Treewidth of the Line Graph of a Complete Graph

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Abstract: In recent articles by Grohe and Marx, the treewidth of the line graph of a complete graph is a critical example—in a certain sense, every graph with large treewidth "contains" $L(K_n)$. However, the treewidth of $L(K_n)$ was not determined exactly. We determine the exact treewidth of the line graph of a complete graph. © 2014 Wiley Periodicals, Inc. J. Graph Theory 00: 1–7, 2014

1. INTRODUCTION

The *treewidth* $\text{tw}(G)$ of a graph G is a graph invariant used to measure how "tree-like" G is. It is of particular importance in structural and algorithmic graph theory; see the surveys [1,5]. The treewidth $\text{tw}(G)$ is the minimum width of a *tree decomposition* of G, which is defined as follows:

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Definition. A *tree decomposition* of a graph G is a pair $(T, \{A_x \subseteq V(G) : x \in V(T)\})$ such that:

- T is a tree.
- $\{A_x \subseteq V(G) : x \in V(T)\}$ is a collection of sets of vertices of G, each called a *bag*, indexed by the nodes of T.
- For all $v \in V(G)$, the nodes of T indexing the bags containing v induce a nonempty (connected) subtree of T.
- For all $vw \in E(G)$, there exists a bag of T containing both v and w.

The *width* of a tree decomposition is the maximum size of a bag of T, minus 1. This minus 1 is added to ensure that every tree has treewidth 1. Similarly, define the pathwidth of a graph G, denoted $\mathrm{pw}(G)$, to be the minimum width of a tree decomposition where the underlying tree is a path. (We call such a tree decomposition a *path decomposition*.) It follows from the definition that $\mathrm{pw}(G) \geqslant \mathrm{tw}(G)$ for all graphs G.

The line-graph $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$, such that two vertices of $L(G)$ are adjacent when the corresponding edges of G are incident at a vertex.

In recent articles by Marx [4] and Grohe and Marx [3], the treewidth of the line graph of a complete graph is a critical example. For a graph G, let $G^{(q)}$ denote the graph created by replacing each vertex of G with a clique of size q and replacing each edge between two vertices with all of the edges between the two new cliques. Marx [4] shows that if $\mathrm{tw}(G) \geqslant k$, then $G^{(p)}$ contains $L(K_k)^{(q)}$ as a minor (for appropriate choices of p and q, depending on k and $|V(G)|$). Then Grohe and Marx [3] show that $\mathrm{tw}(L(K_n)) \geqslant \frac{\sqrt{2}-1}{4}n^2 + O(n)$. In this article, we determine $\mathrm{tw}(L(K_n))$ exactly. As it turns out, the minimum width tree decomposition that we construct is also a path decomposition. Hence, we prove the following result.

Theorem 1.

$$\mathrm{tw}(L(K_n)) = \mathrm{pw}(L(K_n)) = \begin{cases} \binom{\frac{n-1}{2}}{}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right) + n - 2, & \text{if } n \text{ is odd} \\ \left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right) + n - 2, & \text{if } n \text{ is even} \end{cases}$$

Note the following conventions: if S is a subgraph of a graph G and $x \in V(G) - V(S)$, then let $S \cup \{x\}$ denote the subgraph of G with vertex set $V(S) \cup \{x\}$ and edge set $E(S) \cup \{xy : y \in V(S), xy \in E(G)\}$. Similarly, if $u \in V(S)$, let $S - \{u\}$ denote the subgraph with vertex set $V(S) - \{u\}$ and edge set $E(S) - \{uw : w \in V(S) - \{u\}\}$.

2. LINE-BRAMBLES AND THE TREEWIDTH DUALITY THEOREM

A *bramble* of a graph G is a collection $\mathcal{B}$ of connected subgraphs of G such that each pair of subgraphs $X, Y \in \mathcal{B}$ *touch*. Subgraphs X and Y *touch* when they either have at least one vertex in common, or there exists an edge in G with one end in $V(X)$ and the other in $V(Y)$. The *order* of a bramble is the size of the smallest hitting set H, where a *hitting set* of a bramble $\mathcal{B}$ is a set of vertices H such that $H \cap V(X) \neq \emptyset$ for all $X \in \mathcal{B}$. For a given graph G, the *bramble number* $\mathrm{bn}(G)$ is the maximum order of a bramble of G. Brambles are important due to the following theorem of Seymour and Thomas [6]:

Theorem 2. *(Treewidth Duality Theorem) For every graph G,* $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$.

In this article we employ the following standard approach for determining the treewidth and pathwidth of a particular graph G. First construct a bramble of large order, thus proving a lower bound on $\text{tw}(G)$. Then to prove an upper bound, construct a path decomposition of small width. Given that $\text{tw}(G) \leqslant \text{pw}(G)$, this is sufficient to prove Theorem 1.

In order to construct a bramble of the line graph $L(G)$, define the following:

Definition. A *line-bramble* $\mathcal{B}$ of G is a collection of connected subgraphs of G satisfying the following properties:

- For all $X \in \mathcal{B}$, $|V(X)| \geqslant 2$.
- For all $X, Y \in \mathcal{B}$, $V(X) \cap V(Y) \neq \emptyset$.

Define a *hitting set* for a line-bramble $\mathcal{B}$ to be a set of edges $H \subseteq E(G)$ that intersects each $X \in \mathcal{B}$. Then define the *order* of $\mathcal{B}$ to be the size of the minimum hitting set H of $\mathcal{B}$.

Lemma 3. *Given a line-bramble $\mathcal{B}$ of G, there is a bramble $\mathcal{B}'$ of $L(G)$ of the same order.*

Proof. Given a line-bramble $\mathcal{B}$, define $\mathcal{B}' := \{L(G)[E(X)] : X \in \mathcal{B}\}$. Let $X \in \mathcal{B}$. Since X is connected and $|V(X)| \geqslant 2$, the subgraph X contains an edge. So $E(X)$ induces a nonempty connected subgraph of $L(G)$. Consider $E(X)$ and $E(Y)$ in $\mathcal{B}'$. Thus $V(X) \cap V(Y) \neq \emptyset$. Let v be a vertex in $V(X) \cap V(Y)$. Then there exists some $xv \in E(X)$ and $vy \in E(Y)$, and thus in $L(G)$ there is an edge between the vertex xv and the vertex vy. Hence $E(X)$ and $E(Y)$ touch, and so $\mathcal{B}'$ is a bramble of $L(G)$. All that remains is to ensure $\mathcal{B}$ and $\mathcal{B}'$ have the same order. If H is a minimum hitting set for $\mathcal{B}$, then H is also a set of vertices in $L(G)$ that intersects a vertex in each $E(X) \in \mathcal{B}'$. So H is a hitting set for $\mathcal{B}'$ of the same size. Conversely, if H' is a minimum hitting set of $\mathcal{B}'$, then H' is a set of edges in G that contains an edge in each $X \in \mathcal{B}$. So H' is a hitting set for $\mathcal{B}$. Thus, the orders of $\mathcal{B}$ and $\mathcal{B}'$ are equal. ■

Hence, in order to determine a lower bound on the bramble number $\text{bn}(L(G))$, it is sufficient to construct a line-bramble of G of large order. We will now define a particular line-bramble for any graph G with $|V(G)| \geqslant 3$.

Definition. Given a graph G and a vertex $v \in V(G)$, the *canonical line-bramble for* v of G is the set of connected subgraphs X of G such that either $|V(X)| > \frac{|V(G)|}{2}$, or $|V(X)| = \frac{|V(G)|}{2}$ and X contains v. Note that if $|V(G)|$ is odd, then no elements of the second type occur.

Lemma 4. *For every graph G with $|V(G)| \geqslant 3$ and for all $v \in V(G)$, the canonical line-bramble for v, denoted by $\mathcal{B}$, is a line-bramble of G.*

Proof. By definition, each element of $\mathcal{B}$ is a connected subgraph. Since $|V(G)| \geqslant 3$, each element of $\mathcal{B}$ contains at least two vertices. All that remains to show is that each pair of subgraphs X,Y in $\mathcal{B}$ intersect in at least one vertex. If $|V(X)| = |V(Y)| = \frac{|V(G)|}{2}$, then X and Y intersect at v. Otherwise, without loss of generality, $|V(X)| > \frac{|V(G)|}{2}$ and $|V(Y)| \geqslant \frac{|V(G)|}{2}$. If $V(X) \cap V(Y) = \emptyset$, then $|V(X) \cup V(Y)| = |V(X)| + |V(Y)| > |V(G)|$, which is a contradiction. ■

Let $v \in V(G)$ be an arbitrary vertex and let H be a minimum hitting set of $\mathcal{B}$, the canonical line-bramble for v. Consider the graph $G - H$. Since H is a set of edges,

$V(G-H) = V(G)$. Then each component of $G-H$ contains at most $\frac{|V(G)|}{2}$ vertices, otherwise some component of $G-H$ contains an element of $\mathcal{B}$ that does not contain an edge of H. Similarly, if a component contains $\frac{|V(G)|}{2}$ vertices, it cannot contain the vertex v. Thus, our hitting set H must be large enough to separate G into such components. The next lemma follows directly:

Lemma 5. *Let G be a graph with $|V(G)| \geqslant 3$, let v be a vertex of G, and let $\mathcal{B}$ be the canonical line-bramble for v. Then $H \subseteq E(G)$ is a hitting set of $\mathcal{B}$ if and only if every component of $G-H$ has at most $\frac{|V(G)|}{2}$ vertices, and v is not in a component of $G-H$ that contains exactly $\frac{|V(G)|}{2}$ vertices.*

Note the similarity between this characterization and the *bisection width* of a graph (see [2], for example), which is the minimum number of edges between any $A, B \subset V(G)$ where $A \cap B = \emptyset$ and $|A| = \lfloor \frac{|V(G)|}{2} \rfloor$ and $|B| = \lceil \frac{|V(G)|}{2} \rceil$. (Later we show that most of our components have maximum or almost maximum allowable order.) Given that the components of $G-H$ are what is important, we can also prove the following lemma.

Lemma 6. *Let G be a graph with $|V(G)| \geqslant 3$, let v be a vertex of G, and let $\mathcal{B}$ be the canonical line-bramble for v. If H is a minimum hitting set for $\mathcal{B}$, then no edge of H has both endpoints in the same component of $G-H$.*

Proof. For the sake of a contradiction assume that both endpoints of an edge $e \in H$ are in the same component of $G-H$. Then consider the set $H-e$. By Lemma 5, $H-e$ is a hitting set of $\mathcal{B}$, since the vertex sets of the components of $G-H$ have not changed. But $H-e$ is smaller than the minimum hitting set H, a contradiction. ■

3. PROOF OF THEOREM 1

Let $G := K_n$. When $n \leqslant 2$, Theorem 1 holds trivially, so assume $n \geqslant 3$. First, we determine a lower bound on the treewidth by considering a canonical line-bramble for v, denoted $\mathcal{B}$. Given that K_n is regular, it suffices to choose a vertex v of K_n arbitrarily.

If H is a minimum hitting set of a canonical line-bramble $\mathcal{B}$, label the components of $G-H$ as $Q_1, \ldots, Q_p$ such that $|V(Q_1)| \geqslant |V(Q_2)| \geqslant \ldots \geqslant |V(Q_p)|$. We refer to this as labeling the components *descendingly*.

Consider a pair of components (Q_i, Q_j) where $i < j$ and the components are labeled descendingly. Call this a *good pair* if one of the following conditions hold:

1. $|V(Q_i)| < \frac{n}{2} - 1$,
2. n is even, $|V(Q_i)| = \frac{n}{2} - 1$, $V(Q_j) \neq \{v\}$, and $v \notin V(Q_i)$.

Lemma 7. *Let G be a complete graph with $n \geqslant 3$ vertices, let v be a vertex of G, let $\mathcal{B}$ be the canonical line-bramble for v, and let H be a minimum hitting set of $\mathcal{B}$. If $Q_1, \ldots, Q_p$ are the components of $G-H$ labeled descendingly, then $Q_1, \ldots, Q_p$ does not contain a good pair.*

Proof. Say (Q_i, Q_j) is a good pair. Let x be a vertex of Q_j, such that if (Q_i, Q_j) is of the second type, then $x \neq v$. Let H' be the set of edges obtained from H by removing the edges from x to Q_i and adding the edges from x to Q_j. Then the components for $G-H'$ are $Q_1, \ldots, Q_{i-1}, Q_i \cup \{x\}, Q_{i+1}, \ldots, Q_{j-1}, Q_j - \{x\}, Q_{j+1}, \ldots Q_p$. By Lemma 5, to ensure H' is a hitting set, it suffices to ensure that $V(Q_i) \cup \{x\}$ is sufficiently small, since all other components are the same as in $G-H$, or smaller. If (Q_i, Q_j) is of the first type, then $|V(Q_i) \cup \{x\}| = |V(Q_i)| + 1 < \frac{n}{2}$. If (Q_i, Q_j) is of the second type, then $|V(Q_i) \cup \{x\}| = \frac{n}{2}$, but it does not contain v. Thus, by Lemma 5, H' is a hitting set. However, $|H'| = |H| - |V(Q_i)| + |V(Q_j)| - 1 \leqslant |H| - 1$, which contradicts that H is a minimum hitting set. ■

Lemma 8. *Let $G, v, \mathcal{B}$ and H be as in Lemma 7. Then $G-H$ has exactly three components.*

Proof. Recall by Lemma 5, there is an upper bound on the order of the components of $G-H$. First, show that $G-H$ has at least three components. If $G-H$ has only one component, clearly this component is too large. If $G-H$ has two components and n is odd, then one of the components must have more than $\frac{n}{2}$ vertices. If $G-H$ has two components and n is even, it is possible that both components have exactly $\frac{n}{2}$ vertices; however, one of these components must contain v. Thus $G-H$ has at least three components. Now, assume $G-H$ has at least four components and label the components of $G-H$ descendingly. We show that these components contain a good pair, contradicting Lemma 7.

If n is odd, there is a good pair of the first type when any two components have less than $\frac{n-1}{2}$ vertices. Thus, at least three components have order at least $\frac{n-1}{2}$. Then $|V(G)| \geqslant 3(\frac{n-1}{2}) + 1 > n$ when $n \geqslant 2$, which is a contradiction.

If n is even, there is a good pair of the first type when any two components have less than $\frac{n}{2} - 1$ vertices. Similarly to the previous case, $|V(G)| \geqslant 3(\frac{n}{2} - 1) + 1 > n$, again a contradiction when $n > 4$. If $n = 4$ then each component is a single vertex. Take Q_i, Q_j to be two of these components, neither of which contain the vertex v. Then (Q_i, Q_j) is a good pair of the second type. Hence $G-H$ does not have more than three components, and as such it has exactly three components. ■

Lemma 9. *Let $G, v, \mathcal{B}$ and H be as in Lemma 7, and let the components of $G-H$ be labeled descendingly. If n is odd then $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$ and $|V(Q_3)| = 1$. If n is even then $|V(Q_1)| = \frac{n}{2}$, $|V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$.*

Proof. Lemma 8 shows that $G-H$ has exactly three components. By Lemma 7, (Q_2, Q_3) is not a good pair. Hence $|V(Q_1)| \geqslant |V(Q_2)| \geqslant \frac{n-1}{2}$ when n is odd, and $|V(Q_1)| \geqslant |V(Q_2)| \geqslant \frac{n}{2} - 1$ when n is even, or else there is a good pair of the first type. When n is odd, it follows from Lemma 5 that $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$, and so $|V(Q_3)| = 1$. When n is even, however, $\frac{n}{2} - 1 \leqslant |V(Q_1)|, |V(Q_2)| \leqslant \frac{n}{2}$. Since Q_3 is not empty, it follows that $|V(Q_3)| = 1$ or 2. If $|V(Q_3)| = 1$, then $|V(Q_1)| = \frac{n}{2}$, $|V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$, as required. Otherwise, $|V(Q_1)|, |V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 2$. But then at least one of Q_1, Q_2 does not contain v, and $V(Q_3) \neq \{v\}$. Thus either (Q_1, Q_3) or (Q_2, Q_3) is a good pair of the second type, contradicting Lemma 7. ■

Lemma 10. *Let $G, v, \mathcal{B}$ and H be as in Lemma 7. Then $|H| = (\frac{n-1}{2})(\frac{n-1}{2}) + (n-1)$ when n is odd, and $|H| = (\frac{n-2}{2})(\frac{n}{2}) + (n-1)$ when n is even.*

Proof. From Lemma 9 we know the order of the components of $G-H$. By Lemma 6, H is exactly the set of all edges between each pair of components, and since G is complete there is an edge for each pair of vertices. From this it is easy to calculate $|H|$. ■

Lemma 10 and the Treewidth Duality Theorem imply:

Corollary 11. *Let G be a complete graph with $n \geqslant 3$ vertices. Then*

$$\begin{aligned}\mathrm{pw}(L(G)) &\geqslant \mathrm{tw}(L(G)) \\ &= \mathrm{bn}(L(G)) - 1 \geqslant \begin{cases} \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right) + (n-2), & \text{if } n \text{ is odd} \\ \left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right) + (n-2), & \text{if } n \text{ is even.} \end{cases}\end{aligned}$$

Now, to obtain an upper bound on $\mathrm{pw}(L(G))$, construct a path decomposition of $L(G)$. First, label the vertices of G by $1, \ldots, n$. Let T be an n-node path, also labeled by $1, \ldots, n$. The bag A_i, for the node labeled i, is defined such that $A_i = \{ij \in E(G) : j \in V(G)\} \cup \{uw : u < i < w\}$. For a given A_i, call the edges of $\{ij \in E(G) : j \in V(G)\}$ *initial edges* and call the edges of $\{uw : u < i < w\}$ *crossover edges*. (Note here these edges of G are really acting as vertices of $L(G)$, but refer to them as edges for simplicity.)

Lemma 12. *Let G be a complete graph with $n \geqslant 3$ vertices. Then $(T, \{A_1, \ldots, A_n\})$ is a path decomposition for $L(G)$ of width*

$$\begin{cases} \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right) + (n-2), & \text{if } n \text{ is odd} \\ \left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right) + (n-2), & \text{if } n \text{ is even.} \end{cases}$$

Proof. Each edge uw of G appears in A_u and A_w as an initial edge. Observe that uw is in A_i if and only if $u \leqslant i \leqslant w$, so the nodes indexing the bags containing uw form a connected subtree of T. Finally, all of the edges incident at the vertex u appear in A_u, and the same holds for w, so if two edges of G are adjacent in $L(G)$, they share a bag.

Now determine the size of A_i. The bag A_i contains $n-1$ initial edges and $(i-1)(n-i)$ crossover edges. So $|A_i| = (n-1) + (i-1)(n-i)$. This is maximized when $i = \frac{n+1}{2}$ if n is odd, and when $i = \frac{n}{2}$ or $\frac{n+2}{2}$ if n is even. From this it is possible to calculate the largest bag size, and hence the width of T. ■

Lemma 12 gives an upper bound on $\mathrm{pw}(L(K_n))$ and also on $\mathrm{tw}(L(K_n))$. This, combined with the lower bound in Corollary 11, completes the proof of Theorem 1.

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