.5}(\chi'(G)-3)+1$.

As explained in Section 2, this corollary is related to and implies a result by Kawai and Shibata [15].

1.2. Graph Minors

A connected graph with two or more vertices is *biconnected* if it cannot be disconnected by removing a vertex. By *contracting* an edge we mean identifying its end vertices and deleting possible resulting loops. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An H*-minor* is a minor of G that is isomorphic to H. The *Hadwiger number* $\eta(G)$ of G is the maximum integer t such that G contains a K_t-minor. Denote by $\delta(G)$ the minimum degree of G. The *degeneracy* $d(G)$ of G is the maximum $\delta(H)$ over the subgraphs H of G. We prove the following.

Theorem 1.3. *Let* $\ell \geqslant 1$, *and G be a graph such that* $\mathbb{L}_\ell(G)$ *contains at least one edge. Then* $\eta(\mathbb{L}_\ell(G)) \geqslant \max\{\eta(G), d(G)\}$.

By definition $\mathbb{L}(G)$ is the underlying simple graph of $\mathbb{L}_1(G)$. And $\mathbb{L}_\ell(G) = \mathbb{P}_\ell(G)$ if $girth(G) > \{\ell, 2\}$. Thus Theorem 1.3 can be applied to path graphs.

Corollary 1.4. *Let* $\ell \geqslant 1$, *and G be a graph of girth at least* $\ell + 1$ *such that* $\mathbb{P}_\ell(G)$ *contains at least one edge. Then* $\eta(\mathbb{P}_\ell(G)) \geqslant \max\{\eta(G), d(G)\}$.

As a far-reaching generalization of the four-color theorem, in 1943, Hugo Hadwiger [10] conjectured the following.

Hadwiger's conjecture: $\eta(G) \geqslant \chi(G)$ for every graph G.

Hadwiger's conjecture was proved by Robertson, Seymour, and Thomas [17] for $\chi(G) \leqslant 6$. The conjecture for line graphs, or equivalently for 1-link graphs, was proved by Reed and Seymour [16]. We prove the following.

Theorem 1.5. *Hadwiger's conjecture is true for* $\mathbb{L}_\ell(G)$ *in the following cases:*

(1) $\ell \geqslant 1$ *and G is biconnected.*
(2) $\ell \geqslant 2$ *is an even integer.*
(3) $d(G) \geqslant 3$ *and* $\ell > 2\log_{1.5}\frac{\Delta(G)-2}{d(G)-2} + 3$.
(4) $\Delta(G) \geqslant 3$ *and* $\ell > 2\log_{1.5}(\Delta(G)-2) - 3.83$.
(5) $\Delta(G) \leqslant 5$.

The corresponding results for path graphs are listed below.

Corollary 1.6. *Let G be a graph of girth at least* $\ell + 1$. *Then Hadwiger's conjecture holds for* $\mathbb{P}_\ell(G)$ *in the cases of Theorem 1.5 (1)–(5).*

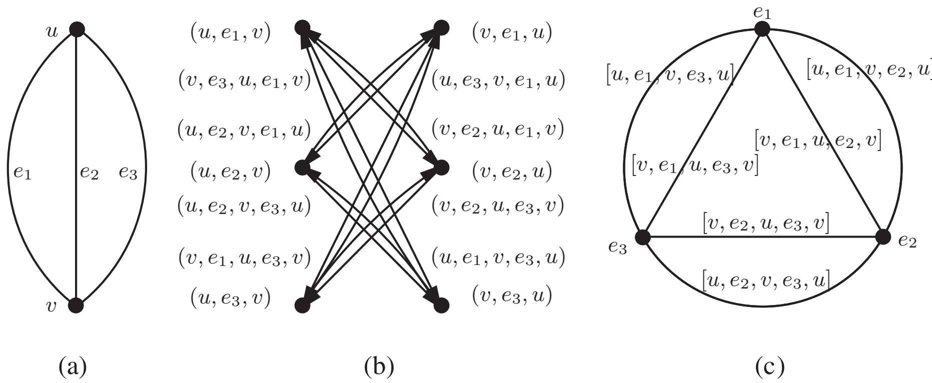

FIGURE 1. (a) D_3, (b) $\mathbb{A}_1(D_3)$, (c) $\mathbb{L}_1(D_3)$.

2. DEFINITIONS AND TERMINOLOGY

We now give some formal definitions. A graph G is *null* if $V(G) = \emptyset$, and *non-null* otherwise. A non-null graph G is *empty* if $E(G) = \emptyset$, and nonempty otherwise. A *unit* is a vertex or an edge. The subgraph of G induced by $V \subseteq V(G)$ is the maximal subgraph of G with vertex set V. And in this case, the subgraph is called an *induced subgraph* of G. We may not distinguish between V and its induced subgraph. For $\emptyset \neq E \subseteq E(G)$, the subgraph of G induced by $E \cup V$ is the minimal subgraph of G with edge set E, and vertex set including V. The *diameter diam*(G) of G is $+\infty$ if G is disconnected, and the maximum distance between two vertices of G otherwise.

Let G be a graph, and H be a subgraph of G. Let $\mathcal{V}$ be a partition of $V(H)$ such that every $V \in \mathcal{V}$ induces a connected subgraph of H. Let M be the graph obtained from H by contracting each $V \in \mathcal{V}$ into a vertex. Then M is a minor of G. And V is called a *branch set* of M.

For more accurate analysis, we need to define *ℓ-arcs*. An *ℓ-arc* (or *∗-arc* if we ignore the length) of G is an alternating sequence $\vec{L} := (v_0, e_1, \ldots, e_\ell, v_\ell)$ of units of G such that the end vertices of $e_i \in E(G)$ are v_{i-1} and v_i for $i \in [\ell]$, and that $e_i \neq e_{i+1}$ for $i \in [\ell - 1]$. The *direction* of $\vec{L}$ is its vertex sequence $(v_0, v_1, \ldots, v_\ell)$. In algebraic graph theory, ℓ-arcs in simple graphs have been widely studied [3, 19, 20, 22]. Note that $\vec{L}$ and its *reverse* $-\vec{L} := (v_\ell, e_\ell, \ldots, e_1, v_0)$ are different unless $\ell = 0$. The ℓ-link (or *∗-link* if the length is ignored) $L := [v_0, e_1, \ldots, e_\ell, v_\ell]$ is obtained by taking $\vec{L}$ and $-\vec{L}$ as a single object. For $0 \leqslant i \leqslant j \leqslant \ell$, the $(j-i)$-arc $\vec{L}(i, j) := (v_i, e_{i+1}, \ldots, e_j, v_j)$ and the $(j-i)$-link $\vec{L}[i, j] := [v_i, e_{i+1}, \ldots, e_j, v_j]$ are called *segments* of $\vec{L}$ and L, respectively. We may write $\vec{L}(j, i) := -\vec{L}(i, j)$, and $\vec{L}[j, i] := \vec{L}[i, j]$. These segments are called *middle segments* if $i + j = \ell$. L is called an *ℓ-cycle* if $\ell \geqslant 2$, $v_0 = v_\ell$ and $\vec{L}[0, \ell - 1]$ is an $(\ell - 1)$-path. Denote by $\vec{\mathscr{L}}_\ell(G)$ and $\mathscr{C}_\ell(G)$ the sets of ℓ-arcs and ℓ-cycles of G, respectively. Usually, $\vec{e}_i := (v_{i-1}, e_i, v_i)$ is called an *arc* for short. In particular, $v_0, v_\ell, e_1, e_\ell, \vec{e}_1$, and $\vec{e}_\ell$ are called the *tail vertex*, *head vertex*, *tail edge*, *head edge*, *tail arc*, and *head arc* of $\vec{L}$, respectively.

Godsil and Royle [9] defined the *ℓ-arc graph* $\mathbb{A}_\ell(G)$ to be the digraph with vertex set $\vec{\mathscr{L}}_\ell(G)$, such that there is an arc, labeled by $\vec{Q}$, from $\vec{Q}(0, \ell)$ to $\vec{Q}(1, \ell + 1)$ in $\mathbb{A}_\ell(G)$ for every $\vec{Q} \in \vec{\mathscr{L}}_{\ell+1}(G)$. The *t-dipole graph* D_t is the graph consists of two vertices and $t \geqslant 1$ edges between them. (See Figure 1 a for D_3, and Figure 1 b the 1-arc graph of D_3.)

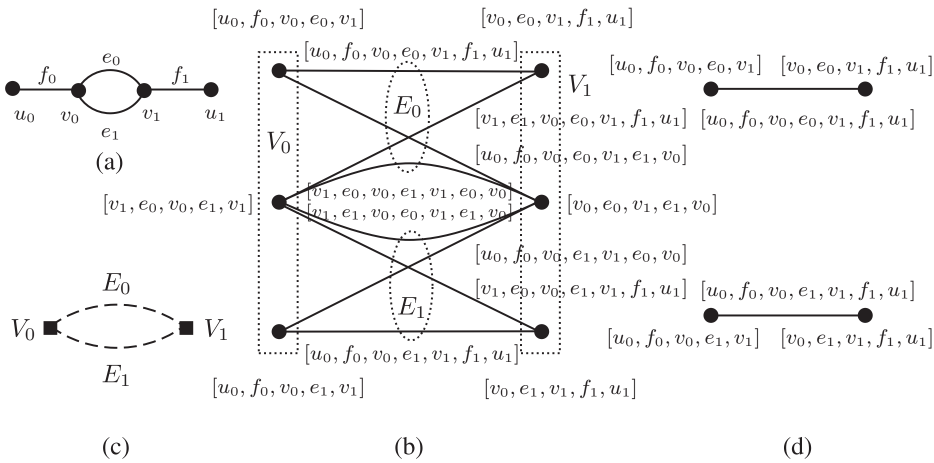

FIGURE 2. (a) G, (b) $H := \mathbb{L}_2(G)$, (c) $H_{(\mathcal{V},\mathcal{E})}$, (d) $\mathbb{P}_2(G)$.

The *ℓth iterated line digraph* $\mathbb{A}^\ell(G)$ is $\mathbb{A}_1(G)$ if $\ell = 1$, and $\mathbb{A}_1(\mathbb{A}^{\ell-1}(G))$ if $\ell \geqslant 2$ (see [2]). Examples of undirected graphs constructed from ℓ-arcs can be found in [12, 13].

Shunting of ℓ-arcs was introduced by Tutte [21]. We extend this notion to ℓ-links. For $\ell, s \geqslant 0$, and $\vec{Q} \in \vec{\mathscr{L}}_{\ell+s}(G)$, let $L_i := \vec{Q}[i, \ell+i]$ for $i \in [s] \cup \{0\}$, and $Q_i := \vec{Q}[i-1, \ell+i]$ for $i \in [s]$. Let $Q^{[\ell]} := [L_0, Q_1, L_1, \ldots, L_{s-1}, Q_s, L_s]$. We say L_0 can be *shunted* to L_s through $\vec{Q}$ or Q. $Q^{\{\ell\}} := \{L_0, L_1, \ldots, L_s\}$ is the set of *images* during this shunting. For $L, R \in \mathscr{L}_\ell(G)$, we say L can be *shunted* to R if there are ℓ-links $L = L_0, L_1, \ldots, L_s = R$ such that L_{i-1} can be shunted to L_i through some $*$-arc $\vec{Q}_i$ for $i \in [s]$. In Figure 2, $[u_0, f_0, v_0, e_0, v_1]$ can be shunted to $[v_1, e_0, v_0, e_1, v_1]$ through $(u_0, f_0, v_0, e_0, v_1, f_1, u_1)$ and $(u_1, f_1, v_1, e_0, v_0, e_1, v_1)$.

For $L, R \in \mathscr{L}_\ell(G)$ and $\mathscr{Q} \subseteq \mathscr{L}_{\ell+1}(G)$, denote by $\mathscr{Q}(L, R)$ the set of $Q \in \mathscr{Q}$ such that L can be shunted to R through Q. We show in Section 3 that $|\mathscr{Q}(L, R)|$ is 0 or 1 if G is simple, and can be up to 2 if $\ell \geqslant 1$ and G contains parallel edges. A more formal definition of ℓ-link graphs is given below.

Definition 2.1. *Let $\mathscr{L} \subseteq \mathscr{L}_\ell(G)$, and $\mathscr{Q} \subseteq \mathscr{L}_{\ell+1}(G)$. The partial ℓ-link graph $\mathbb{L}(G, \mathscr{L}, \mathscr{Q})$ of G, with respect to $\mathscr{L}$ and $\mathscr{Q}$, is the graph with vertex set $\mathscr{L}$, such that $L, R \in \mathscr{L}$ are joined by exactly $|\mathscr{Q}(L, R)|$ edges. In particular, $\mathbb{L}_\ell(G) = \mathbb{L}(G, \mathscr{L}_\ell(G), \mathscr{L}_{\ell+1}(G))$ is the ℓ-link graph of G.*

Remark. We assign exclusively to each edge of $\mathbb{L}_\ell(G)$ between $L, R \in \mathscr{L}_\ell(G)$ a $Q \in \mathscr{L}_{\ell+1}(G)$ such that L can be shunted to R through Q, and refer to this edge simply as Q. In this sense, $Q^{[\ell]} := [L, Q, R]$ is a 1-link of $\mathbb{L}_\ell(G)$.

For example, the 1-link graph of D_3 can be seen in Figure 1 c. A 2-link graph is given in Figure 2 b, and a 2-path graph is depicted in Figure 2 d.

Reed and Seymour [16] pointed out that proving Hadwiger's conjecture for line graphs of multigraphs is more difficult than for that of simple graphs. This motivates us to work on the ℓ-link graphs of multigraphs. Diestel [7, page 28] explained that, in some situations, it is more natural to develop graph theory for multigraphs. We allow parallel edges in ℓ-link graphs in order to investigate the structure of $\mathbb{L}_\ell(G)$ by studying the shunting of

ℓ-links in G regardless of whether G is simple. The observation below follows from the definitions.

Observation 2.2. *$\mathbb{L}_0(G) = G$, $\mathbb{P}_1(G) = \mathbb{L}(G)$, and $\mathbb{P}_\ell(G)$ is the underlying simple graph of $\mathbb{L}_\ell(G)$ for $\ell \in \{0, 1\}$. For $\ell \geqslant 2$, $\mathbb{P}_\ell(G) = \mathbb{L}(G, \mathscr{P}_\ell(G), \mathscr{P}_{\ell+1}(G) \cup \mathscr{C}_{\ell+1}(G))$ is an induced subgraph of $\mathbb{L}_\ell(G)$. If G is simple, then $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$ for $\ell \in \{0, 1, 2\}$. Further, $\mathbb{P}_\ell(G) = \mathbb{L}_\ell(G)$ if girth$(G) > \max\{\ell, 2\}$.*

Let $\vec{Q} \in \vec{\mathscr{L}}_{\ell+s}(G)$, and $[L_0, Q_1, L_1, \dots, L_{s-1}, Q_s, L_s] := Q^{[\ell]}$. From the remark above, for $i \in [s]$, Q_i is an edge of $H := \mathbb{L}_\ell(G)$ between $L_{i-1}, L_i \in V(H)$. So $Q^{[\ell]}$ is an s-link of H. In Figure 2 b, $[u_0, f_0, v_0, e_0, v_1, e_1, v_0, e_0, v_1]^{[2]} = [[u_0, f_0, v_0, e_0, v_1], [u_0, f_0, v_0, e_0, v_1, e_1, v_0], [v_0, e_0, v_1, e_1, v_0], [v_0, e_0, v_1, e_1, v_0, e_0, v_1], [v_1, e_1, v_0, e_0, v_1]]$ is a 2-path of H.

We say H is *homomorphic* to G, written $H \to G$, if there is an injection $\alpha : V(H) \cup E(H) \to V(G) \cup E(G)$ such that for $w \in V(H)$, $f \in E(H)$ and $[u, e, v] \in \mathscr{L}_1(H)$, their images $w^\alpha \in V(G)$, $f^\alpha \in E(G)$ and $[u^\alpha, e^\alpha, v^\alpha] \in \mathscr{L}_1(G)$. In this case, α is called a *homomorphism* from H to G. The definition here is a generalisation of the one for simple graphs by Godsil and Royle [9, page 6]. A bijective homomorphism is an *isomorphism*. By Hell and Nešetřil [11], $\chi(H) \leqslant \chi(G)$ if $H \to G$. For instance, $\vec{L} \mapsto L$ for $\vec{L} \in \vec{\mathscr{L}}_\ell(G) \cup \vec{\mathscr{L}}_{\ell+1}(G)$ can be seen as a homomorphism from $\mathbb{A}_\ell(G)$ to $\mathbb{L}_\ell(G)$. By Bang-Jensen and Gutin [1], $\mathbb{A}_\ell(G)$ is isomorphic to $\mathbb{A}^\ell(G)$. So $\chi(\mathbb{A}^\ell(G)) = \chi(\mathbb{A}_\ell(G)) \leqslant \chi(\mathbb{L}_\ell(G)) = \chi_\ell(G)$. We emphasize that $\chi(\mathbb{A}^\ell(G))$ might be much less than $\chi_\ell(G)$. For example, as depicted in Figure 1, when $t \geqslant 3$, $\chi(\mathbb{A}^\ell(D_t)) = 2 < t = \chi_\ell(D_t)$. Kawai and Shibata proved that $\mathbb{A}^\ell(G)$ is 3-colorable for large enough ℓ. By the analysis above, Corollary 1.2 implies this result.

A graph homomorphism from H is usually represented by a vertex partition $\mathcal{V}$ and an edge partition $\mathcal{E}$ of H such that (a) each part of $\mathcal{V}$ is an independent set of H, and (b) each part of $\mathcal{E}$ is incident to exactly two parts of $\mathcal{V}$. In this situation, for different $U, V \in \mathcal{V}$, define $\mu(U, V)$ to be the number of parts of $\mathcal{E}$ incident to both U and V. The *quotient graph* $H_{(\mathcal{V},\mathcal{E})}$ of H is defined to be the graph with vertex set $\mathcal{V}$, and for every pair of different $U, V \in \mathcal{V}$, there are exactly $\mu(U, V)$ edges between them. To avoid ambiguity, for $V \in \mathcal{V}$ and $E \in \mathcal{E}$, we use $V_\mathcal{V}$ and $E_\mathcal{E}$ to denote the corresponding vertex and edge of $H_{(\mathcal{V},\mathcal{E})}$, which defines a graph homomorphism from H to $H_{(\mathcal{V},\mathcal{E})}$. Sometimes, we only need the underlying simple graph $H_\mathcal{V}$ of $H_{(\mathcal{V},\mathcal{E})}$.

For $\ell \geqslant 2$, there is a natural partition in an ℓ-link graph. For each $R \in \mathscr{L}_{\ell-2}(G)$, let $\mathscr{L}_\ell(G, R)$, or $\mathscr{L}_\ell(R)$ for short, be the set of ℓ-links of G with middle segment R. Clearly, $\mathcal{V}_\ell(G) := \{\mathscr{L}_\ell(R) \neq \emptyset | R \in \mathscr{L}_{\ell-2}(G)\}$ is a vertex partition of $\mathbb{L}_\ell(G)$. And $\mathcal{E}_\ell(G) := \{\mathscr{L}_{\ell+1}(R) \neq \emptyset | R \in \mathscr{L}_{\ell-1}(G)\}$ is an edge partition of $\mathbb{L}_\ell(G)$. Consider the 2-link graph H in Figure 2 b. The vertex and edge partitions of H are indicated by the dotted rectangles and ellipses, respectively. The corresponding quotient graph is given in Figure 2 c.

Special partitions are required to describe the structure of ℓ-link graphs. Let H be a graph admitting partitions $\mathcal{V}$ of $V(H)$ and $\mathcal{E}$ of $E(H)$ that satisfy (a) and (b) above. $(\mathcal{V}, \mathcal{E})$ is called an *almost standard partition* of H if further:

(c) each part of $\mathcal{E}$ induces a complete bipartite subgraph of H,
(d) each vertex of H is incident to at most two parts of $\mathcal{E}$,
(e) for each $V \in \mathcal{V}$, and different $E, F \in \mathcal{E}$, V contains at most one vertex incident to both E and F.

If $\ell \geqslant 2$ is an even integer, and G is a simple graph, then $\mathbb{L}_\ell(G)$ is isomorphic to the $(2, \ell/2)$-*double star graph* of G introduced by Jia [12]. While this article focuses on the combinatorial properties including connectedness, coloring, and minors of $\mathbb{L}_\ell(G)$, a series of companion papers have been composed to contribute to the recognition and determination problems and algorithms. For example, a joint work by Ellingham and Jia [8] shows that, for a given graph H, there is at most one pair (G, ℓ), where $\ell \geqslant 2$, and G is a simple graph of minimum degree at least 3, such that $\mathbb{L}_\ell(G)$ is isomorphic to H. Moreover, such a pair can be determined from H in linear time.

3. GENERAL STRUCTURE OF ℓ-LINK GRAPHS

We begin by determining some basic properties of ℓ-link graphs, including their multiplicity and connectedness. The work in this section forms the basis for our main results on coloring and minors of ℓ-link graphs.

Let us first fix some concepts by two observations.

Observation 3.1. *The number of edges of $\mathbb{L}_\ell(G)$ is equal to the number of vertices of $\mathbb{L}_{\ell+1}(G)$. In particular, if G is r-regular for some $r \geqslant 2$, then this number is $|E(G)|(r-1)^\ell$. If further $\ell \geqslant 1$, then $\mathbb{L}_\ell(G)$ is $2(r-1)$-regular.*

Proof. Let G be r-regular, $n := |V(G)|$ and $m := |E(G)|$. We prove that $|\mathscr{L}_{\ell+1}(G)| = m(r-1)^\ell$ by induction on ℓ. It is trivial for $\ell = 0$. For $\ell = 1$, $|\mathscr{L}_2([v])| = \binom{r}{2}$, and hence $|\mathscr{L}_2(G)| = \binom{r}{2} n = m(r-1)$. Inductively assume $|\mathscr{L}_{\ell-1}(G)| = m(r-1)^{\ell-2}$ for some $\ell \geqslant 2$. For each $R \in \mathscr{L}_{\ell-1}(G)$, we have $|\mathscr{L}_{\ell+1}(R)| = (r-1)^2$ since $r \geqslant 2$. Thus $|\mathscr{L}_{\ell+1}(G)| = |\mathscr{L}_{\ell-1}(G)|(r-1)^2 = m(r-1)^\ell$ as desired. The other assertions follow from the definitions. ■

Observation 3.2. *Let $n, m \geqslant 2$. If $\ell \geqslant 1$ is odd, then $\mathbb{L}_\ell(K_{n,m})$ is $(n+m-2)$-regular with order $nm[(n-1)(m-1)]^{\frac{\ell-1}{2}}$. If $\ell \geqslant 2$ is even, then $\mathbb{L}_\ell(K_{n,m})$ has average degree $\frac{4(n-1)(m-1)}{n+m-2}$, and order $\frac{1}{2}nm(n+m-2)[(n-1)(m-1)]^{\frac{\ell}{2}-1}$.*

Proof. Let $\ell \geqslant 1$ be odd, and L be an ℓ-link of $K_{n,m}$ with middle edge incident to a vertex u of degree n in $K_{n,m}$. It is not difficult to see that L can be shunted in one step to $n-1$ ℓ-links whose middle edge is incident to u. By symmetry, each vertex of $\mathbb{L}_\ell(K_{n,m})$ is incident to $(n-1)+(m-1) = n+m-2$ edges. Now we prove $|\mathscr{L}_\ell(K_{n,m})| = nm[(n-1)(m-1)]^{\frac{\ell-1}{2}}$ by induction on ℓ. Clearly, $|\mathscr{L}_1(K_{n,m})| = |E(K_{n,m})| = nm$. Inductively assume $|\mathscr{L}_{\ell-2}(K_{n,m})| = nm[(n-1)(m-1)]^{\frac{\ell-3}{2}}$ for some $\ell \geqslant 3$. For each $R \in \mathscr{L}_{\ell-2}(K_{n,m})$, we have $|\mathscr{L}_\ell(R)| = (n-1)(m-1)$. So $|\mathscr{L}_\ell(K_{n,m})| = |\mathscr{L}_{\ell-2}(K_{n,m})|(n-1)(m-1) = nm[(n-1)(m-1)]^{\frac{\ell-1}{2}}$ as desired. The even ℓ case is similar. ■

3.1. Loops and Multiplicity

Our next observation is a prerequisite for the study of the chromatic number since it indicates that ℓ-link graphs are loopless.

Observation 3.3. *For each $(\ell+1)$-arc $\vec{Q}$, we have $\vec{Q}[0, \ell] \neq \vec{Q}[1, \ell+1]$.*

Proof. Let G be a graph, and $\vec{Q} := (v_0, e_1, \ldots, e_{\ell+1}, v_{\ell+1}) \in \vec{\mathscr{L}}_{\ell+1}(G)$. Since G is loopless, $v_0 \neq v_1$ and hence $\vec{Q}[0, 0] \neq \vec{Q}[1, 1]$. So the statement holds for $\ell = 0$. Moreover, $\vec{Q}(0, \ell) \neq \vec{Q}(1, \ell + 1)$. Now let $\ell \geqslant 1$. Suppose for a contradiction that $\vec{Q}(0, \ell) = -\vec{Q}(1, \ell + 1)$. Then $v_i = v_{\ell+1-i}$ and $e_{i+1} = e_{\ell+1-i}$ for $i \in \{0, 1, \ldots, \ell\}$. If $\ell = 2s$ for some integer $s \geqslant 1$, then $v_s = v_{s+1}$, contradicting that G is loopless. If $\ell = 2s + 1$ for some integer $s \geqslant 0$, then $e_{s+1} = e_{s+2}$, contradicting the definition of a $*$-arc. ■

The following statement indicates that, for each $\ell \geqslant 1$, $\mathbb{L}_\ell(G)$ is simple if G is simple, and has multiplicity exactly 2 otherwise.

Observation 3.4. *Let G be a graph, $\ell \geqslant 1$, and $L_0, L_1 \in \mathscr{L}_\ell(G)$. Then L_0 can be shunted to L_1 through two $(\ell + 1)$-links of G if and only if G contains a 2-cycle $O := [v_0, e_0, v_1, e_1, v_0]$, such that one of the following cases holds:*

(1) *$\ell \geqslant 1$ is odd, and $L_i = [v_i, e_i, v_{1-i}, e_{1-i}, \ldots, v_i, e_i, v_{1-i}] \in \mathscr{L}_\ell(O)$ for $i \in \{0, 1\}$. In this case, $[v_i, e_i, v_{1-i}, e_{1-i}, \ldots, v_{1-i}, e_{1-i}, v_i] \in \mathscr{L}_{\ell+1}(O)$, for $i \in \{0, 1\}$, are the only two $(\ell + 1)$-links available for the shunting.*
(2) *$\ell \geqslant 2$ is even, and $L_i = [v_i, e_i, v_{1-i}, e_{1-i}, \ldots, v_{1-i}, e_{1-i}, v_i] \in \mathscr{L}_\ell(O)$ for $i \in \{0, 1\}$. In this case, $[v_i, e_i, v_{1-i}, e_{1-i}, \ldots, v_i, e_i, v_{1-i}] \in \mathscr{L}_{\ell+1}(O)$, for $i \in \{0, 1\}$, are the only two $(\ell + 1)$-links available for the shunting.*

Proof. ($\Leftarrow$) is trivial. For ($\Rightarrow$), since L_0 can be shunted to L_1, there exists $\vec{L} := (v_0, e_0, v_1, \ldots, v_\ell, e_\ell, v_{\ell+1}) \in \vec{\mathscr{L}}_{\ell+1}(G)$ such that $L_i = \vec{L}[i, \ell + i]$ for $i \in \{0, 1\}$. Let $\vec{R} \in \vec{\mathscr{L}}_{\ell+1}(G) \setminus \{\vec{L}\}$ such that $L_i = \vec{R}[i, \ell + i]$. Then $\vec{L}(i, \ell + i)$ equals $\vec{R}(i, \ell + i)$ or $\vec{R}(\ell + i, i)$. Suppose for a contradiction that $\vec{L}(0, \ell) = \vec{R}(0, \ell)$. Then $\vec{L}(1, \ell) = \vec{R}(1, \ell)$. Since $\vec{L} \neq \vec{R}$, we have $\vec{L}(1, \ell + 1) \neq \vec{R}(1, \ell + 1)$. Thus $\vec{L}(1, \ell + 1) = \vec{R}(\ell + 1, 1)$, and hence $\vec{L}(2, \ell + 1) = \vec{R}(\ell, 1) = \vec{L}(\ell, 1)$, contradicting Observation 3.3. So $\vec{L}(0, \ell) = \vec{R}(\ell, 0)$. Similarly, $\vec{L}(1, \ell + 1) = \vec{R}(\ell + 1, 1)$. Consequently, $\vec{L}(0, \ell - 1) = \vec{R}(\ell, 1) = \vec{L}(2, \ell + 1)$; that is, $v_j = v_0$ and $e_j = e_0$ if $j \in [0, \ell]$ is even, while $v_j = v_1$ and $e_j = e_1$ if $j \in [0, \ell + 1]$ is odd. ■

3.2. Connectedness

This subsection characterizes when $\mathbb{L}_\ell(G)$ is connected. Let $L := [v_0, e_1, \ldots, e_\ell, v_\ell]$ be an ℓ-link of G, and $m := \lceil \frac{\ell}{2} \rceil$. The *middle unit* c_L of L is defined to be v_m if ℓ is even, and e_m if ℓ is odd. Denote by $G(\ell)$ the subgraph of G induced by the middle units of ℓ-links of G.

The lemma below is important in dealing with the connectedness of ℓ-link graphs. Before stating it, we define a *conjunction* operation, which is an extension of an operation by Biggs [3, Chapter 17]. Let $\vec{L} := (v_0, e_1, v_1, \ldots, e_\ell, v_\ell) \in \vec{\mathscr{L}}_\ell(G)$ and $\vec{R} := (u_0, f_1, u_1, \ldots, f_s, u_s) \in \vec{\mathscr{L}}_s(G)$ such that $v_\ell = u_0$ and $e_\ell \neq f_1$. The *conjunction* of $\vec{L}$ and $\vec{R}$ is $(\vec{L}.\vec{R}) := (v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s) \in \vec{\mathscr{L}}_{\ell+s}(G)$ or $[\vec{L}.\vec{R}] := [v_0, e_1, \ldots, e_\ell, v_\ell = u_0, f_1, \ldots, f_s, u_s] \in \mathscr{L}_{\ell+s}(G)$.

Lemma 3.5. *Let $\ell, s \geqslant 0$, and G be a connected graph. Then $G(\ell)$ is connected. And each s-link of $G(\ell)$ is a middle segment of a $(2\lfloor \frac{\ell}{2} \rfloor + s)$-link of G. Moreover, for ℓ-links L and R of G, there is an ℓ-link L' with middle unit c_L, and an ℓ-link R' with middle unit c_R, such that L' can be shunted to R'.*

Proof. For $\ell \in \{0, 1\}$, since G is connected, $G(\ell) = G$ and the lemma holds. Let $\ell := 2m \geqslant 2$ be even. Then $u, v \in V(G(\ell))$ if and only if they are middle vertices of some $\vec{L}, \vec{R} \in \vec{\mathscr{L}}_\ell(G)$, respectively. Since G is connected, there exists some $\vec{P} := (u = v_0, e_1, \ldots, e_s, v_s = v) \in \vec{\mathscr{L}}_s(G)$. By Observation 3.3, $\vec{L}[m-1, m] \neq \vec{L}[m, m+1]$. For such an s-arc $\vec{P}$, without loss of generality, $e_1 \neq \vec{L}[m-1, m]$, and similarly, $e_s \neq \vec{R}[m, m+1]$. Then $\vec{P}$ is a middle segment of $\vec{Q} := (\vec{L}(0, m).\vec{P}.\vec{R}(m, 2m)) \in \vec{\mathscr{L}}_{\ell+s}(G)$. So $L' := \vec{Q}[0, \ell]$ can be shunted to $R' := \vec{Q}[s, \ell + s]$ through $\vec{Q}$. Moreover, for each $i \in \{0, \ldots, s\}$, v_i is the middle vertex of $\vec{Q}[i, \ell + i] \in \mathscr{L}_\ell(G)$. Hence $\vec{P}$ is an s-arc of $G(\ell)$ from u to v. So $G(\ell)$ is connected. The odd ℓ case is similar. ■

Sufficient conditions for $\mathbb{A}_\ell(G)$ to be strongly connected can be found in [9, page 76]. The following corollary of Lemma 3.5 reveals a strong relationship between the shunting of ℓ-links and the connectedness of ℓ-link graphs.

Corollary 3.6. *For a connected graph G, $\mathbb{L}_\ell(G)$ is connected if and only if every pair of ℓ-links of G with the same middle unit can be shunted to each other.*

Proof. On the one hand, if $\mathbb{L}_\ell(G)$ is connected, then every pair of ℓ-links of G can be shunted to each other. On the other hand, let L and R be two ℓ-links of G. Since G is connected, by Lemma 3.5, there are ℓ-links L' and R' with $c_{L'} = c_L$ and $c_{R'} = c_R$ such that L' can be shunted to R'. Hence if L can be shunted to L' and R can be shunted to R', then L can be shunted to R. So if every pair of ℓ-links of G with the same middle unit can be shunted to each other, then $\mathbb{L}_\ell(G)$ is connected. ■

We now present our main result of this section, which plays a key role in dealing with the graph minors of ℓ-link graphs in Section 5.

Lemma 3.7. *Let G be a graph, and X be a connected subgraph of $G(\ell)$. Then for every pair of ℓ-links L and R of X, L can be shunted to R under the restriction that in each step, the middle unit of the image of L belongs to X.*

Proof. First we consider the case that c_L is in R. Then there is a common segment Q of L and R of maximum length containing c_L. Without loss of generality, assign directions to L and R such that $\vec{L} = (\vec{L}_0.\vec{Q}.\vec{L}_1)$ and $\vec{R} = (\vec{R}_1.\vec{Q}.\vec{R}_0)$, where $\vec{L}_i \in \vec{\mathscr{L}}_{\ell_i}(X)$ and $\vec{R}_i \in \vec{\mathscr{L}}_{s_i}(X)$ for $i \in \{0, 1\}$ such that $s_1 \geqslant s_0$. Then $\ell \geqslant \ell_0 + \ell_1 = s_0 + s_1 \geqslant s_1$. Let x be the head vertex and e be the head edge of $\vec{L}$. Since c_L is in Q, $\ell_0 \leqslant \ell/2$. Since X is a subgraph of $G(\ell)$, by Lemma 3.5, there exists $\vec{L}_2 \in \vec{\mathscr{L}}_{\ell_0}(G)$ with tail vertex x and tail edge different from e. Let y be the tail vertex and f be the tail edge of $\vec{R}$. Then there exits $\vec{R}_2 \in \vec{\mathscr{L}}_{s_0}(G)$ with head vertex y and head edge different from f. We can shunt L to R first through $(\vec{L}.\vec{L}_2) \in \vec{\mathscr{L}}_{\ell+\ell_0}(G)$, then $-(\vec{R}_2.\vec{R}_1.\vec{Q}.\vec{L}_1.\vec{L}_2) \in \vec{\mathscr{L}}_{\ell+\ell_0+\ell_1}(G)$, and finally $(\vec{R}_2.\vec{R}) \in \vec{\mathscr{L}}_{\ell+s_0}(G)$. Since $\ell_0 \leqslant \ell/2$ and $s_0 \leqslant s_1 \leqslant \ell/2$, the middle unit of each image is inside L or R.

Second, we consider the case that c_L is not in R. Then there exists a segment Q of L of maximum length that contains c_L, and is edge-disjoint with R. Since X is connected, there exists a shortest $*$-arc $\vec{P}$ from a vertex v of R to a vertex u of L. Then P is edge-disjoint with Q because of its minimality. Without loss of generality, assign directions to L and R such that u separates $\vec{L}$ into $(\vec{L}_0.\vec{L}_1)$ with c_L on L_1, and v separates $\vec{R}$ into $(\vec{R}_1.\vec{R}_0)$, where L_i is of length ℓ_i while R_i is of length s_i for $i \in \{0, 1\}$, such that $s_1 \geqslant s_0$. Then $\ell_0, s_0 \leqslant \ell/2$. Let x be the head vertex and e be the head edge of $\vec{L}$. Since $\ell_0 \leqslant \ell/2$ and X is a subgraph of $G(\ell)$, by Lemma 3.5, there exists an ℓ_0-arc $\vec{L}_2$ of G with tail vertex x

and tail edge different from e. Let y be the tail vertex and f be the tail edge of $\vec{R}$. Then there exits an s_0-arc $\vec{R}_2$ of G with head vertex y and head edge different from f. Now we can shunt L to R through $(\vec{L}.\vec{L}_2)$, $-(\vec{R}_2.\vec{R}_1.\vec{P}.\vec{L}_1.\vec{L}_2)$ and $(\vec{R}_2.\vec{R})$ consecutively. One can check that in this process the middle unit of each image belongs to L, P, or R. ■

From Lemma 3.7, the set of ℓ-links of a connected $G(\ell)$ serves as a "hub" in the shunting of ℓ-links of G. More explicitly, for $L, R \in \mathscr{L}_\ell(G)$, if we can shunt L to $L' \in \mathscr{L}_\ell(G(\ell))$, and R to $R' \in \mathscr{L}_\ell(G(\ell))$, then L can be shunted to R since L' can be shunted to R'. Thus we have the following corollary that provides a more efficient way to test the connectedness of ℓ-link graphs.

Corollary 3.8. *Let G be a graph such that $G(\ell)$ contains at least one ℓ-link. Then $\mathbb{L}_\ell(G)$ is connected if and only if $G(\ell)$ is connected, and each ℓ-link of G can be shunted to an ℓ-link of $G(\ell)$.*

4. CHROMATIC NUMBER OF ℓ-LINK GRAPHS

In this section, we reveal a recursive structure of an ℓ-link graph H, which leads to an upper bound for the chromatic number of H. To achieve this, we need to show that when $\ell \geqslant 2$, H admits an almost standard partition defined in Section 2.

Lemma 4.1. *Let G be a graph and $\ell \geqslant 2$ be an integer. Then $(\mathcal{V}, \mathcal{E}) := (\mathcal{V}_\ell(G), \mathcal{E}_\ell(G))$ is an almost standard partition of $H := \mathbb{L}_\ell(G)$. Further, $H_{(\mathcal{V},\mathcal{E})}$ is isomorphic to an induced subgraph of $\mathbb{L}_{\ell-2}(G)$.*

Proof. First we verify that $(\mathcal{V}, \mathcal{E})$ satisfies conditions (a)–(e) in the definition of an almost standard partition in Section 2.

(a) We prove that, for each $R \in \mathscr{L}_{\ell-2}(G)$, $V := \mathscr{L}_\ell(R) \in \mathcal{V}$ is an independent set of H. Suppose not. Then there are $\vec{L}, \vec{L}' \in \vec{\mathscr{L}}_\ell(G)$ such that $L, L' \in V$, and L can be shunted to L' in one step. Then $R = \vec{L}[1, \ell-1]$ can be shunted to $R = \vec{L}'[1, \ell-1]$ in one step, contradicting Observation 3.3.
(b) Here we show that each $E \in \mathcal{E}$ is incident to exactly two parts of $\mathcal{V}$. By definition there exists $P \in \mathscr{L}_{\ell-1}(G)$ with $\mathscr{L}_{\ell+1}(P) = E$. Let $\{L, R\} := P^{\{\ell-2\}}$. Then $\mathscr{L}_\ell(L)$ and $\mathscr{L}_\ell(R)$ are the only two parts of $\mathcal{V}$ incident to E.
(c) We explain that each $E \in \mathcal{E}$ is the edge set of a complete bipartite subgraph of H. By definition there exists $\vec{P} \in \vec{\mathscr{L}}_{\ell-1}(G)$ with $\mathscr{L}_{\ell+1}(P) = E$. Let $A := \{[\vec{e}.\vec{P}] \in \mathscr{L}_\ell(G)\}$ and $B := \{[\vec{P}.\vec{f}] \in \mathscr{L}_\ell(G)\}$. One can check that E induces a complete bipartite subgraph of H with bipartition $A \cup B$.
(d) We prove that each $v \in V(H)$ is incident to at most two parts of $\mathcal{E}$. By definition there exists $Q \in \mathscr{L}_\ell(G)$ with $Q = v$. Then the set of edge parts of $\mathcal{E}$ incident to v is $\{\mathscr{L}_{\ell+1}(L) \neq \emptyset | L \in Q^{\{\ell-1\}}\}$ with cardinality at most 2.
(e) Let v be a vertex of $V \in \mathcal{V}$ incident to different $E, F \in \mathcal{E}$. We explain that v is uniquely determined by V, E, and F.

By the analysis above, $(\mathcal{V}, \mathcal{E})$ is an almost standard partition of H.

By definition there exists $\vec{P} \in \vec{\mathscr{L}}_{\ell-2}(G)$ such that $V = \mathscr{L}_\ell(P)$. There also exists $Q := [\vec{e}_1.\vec{P}.\vec{e}_\ell] \in \mathscr{L}_\ell(P)$ such that $v = Q$. Besides, there are $L, R \in \mathscr{L}_{\ell-1}(G)$ such that $E = \mathscr{L}_{\ell+1}(L)$ and $F = \mathscr{L}_{\ell+1}(R)$. Then $\{L, R\} = Q^{\{\ell-1\}}$ since $L \neq R$. Note that Q is uniquely determined by $Q^{\{\ell-1\}}$ and $c_Q = c_P$. Thus it is uniquely determined by $E = \mathscr{L}_{\ell+1}(L)$, $F = \mathscr{L}_{\ell+1}(R)$, and $V = \mathscr{L}_\ell(P)$.

Now we show that $H_{(\mathcal{V},\mathcal{E})}$ is isomorphic to an induced subgraph of $\mathbb{L}_{\ell-2}(G)$. Let X be the subgraph of $\mathbb{L}_{\ell-2}(G)$ of vertices $L \in \mathscr{L}_{\ell-2}(G)$ such that $\mathscr{L}_\ell(L) \neq \emptyset$, and edges $Q \in \mathscr{L}_{\ell-1}(G)$ such that $\mathscr{L}_{\ell+1}(Q) \neq \emptyset$. One can check that X is an induced subgraph of $\mathbb{L}_{\ell-2}(G)$. An isomorphism from $H_{(\mathcal{V},\mathcal{E})}$ to X can be defined as the injection sending $\mathscr{L}_\ell(L) \neq \emptyset$ to L, and $\mathscr{L}_{\ell+1}(Q) \neq \emptyset$ to Q. ■

Below we give an interesting algorithm for coloring a class of graphs.

Lemma 4.2. *Let H be a graph with a t-coloring such that each vertex of H is adjacent to at most $r \geqslant 0$ differently colored vertices. Then $\chi(H) \leqslant \lfloor \frac{tr}{r+1} \rfloor + 1$.*

Proof. The result is trivial for $t = 0$ since, in this case, $\chi(H) = 0$. If $r + 1 \geqslant t \geqslant 1$, then $\lfloor \frac{tr}{r+1} \rfloor = \lfloor t - \frac{t}{r+1} \rfloor = t - 1$, and the lemma holds since $t \geqslant \chi(H)$.

Now assume $t \geqslant r + 2 \geqslant 2$. Let $U_1, U_2, \ldots, U_t$ be the color classes of the given coloring. For $i \in [t]$, denote by i the color assigned to vertices in U_i. Run the following algorithm: For $j = 1, \ldots, t$, and for each $u \in U_{t-j+1}$, let $s \in [t]$ be the minimum integer that is not the color of a neighbor of u in H; if $s < t - j + 1$, then recolor u by s.

In the algorithm above, denote by C_i the set of colors used by the vertices in U_i for $i \in [t]$. Let $k := \lfloor \frac{t-1}{r+1} \rfloor$. Then $t - 1 \geqslant k(r + 1) \geqslant k \geqslant 1$. We claim that after $j \in [0, k]$ steps, $C_{t-i+1} \subseteq [ir + 1]$ for $i \in [j]$, and $C_i = \{i\}$ for $i \in [t - j]$. This is trivial for $j = 0$. Inductively assume it holds for some $j \in [0, k - 1]$. In the $(j + 1)$th step, we change the color of each $u \in U_{t-j}$ from $t - j$ to the minimum $s \in [t]$ that is not used by the neighborhood of u. It is enough to show that $s \leqslant (j + 1)r + 1$.

First suppose that all neighbors of u are in $\bigcup_{i\in[t-j-1]} U_i$. By the analysis above, $t - j - 1 \geqslant t - k \geqslant kr + 1 \geqslant r + 1$. So at least one part of $\mathcal{S} := \{U_i | i \in [t - j - 1]\}$ contains no neighbor of u. From the induction hypothesis, $C_i = \{i\}$ for $i \in [t - j - 1]$. Hence at least one color in $[r + 1]$ is not used by the neighborhood of u; that is, $s \leqslant r + 1 \leqslant (j + 1)r + 1$.

Now suppose that u has at least one neighbor in $\bigcup_{i\in[t-j+1,t]} U_i$. By the induction hypothesis, $\bigcup_{i\in[t-j+1,t]} C_i \subseteq [jr + 1]$. At the same time, u has neighbors in at most $r - 1$ parts of $\mathcal{S}$. So the colors possessed by the neighborhood of u are contained in $[jr + 1 + r - 1] = [(j + 1)r]$. Thus $s \leqslant (j + 1)r + 1$. This proves our claim.

The claim above indicates that, after the kth step, $C_{t-i+1} \subseteq [ir + 1]$ for $i \in [k]$, and $C_i = \{i\}$ for $i \in [t - k]$. Hence we have a $(t - k)$-coloring of H since $t - k \geqslant kr + 1$. Therefore, $\chi(H) \leqslant t - k = \lceil \frac{tr+1}{r+1} \rceil = \lfloor \frac{tr}{r+1} \rfloor + 1$. ■

Lemma 4.1 indicates that $\mathbb{L}_\ell(G)$ is homomorphic to $\mathbb{L}_{\ell-2}(G)$ for $\ell \geqslant 2$. So by [6, Proposition 1.1], $\chi_\ell(G) \leqslant \chi_{\ell-2}(G)$. By Lemma 4.1, every vertex of $\mathbb{L}_\ell(G)$ has neighbors in at most two parts of $\mathcal{V}_\ell(G)$, which enables us to improve the upper bound on $\chi_\ell(G)$.

Lemma 4.3. *Let G be a graph, and $\ell \geqslant 2$. Then $\chi_\ell(G) \leqslant \lfloor \frac{2}{3}\chi_{\ell-2}(G) \rfloor + 1$.*

Proof. By Lemma 4.1, $(\mathcal{V}, \mathcal{E}) := (\mathcal{V}_\ell(G), \mathcal{E}_\ell(G))$ is an almost standard partition of $H := \mathbb{L}_\ell(G)$. So each vertex of H has neighbors in atmost two parts of $\mathcal{V}$. Further, $H_\mathcal{V}$ is a subgraph of $\mathbb{L}_{\ell-2}(G)$. So $\chi_\ell(G) \leqslant \chi := \chi(H_\mathcal{V}) \leqslant \chi_{\ell-2}(G)$.

We now construct a χ-coloring of H such that each vertex of H is adjacent to at most two differently colored vertices. By definition $H_\mathcal{V}$ admits a χ-coloring with color classes $K_1, \ldots, K_\chi$. For $i \in [\chi]$, assign the color i to each vertex of H in $U_i := \bigcup_{V_\mathcal{V}\in K_i} V$. One can check that this is a desired coloring. In Lemma 4.3, letting $t = \chi$ and $r = 2$ yields that $\chi_\ell(G) \leqslant \lfloor \frac{2}{3}\chi \rfloor + 1$. Recall that $\chi \leqslant \chi_{\ell-2}(G)$. Thus the lemma follows. ■

As shown below, Lemma 4.3 can be applied recursively to produce an upper bound for $\chi_\ell(G)$ in terms of $\chi(G)$ or $\chi'(G)$.

Proof of Theorem 1.1. When $\ell \in \{0, 1\}$, it is trivial for (1)(2) and (4). By [7, Proposition 5.2.2], $\chi_0 = \chi \leqslant \Delta + 1$. So (3) holds. Now let $\ell \geqslant 2$. By Lemma 4.1, $H := \mathbb{L}_\ell(G)$ admits an almost standard partition $(\mathcal{V}, \mathcal{E}) := (\mathcal{V}_\ell(G), \mathcal{E}_\ell(G))$, such that $H_{(\mathcal{V},\mathcal{E})}$ is an induced subgraph of $\mathbb{L}_{\ell-2}(G)$. By definition each part of $\mathcal{V}$ is an independent set of H. So $H \to \mathbb{L}_{\ell-2}(G)$, and $\chi_\ell \leqslant \chi_{\ell-2}$. This proves (4). Moreover, each vertex of H has neighbors in at most two parts of $\mathcal{V}$. By Lemma 4.3, $\chi_\ell := \chi_\ell(G) \leqslant \frac{2\chi_{\ell-2}}{3} + 1$. Continue the analysis, we have $\chi_\ell \leqslant \chi_{\ell-2i}$, and $\chi_\ell - 3 \leqslant (\frac{2}{3})^i(\chi_{\ell-2i} - 3)$ for $1 \leqslant i \leqslant \lfloor \ell/2 \rfloor$. Therefore, if ℓ is even, then $\chi_\ell \leqslant \chi_0 = \chi \leqslant \Delta + 1$, and $\chi_\ell - 3 \leqslant (\frac{2}{3})^{\ell/2}(\chi - 3)$. Thus (1) holds. Now let $\ell \geqslant 3$ be odd. Then $\chi_\ell \leqslant \chi_1 = \chi'$, and $\chi_\ell - 3 \leqslant (\frac{2}{3})^{\frac{\ell-1}{2}}(\chi' - 3)$. This verifies (2). As a consequence, $\chi_\ell \leqslant \chi_3 \leqslant \frac{2}{3}(\chi' - 3) + 3 = \frac{2}{3}\chi' + 1$. By Shannon [18], $\chi' \leqslant \frac{3}{2}\Delta$. So $\chi_\ell \leqslant \Delta + 1$, and hence (3) holds. ■

The following corollary of Theorem 1.1 implies that Hadwiger's conjecture is true for $\mathbb{L}_\ell(G)$ if G is regular and $\ell \geqslant 4$.

Corollary 4.4. *Let G be a graph with $\Delta := \Delta(G) \geqslant 3$. Then $\chi_\ell(G) \leqslant 3$ for all $\ell > 2\log_{1.5}(\Delta - 2) + 3$. Further, Hadwiger's conjecture holds for $\mathbb{L}_\ell(G)$ if $\ell > 2\log_{1.5}(\Delta - 2) - 3.83$, or $d := d(G) \geqslant 3$ and $\ell > 2\log_{1.5}\frac{\Delta-2}{d-2} + 3$.*

Proof. By Theorem 1.1, for each $t \geqslant 3$, $\chi_\ell := \chi_\ell(G) \leqslant t$ if $(\frac{2}{3})^{\ell/2}(\Delta - 2) < t - 2$ and $(\frac{2}{3})^{\frac{\ell-1}{2}}(\frac{3}{2}\Delta - 3) < t - 2$. Solving these inequalities gives $\ell > 2\log_{1.5}(\Delta - 2) - 2\log_{1.5}(t - 2) + 3$. Thus $\chi_\ell \leqslant 3$ if $\ell > 2\log_{1.5}(\Delta - 2) + 3$. So the first statement holds. By Robertson et al. [17] and Theorem 1.3, Hadwiger's conjecture holds for $\mathbb{L}_\ell(G)$ if $\ell \geqslant 1$ and $\chi_\ell \leqslant \max\{6, d\}$. Letting $t = 6$ gives that $\ell > 2\log_{1.5}(\Delta - 2) - 4\log_{1.5} 2 + 3$. Letting $t = d \geqslant 3$ gives that $\ell > 2\log_{1.5}\frac{\Delta-2}{d-2} + 3$. So the corollary holds since $4\log_{1.5} 2 - 3 > 3.83$. ■

Proof of Theorem 1.5(3)(4)(5). (3) and (4) follow from Corollary 4.4. Now consider (5). By Reed and Seymour [16], Hadwiger's conjecture holds for $\mathbb{L}_1(G)$. If $\ell \geqslant 2$ and $\Delta \leqslant 5$, by Theorem 1.1(3), $\chi_\ell(G) \leqslant 6$. In this case, Hadwiger's conjecture holds for $\mathbb{L}_\ell(G)$ by Robertson et al. [17]. ■

5. COMPLETE MINORS OF ℓ-LINK GRAPHS

It has been proved in the last section that Hadwiger's conjecture is true for $\mathbb{L}_\ell(G)$ if ℓ is large enough. In this section, we further investigate the minors, especially the complete minors, of ℓ-link graphs. To see the intuition of our method, let v be a vertex of degree t in a graph G. Then $\mathbb{L}_1(G)$ contains a K_t-subgraph whose vertices correspond to the edges of G incident to v. For $\ell \geqslant 2$, roughly speaking, we extend v to a subgraph X of diameter less than ℓ, and extend each edge incident to v to an ℓ-link of G starting from a vertex of X. By studying the shunting of these ℓ-links, we find a K_t-minor in $\mathbb{L}_\ell(G)$.

Let $[u, e, v]$ be a 1-link of G. Since G is undirected, e has no direction. But we can choose a direction, say u to v, for e to get an arc $\vec{e} := (u, e, v)$ of G. For subgraphs X, Y

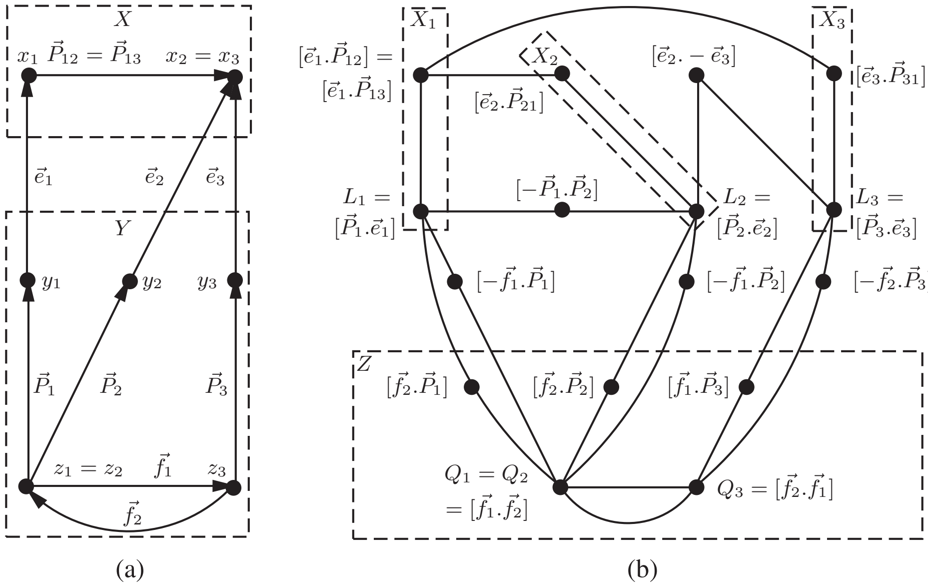

FIGURE 3. (a) G, (b) $\mathbb{L}_2(G)$.

of G, let $E(X, Y)$ be the set of edges of G between $V(X)$ and $V(Y)$, and $\vec{E}(X, Y)$ be the set of arcs of G from $V(X)$ to $V(Y)$. Figure 3 illustrates the proofs of Lemmas 5.1 and 5.2.

Lemma 5.1. *Let $\ell \geqslant 1$ be an integer, G be a graph, and X be a subgraph of G with $diam(X) < \ell$ such that $Y := G - V(X)$ is connected. If $t := |E(X, Y)| \geqslant 2$, then $\mathbb{L}_\ell(G)$ contains a K_t-minor.*

Proof. Let $\vec{e}_1, \ldots, \vec{e}_t$ be distinct arcs in $\vec{E}(Y, X)$. Say $\vec{e}_i = (y_i, e_i, x_i)$ for $i \in [t]$. Since $diam(X) < \ell$, there is a dipath $\vec{P}_{ij}$ of X from x_i to x_j of length $\ell_{ij} \leqslant \ell - 1$ such that $P_{ij} = P_{ji}$. Since Y is connected, it contains a dipath $\vec{Q}_{ij}$ from y_i to y_j. Since $t \geqslant 2$, $O_i := [\vec{P}_{i\,i'}. - \vec{e}_{i'}.\vec{Q}_{i'\,i}.\vec{e}_i]$ is a cycle of G, where $i' := (i \bmod t) + 1$. Thus $H := \mathbb{L}_\ell(G)$ contains a cycle $\mathbb{L}_\ell(O_1)$, and hence a K_2-minor. Now let $t \geqslant 3$, and $\vec{L}_i \in \vec{\mathscr{L}}_\ell(O_i)$ with head arc $\vec{e}_i$. Then $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]} \in \mathscr{L}_{\ell_{ij}}(H)$. And the union of the units of $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]}$ over $j \in [t]$ is a connected subgraph X_i of H. In the remainder of the proof, for distinct $i, j \in [t]$, we show that X_i and X_j are disjoint. Further, we construct a path in H between X_i and X_j that is internally disjoint with its counterparts, and has no inner vertex in any of $V(X_1), \ldots, V(X_t)$. Then by contracting each X_i into a vertex, and each path into an edge, we obtain a K_t-minor of H.

First of all, assume for a contradiction that there are different $i, j \in [t]$ such that X_i and X_j share a common vertex that corresponds to an ℓ-link R of G. Then by definition, there exists some $p \in [t]$ such that R can be obtained by shunting L_i along $(\vec{L}_i.\vec{P}_{ip})$ by some $s_i \leqslant \ell_{ip}$ steps. So $R = [\vec{L}_i(s_i, \ell).\vec{P}_{ip}(0, s_i)]$. Similarly, there are $q \in [t]$ and $s_j \leqslant \ell_{jq}$ such that $R = [\vec{L}_j(s_j, \ell).\vec{P}_{jq}(0, s_j)]$. Recall that $E(X) \cap E(X, Y) = E(Y) \cap E(X, Y) = \emptyset$. So $e_i = \vec{L}_i[\ell - 1, \ell]$ and $e_j = \vec{L}_j[\ell - 1, \ell]$ belong to both L_i and L_j. By the definition of O_i, this happens if and only if $i = j'$ and $j = i'$, which is impossible since $t \geqslant 3$.

Second, for distinct $i, j \in [t]$, we define a path of H between X_i and X_j. Clearly, L_i can be shunted to L_j through $\vec{R}'_{ij} := (\vec{L}_i.\vec{P}_{ij}. - \vec{L}_j)$ in G. In this shunting, $L'_i := [\vec{L}_i(\ell_{ij}, \ell).\vec{P}_{ij}]$ is the last image corresponding to a vertex of X_i, while $L'_j := [\vec{P}_{ij}.\vec{L}_j(\ell, \ell_{ij})]$ is the first image corresponding to a vertex of X_j. Further, L'_i can be shunted to L'_j through $\vec{R}_{ij} := (\vec{L}_i(\ell_{ij}, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell_{ij})) \in \mathscr{L}_{2\ell-\ell_{ij}}(G)$, which is a subsequence of $\vec{R}'_{ij}$. Then $R^{[\ell]}_{ij}$ is an $(\ell - \ell_{ij})$-path of H between X_i and X_j. We show that for each $p \in [t]$, X_p contains no inner vertex of $R^{[\ell]}_{ij}$. When $\ell - \ell_{ij} = 1$, $R^{[\ell]}_{ij}$ contains no inner vertex. Now assume $\ell - \ell_{ij} \geqslant 2$. Each inner vertex of $R^{[\ell]}_{ij}$ corresponds to some $Q_{ij} := [\vec{L}_i(s_i, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell + \ell_{ij} - s_i)] \in \mathscr{L}_\ell(G)$, where $\ell_{ij} + 1 \leqslant s_i \leqslant \ell - 1$. Assume for a contradiction that for some $p \in [t]$, X_p contains a vertex corresponding to Q_{ij}. By definition there exists $q \in [t]$ such that $Q_{ij} = [\vec{L}_p(s_p, \ell).\vec{P}_{pq}(0, s_p)]$, where $0 \leqslant s_p \leqslant \ell_{pq}$. Without loss of generality, $(\vec{L}_i(s_i, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell + \ell_{ij} - s_i)) = (\vec{L}_p(s_p, \ell).\vec{P}_{pq}(0, s_p))$. Since e_j and e_p are not in P_{pq}, hence $\vec{e}_j$ belongs to $-\vec{L}_p$ and $\vec{e}_p$ belongs to $-\vec{L}_j$. By the definition of $\vec{L}_i$, this happens only when $j = p'$ and $p = j'$, contradicting $t \geqslant 3$.

We now show that $R^{[\ell]}_{ij}$ and $R^{[\ell]}_{pq}$ are internally disjoint, where $i \neq j$, $p \neq q$ and $\{i, j\} \neq \{p, q\}$. Suppose not. Then by the analysis above, there are s_i and s_p with $\ell_{ij} + 1 \leqslant s_i \leqslant \ell - 1$ and $\ell_{pq} + 1 \leqslant s_p \leqslant \ell - 1$ such that $Q_{ij} = Q_{pq}$. Without loss of generality, $(\vec{L}_i(s_i, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell + \ell_{ij} - s_i)) = (\vec{L}_p(s_p, \ell).\vec{P}_{pq}.\vec{L}_q(\ell, \ell + \ell_{pq} - s_p))$. If $s_i = s_p$, then $\vec{e}_i = \vec{e}_p$ and $\vec{e}_j = \vec{e}_q$ since $E(X) \cap E(X, Y) = \emptyset$; that is, $i = p$ and $j = q$, contradicting $\{i, j\} \neq \{p, q\}$. Otherwise, with no loss of generality, $s_i > s_p$. Then $\vec{e}_q$ and $\vec{e}_i$ belong to $\vec{L}_j$ and $\vec{L}_p$, respectively; that is, $i = p$ and $j = q$, again contradicting $\{i, j\} \neq \{p, q\}$.

In summary, $X_1, \ldots, X_t$ are vertex-disjoint connected subgraphs, which are pairwise connected by internally disjoint $*$-links $R^{[\ell]}_{ij}$ of H, such that no inner vertex of $R^{[\ell]}_{ij}$ is in $V(X_1) \cup \cdots \cup V(X_t)$. So by contracting each X_i to a vertex, and $R^{[\ell]}_{ij}$ to an edge, we obtain a K_t-minor of H. ■

Lemma 5.2. *Let $\ell \geqslant 1$, G be a graph, and X be a subgraph of G with $diam(X) < \ell$ such that $Y := G - V(X)$ is connected and contains a cycle. Let $t := |E(X, Y)|$. Then $\mathbb{L}_\ell(G)$ contains a K_{t+1}-minor.*

Proof. Let O be a cycle of Y. Then $H := \mathbb{L}_\ell(G)$ contains a cycle $\mathbb{L}_\ell(O)$ and hence a K_2-minor. Now assume $t \geqslant 2$. Let $\vec{e}_1, \ldots, \vec{e}_t$ be distinct arcs in $\vec{E}(Y, X)$. Say $\vec{e}_i = (y_i, e_i, x_i)$ for $i \in [t]$. Since Y is connected, there is a dipath $\vec{P}_i$ of Y of minimum length $s_i \geqslant 0$ from some vertex z_i of O to y_i. Let $\vec{Q}_i$ be an ℓ-arc of O with head vertex z_i. Then $\vec{L}_i := (\vec{Q}_i.\vec{P}_i.\vec{e}_i)(s_i + 1, \ell + s_i + 1) \in \vec{\mathscr{L}}_\ell(G)$. Since $diam(X) \leqslant \ell - 1$, there is a dipath $\vec{P}_{ij}$ of X of length $\ell_{ij} \leqslant \ell - 1$ from x_i to x_j such that $P_{ij} = P_{ji}$.

Clearly, $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]}$ is an ℓ_{ij}-link of H. And the union of the units of $[\vec{L}_i.\vec{P}_{ij}]^{[\ell]}$ over $j \in [t]$ induces a connected subgraph X_i of H. For different $i, j \in [t]$, let $R_{ij} := [\vec{L}_i(\ell_{ij}, \ell).\vec{P}_{ij}.\vec{L}_j(\ell, \ell_{ij})] = R_{ji} \in \mathscr{L}_{2\ell-\ell_{ij}}(G)$. Then $R^{[\ell]}_{ij}$ is an $(\ell - \ell_{ij})$-path of H between X_i and X_j. As in the proof of Lemma 5.1, it is easy to check that $X_1, \ldots, X_t$ are vertex-disjoint connected subgraphs of H, which are pairwise connected by internally disjoint paths $R^{[\ell]}_{ij}$. Further, no inner vertex of $R^{[\ell]}_{ij}$ is in $V(X_1) \cup \cdots \cup V(X_t)$. So a K_t-minor of H is obtained accordingly.

Finally, let Z be the connected subgraph of H induced by the units of $\mathbb{L}_\ell(O)$ and $[\vec{Q}_i.\vec{P}_i]^{[\ell]}$ over $i \in [t]$. Then Z is vertex-disjoint with X_i and with the paths $R^{[\ell]}_{ij}$. Moreover, Z sends an edge $(\vec{Q}_i.\vec{P}_i.\vec{e}_i)(s_i, \ell + s_i + 1)^{[\ell]}$ to each X_i. Thus H contains a K_{t+1}-minor. ■

In the following, we use the "hub" (described after Lemma 3.7) to construct certain minors in ℓ-link graphs.

Corollary 5.3. *Let $\ell \geqslant 0$, G be a graph, M be a minor of $G(\ell)$ such that each branch set contains an ℓ-link. Then $\mathbb{L}_\ell(G)$ contains an M-minor.*

Proof. Let $X_1, \ldots, X_t$ be the branch sets of an M-minor of $G(\ell)$ such that X_i contains an ℓ-link for each $i \in [t]$. For any connected subgraph Y of $G(\ell)$ contains at least one ℓ-link, let $\mathbb{L}_\ell(G, Y)$ be the subgraph of $H := \mathbb{L}_\ell(G)$ induced by the ℓ-links of G of which the middle units are in Y. Let $H(Y)$ be the union of the components of $\mathbb{L}_\ell(G, Y)$, which contains at least one vertex corresponding to an ℓ-link of Y. By Lemma 3.7, $H(Y)$ is connected.

By definition each edge of M corresponds to an edge e of $G(\ell)$ between two different branch sets, say X_i and X_j. Let Y be the graph consisting of X_i, X_j, and e. Then $H(X_i)$ and $H(X_j)$ are vertex-disjoint since X_i and X_j are vertex-disjoint. By the analysis above, $H(X_i)$ and $H(X_j)$ are connected subgraphs of the connected graph $H(Y)$. Thus there is a path Q of $H(Y)$ joining $H(X_i)$ and $H(X_j)$ only at end vertices. Further, if ℓ is even, then Q is an edge; otherwise, Q is a 2-path whose middle vertex corresponds to an ℓ-link L of Y such that $c_L = e$. This implies that Q is internally disjoint with its counterparts and has no inner vertex in any branch set. Then, by contracting each $H(X_i)$ to a vertex, and Q to an edge, we obtain an M-minor of H. ∎

Now we are ready to give a lower bound for the Hadwiger number of $\mathbb{L}_\ell(G)$.

Proof of Theorem 1.3. Since $H := \mathbb{L}_\ell(G)$ contains an edge, $t := \eta(H) \geqslant 2$. We first show that $t \geqslant d := d(G)$. By definition there exists a subgraph X of G with $\delta(X) = d$. We may assume that $d \geqslant 3$ and $\ell \geqslant 2$. Then X contains an $(\ell - 1)$-arc $\vec{P} := (u, e, \ldots, f, v)$. Since the degree of u in X is at least d, there are $d - 1$ distinct arcs $\vec{e}_1, \ldots, \vec{e}_{d-1}$ of X with head vertex u such that $e_i \neq e$ for $i \in [d-1]$. Similarly, there are $d - 1$ distinct arcs $\vec{f}_1, \ldots, \vec{f}_{d-1}$ of X with tail vertex v such that $f_j \neq f$ for $j \in [d-1]$. Then the ℓ-link $L_i := [\vec{e}_i.\vec{P}]$ can be shunted to the ℓ-link $R_j := [\vec{P}.\vec{f}_j]$ through the $(\ell+1)$-link $Q_{ij} := [\vec{e}_i.\vec{P}.\vec{f}_j]$. So H contains a $K_{d-1,d-1}$-subgraph with bipartition $\{L_i | i \in [d-1]\} \cup \{R_j | j \in [d-1]\}$ and edge set $\{Q_{ij} | i, j \in [d-1]\}$. By Zelinka [25], $K_{d-1,d-1}$ contains a K_d-minor. Thus $t \geqslant d$ as desired.

We now show that $t \geqslant \eta := \eta(G)$. If $\eta = 3$, then G contains a cycle O of length at least 3, and H contains a K_3-minor contracted from $\mathbb{L}_\ell(O)$. Now assume that G is connected with $\eta \geqslant 4$. Repeatedly delete vertices of degree 1 in G until $\delta(G) \geqslant 2$. Then $G = G(\ell)$. Clearly, this process does not reduce the Hadwiger number of G. So G contains branch sets of a K_η-minor covering $V(G)$ (see [24]). If every branch set contains an ℓ-link, then the statement follows from Corollary 5.3. Otherwise, there exists some branch set X with $diam(X) < \ell$. Since $\eta \geqslant 4$, $Y := G - V(X)$ is connected and contains a cycle. Thus by Lemma 5.2, H contains a K_η-minor since $|E(X, Y)| \geqslant \eta - 1$. ∎

Here we prove Hadwiger's conjecture for $\mathbb{L}_\ell(G)$ for even $\ell \geqslant 2$.

Proof of Theorem 1.5(2). Let $d := d(G)$, $\ell \geqslant 2$ be an even integer, and $H := \mathbb{L}_\ell(G)$. By [7, Proposition 5.2.2], $\chi := \chi(G) \leqslant d + 1$. So by Theorem 1.1, $\chi(H) \leqslant \min\{d + 1, \frac{2}{3}d + \frac{5}{3}\}$. If $d \leqslant 4$, then $\chi(H) \leqslant 5$. By Robertson et al. [17], Hadwiger's conjecture holds for H in this case. Otherwise, $d \geqslant 5$. By Theorem 1.3, $\eta(H) \geqslant d \geqslant \frac{2}{3}d + \frac{5}{3} \geqslant \chi(H)$ and the statement follows. ∎

We end this article by proving Hadwiger's conjecture for ℓ-link graphs of biconnected graphs for $\ell \geqslant 1$.

Proof of Theorem 1.5(1). By Reed and Seymour [16], Hadwiger's conjecture holds for $H := \mathbb{L}_\ell(G)$ for $\ell = 1$. By Theorem 1.5(2), the conjecture is true if $\ell \geqslant 2$ is even. So we only need to consider the situation that $\ell \geqslant 3$ is odd. If G is a cycle, then H is a cycle and the conjecture holds [10]. Now let v be a vertex of G with degree $\Delta := \Delta(G) \geqslant 3$. By Theorem 1.1, $\chi(H) \leqslant \Delta + 1$. Since G is biconnected, $Y := G - v$ is connected. By Lemma 5.2, if Y contains a cycle, then $\eta(H) \geqslant \Delta + 1 \geqslant \chi(H)$. Now assume that Y is a tree, which implies that G is K_4-minor free. By Lemma 5.1, $\eta(H) \geqslant \Delta$. By Theorem 1.1, $\chi(H) \leqslant \chi' := \chi'(G)$. So it is enough to show that $\chi' = \Delta$.

Let $U := \{u \in V(Y) |\ \deg_Y(u) \leqslant 1\}$. Then $|U| \geqslant \Delta(Y)$. Let $\hat{G}$ be the underlying simple graph of G, $t := \deg_{\hat{G}}(v) \geqslant 1$ and $\hat{\Delta} := \Delta(\hat{G}) \geqslant t$. Since G is biconnected, $U \subseteq N_G(v)$. So $t \geqslant |U| \geqslant \Delta(Y)$. Let $u \in U$. When $|U| = 1$, $t = \deg_{\hat{G}}(u) = 1$. When $|U| \geqslant 2$, $\deg_{\hat{G}}(u) = 2 \leqslant |U| \leqslant t$. Thus $t = \hat{\Delta}$. Juvan et al. [14] proved that the edge-chromatic number of a K_4-minor free simple graph equals the maximum degree of this graph. So $\hat{\chi}' := \chi'(\hat{G}) = \hat{\Delta}$ since $\hat{G}$ is simple and K_4-minor free. Note that all parallel edges of G are incident to v. So $\chi' = \hat{\chi}' + \deg_G(v) - t = \hat{\Delta} + \Delta - \hat{\Delta} = \Delta$ as desired.■

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